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A HOMOLOGICAL CHARACTERIZATION OF LOCAL COMPLETE INTERSECTIONS

by

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Let R denote a local ring with residue field $k = R/\mathfrak{M}$. Let P_R be the Poincaré series of R i.e. the power series

$$P_R = \sum_{p=0}^{\infty} \dim_k \operatorname{Tor}_p^R(k, k) Z^p.$$

It is known that P_R may be written uniquely as a product of the form

$$P_R = \prod_{i=0}^{\infty} \frac{(1 + Z^{2i+1})^{\varepsilon_{2i}}}{(1 - Z^{2i+2})^{\varepsilon_{2i+1}}}$$

where $\varepsilon_q(R) = \varepsilon_q$ ($q = 0, 1, \dots$) are non-negative integers only depending on R (Assmus, Levin). If R is a local complete intersection (i.e. the \mathfrak{M} -adic completion of R is a factor ring of a regular ring \tilde{R} modulo an \tilde{R} -sequence) then it is known that $\varepsilon_q = 0$ for all $q \geq 2$ (Tate, Zariski). If $\varepsilon_2 = 0$ or $\varepsilon_3 = 0$ then R is a complete intersection. The case $\varepsilon_2 = 0$ is due to Assmus, the case $\varepsilon_3 = 0$ is due to the author. Cf. [5].

The purpose of this note is to prove the following:

THEOREM. *If $\varepsilon_q(R) = 0$ for all sufficiently large q , then R is a local complete intersection.*

NOTATION. The term ‘ R -algebra’ will be used in the sense of Tate [6] i.e. an associative, graded, differential, strictly skew-commutative algebra X over R , with unit element 1, such that the homogeneous components X_q are finitely generated modules over R , $X_0 = 1 \cdot R$ and $X_q = 0$ for $q < 0$.

$Z_+(X)$ (resp. $H_+(X)$) will denote the set of homogeneous cycles (resp. homologyclasses) in X of positive degree.

If X is an R -algebra and s is a homogeneous cycle in X , then $X\langle S; dS = s \rangle$ or briefly $X\langle S \rangle$ denotes the R -algebra obtained from X by the adjunction of a variable S which kills s . Cf. [6].

By the Koszul complex over R generated by elements t_1, \dots, t_n in R we mean the R -algebra obtained from the trivial R -algebra R by the adjunction of variables T_1, \dots, T_n of degree 1 killing t_1, \dots, t_n .

LEMMA 1. Let X be an R -algebra satisfying

- (i) $H_0(X) \approx R/\mathfrak{M}$
- (ii) $Z_+(X) \subset \mathfrak{M}X$.

Let $n = \dim \mathfrak{M}/\mathfrak{M}^2$.

Then for all $\sigma \in H_+(X)$ we have $\sigma^{n+1} = 0$.

PROOF. Let s be a cycle representing σ .

Let \mathfrak{M} be minimally generated by t_1, \dots, t_n . By (ii) there exist $x_1, \dots, x_n \in X$ such that

$$s = \sum_{i=1}^n t_i x_i.$$

By (i) we can choose elements T_1, \dots, T_n of degree 1 such that $dT_i = t_i$ for $i = 1, \dots, n$. s is obviously homologous to the cycle

$$s_0 := \sum_{i=1}^n T_i dx_i.$$

Since $T_i^2 = 0$ for all i we have $s_0^{n+1} = 0$, hence $\sigma^{n+1} = 0$.

DEFINITION. Let X be an R -algebra. Define

$$q(X) = \inf \{r | H_i(X) = 0 \text{ for all } i > r\}$$

($\inf \emptyset = \infty$).

LEMMA 2. Let X be an R -algebra satisfying the assumptions (i) and (ii) of lemma 1. Let s be a homogeneous cycle of positive degree in X and put $Y = X\langle S; dS = s \rangle$. Then

$$q(Y) < \infty \Rightarrow q(X) < \infty$$

PROOF. Let us assume that $q(Y) < \infty$. We will consider two cases. First assume that $\deg S$ is even. In this case we have an exact sequence of complexes

$$0 \rightarrow X \xrightarrow{i} Y \xrightarrow{j} Y \rightarrow 0$$

where i and j are maps of degree 0 and $-\deg S$ respectively. Cf. [6]. Looking at the associated exact homology sequence one sees that $q(X) < \infty$.

Let us now consider that case where $\deg S$ is odd. In this case we have an exact sequence of complexes

$$0 \rightarrow X \xrightarrow{i} Y \xrightarrow{j} X \rightarrow 0$$

where i and j have degrees 0 and $-\deg S$ respectively and where the con-

necting homomorphism d_* in the associated homology triangle

$$\begin{array}{ccc}
 & H(Y) & \\
 i_* \nearrow & & \searrow j_* \\
 H(X) & \xleftarrow{d_*} & H(X)
 \end{array}$$

is, up to sign, multiplication by σ , see the proof of theorem 2 in [6]. Now put $n = \dim \mathfrak{M}/\mathfrak{M}^2$ and $v = \deg \sigma$. Using (1) we obtain for each $r > q(Y)$ an exact sequence

$$H_r(X) \xrightarrow{d_*} H_{r+v}(X) \rightarrow H_{r+v}(Y) = 0$$

Hence $H_{r+v}(Y) = \sigma H_r(X)$ for $r > q(Y)$. It follows that

$$H_{r+(n+1)v}(X) = \sigma^{n+1} H_r(X) \quad \text{for } r > q(Y).$$

By Lemma 1 we have $\sigma^{n+1} = 0$. It follows that $q(X) < \infty$.

PROOF OF THE THEOREM: It is enough to prove the theorem for complete local rings, hence we may assume that there exists a regular ring \tilde{R} and a surjective ringhomomorphism $f: \tilde{R} \rightarrow R$. Put $\mathfrak{A} = \ker f$. We may also assume that \mathfrak{A} is contained in the square of the maximal ideal $\tilde{\mathfrak{M}}$ in \tilde{R} .

Let a_1, \dots, a_c be a maximal \tilde{R} -sequence in \mathfrak{A} and let \mathfrak{A}' be the ideal generated by a_1, \dots, a_c . This sequence can be chosen such that it can be extended to a minimal set of generators for \mathfrak{A} , i.e. the canonical map $\mathfrak{A}' \otimes k \rightarrow \mathfrak{A} \otimes k$ is injective. Put $R' = \tilde{R}/\mathfrak{A}'$ and let $g: R' \rightarrow R$ be the homomorphism induced by $f: \tilde{R} \rightarrow R$.

Now assume that $\varepsilon_q(R) = 0$ for all q sufficiently large. We will show that $\ker g = 0$. It suffices to show that R is an R' -module of finite projective dimension. Indeed, by construction every element in $\ker g$ is a zero-divisor in the ring R' . Hence if $pd_{R'} R < \infty$ it follows from proposition 6.2 in [3] that $\ker g = 0$.

Let \tilde{E} be the Koszul complex generated over \tilde{R} by a minimal set of generators for $\tilde{\mathfrak{M}}$. Since $\ker f \subset \tilde{\mathfrak{M}}^2$, the rings \tilde{R}, R' and R have the same imbedding dimension. Thus, putting $E' := \tilde{E} \otimes_{\tilde{R}} R'$ and $E := \tilde{E} \otimes_{\tilde{R}} R = E' \otimes_{R'} R$, E' and E will be Koszul complexes generated over R' and R by minimal sets of generators for \mathfrak{M}' and \mathfrak{M} respectively.

Let s_1, \dots, s_c be cycles representing a basis for the k -module $H_1(E')$. Put $F' = E' \langle S_1, \dots, S_c; dS_i = s_i \rangle$. Then F' is a minimal R' -free resolution of k . Consider the R -algebra $F := F' \otimes_{R'} R$ which contains the R -algebra E . Since the map $\mathfrak{A}' \otimes k \rightarrow \mathfrak{A} \otimes k$ is injective, the images of s_1, \dots, s_c in $H_1(E)$ can be extended to a basis for $H_1(E)$. Indeed, we have a commutative diagram

$$\begin{array}{ccc}
 H_1(E') & \xrightarrow{\approx} & \mathfrak{A}' \otimes k \\
 \downarrow & & \downarrow \\
 H_1(E) & \xrightarrow{\approx} & \mathfrak{A} \otimes k
 \end{array}$$

where the left vertical map is induced by the obvious map $E' \rightarrow E$. Therefore, by the theorem in [4], F can be extended to a minimal R -algebra resolution X of k of the form $X = F \langle \cdots U_i \cdots \rangle$. Since for $q \geq 2 \varepsilon_q(R)$ is the number of variables of degree $q+1$ adjoined to F in order to obtain X , and since $\varepsilon_q(R) = 0$ for all sufficiently large q , X has in fact the form

$$X = F \langle U_1, \cdots, U_r \rangle = F \langle U_1 \rangle \cdots \langle U_r \rangle.$$

for some $r \geq 0$.

Every sub- R -algebra of X containing F satisfies the assumptions (i) and (ii) of lemma 1. Since X is acyclic we have $q(X) = 0 < \infty$. Hence, using lemma 2, r times we obtain

$$q(F) < \infty.$$

But $H(F) = \text{Tor}^{R'}(k, R)$, hence R has finite projective dimension over R' .

REMARK. In [1] André defines homology groups $H.(A, B, W)$ where B is a commutative algebra over a commutative ring A , and W is a B -module. For a local ring R with residue field k he defines the simplicial dimension of R as follows

$$s\text{-dim } R = \inf \{r | H_i(R, k, k) = 0 \text{ for } i \geq r\} \text{ (inf } \emptyset = \infty).$$

In studying the relationship between $\text{Tor}^R(k \cdot k)$ and $H.(R, k, k)$ we are tempted to conjecture that $\varepsilon_i(R) = \dim_k H_{i+1}(R, k, k)$ for $i \geq 0$. It would follow from this that the only local rings of finite simplicial dimension are the complete intersection.

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