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## EQUIVALENCE OF ALMOST PERIODIC COMPACTIFICATIONS

by

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### 1. Introduction

1.1. Purpose of this note is to give a simple proof of [1], Theorem 5.5 (see Corollary 2.4 below).

Let  $S, T$  be semitopological semigroups ([2], p. 1). We do not assume  $S$  and  $T$  to have an identity. The Banach space of almost periodic (a.p.) functions on  $S$  is denoted by  $A(S)$  and the Banach space of weakly almost periodic functions on  $S$  is denoted by  $W(S)$ . If  $\phi : S \rightarrow T$  is a continuous homomorphism, then the induced mapping  $f \mapsto f \circ \phi$  of  $C(T)$  into  $C(S)$  is denoted by  $\tilde{\phi}$ .

Recall that  $\tilde{\phi}[C(T)] \subset W(S)$  if  $T$  is a compact semitopological semigroup, and that  $\tilde{\phi}[C(T)] \subset A(S)$  if  $T$  is a compact topological semigroup ([1], lemma 5.2).

1.2. An ordered pair  $(\phi, T)$  is called an *almost periodic compactification* (a *weakly almost periodic compactification*) of  $S$ , if the following conditions are satisfied:

- (i)  $T$  is a compact topological (semitopological) Hausdorff semigroup.
- (ii)  $\phi : S \rightarrow T$  is a continuous homomorphism such that for each  $f \in A(S)$  ( $f \in W(S)$ ) there is a unique  $\tilde{f} \in C(T)$  with  $f = \tilde{f} \circ \phi$ .

Note that (ii) is equivalent with

- (ii)'  $\phi : S \rightarrow T$  is a continuous homomorphism with dense image in  $T$ , and  $\tilde{\phi}[C(T)] = A(S)$  ( $\tilde{\phi}[C(T)] = W(S)$ ).

(The proof depends on 1.1 and the fact, that  $T$  is a completely regular topological space.)

1.3. Let  $S$  be a semitopological semigroup. Two a.p. compactifications (w.a.p. compactifications)  $(\phi_1, T_1)$  and  $(\phi_2, T_2)$  are called *equivalent*, if there is a topological isomorphism  $\psi$  of  $T_1$  onto  $T_2$  such that  $\psi \circ \phi_1 = \phi_2$ .

We shall prove, that two a.p. compactifications (w.a.p. compactifications) of a semitopological semigroup are equivalent ([1], Corollary 5.6); in fact, this is an easy corollary of the 'universal property' of the a.p. and

w.a.p. compactifications, explained in Theorem 2.2 of this note. From this, Theorem 5.5 of [1] is also easily derived.

All semitopological semigroups are not supposed to have an identity (so the existence of the compactifications is not a priori guaranteed).

## 2. The main theorem

**2.1. LEMMA.** *Let  $X$  be a uniform space,  $Y$  a compact Hausdorff topological space. A function  $f : X \rightarrow Y$  is uniformly continuous ( $Y$  with its unique uniformity) if and only if  $g \circ f : X \rightarrow [0, 1]$  is uniformly continuous for all continuous  $g : Y \rightarrow [0, 1]$ .*

**PROOF:** It is known that  $Y$  may be regarded as a (closed) subset of a topological product of copies of the interval  $[0, 1]$  such that the restrictions to  $Y$  of the canonical projections are the continuous functions of  $Y$  into  $[0, 1]$ . The lemma now follows from [3], Ch. 6, Theorem 10.

**2.2. THEOREM.** *Let  $(\phi, S^a)$  be an a.p. compactification of  $S$  and  $(\psi, S^w)$  a weakly a.p. compactification. Then  $(\phi, S^a)$  and  $(\psi, S^w)$  have the following 'universal' property:*

*If  $T$  is a topological (resp. semitopological) compact Hausdorff semigroup and  $\xi : S \rightarrow T$  a continuous homomorphism, then there exists a unique continuous homomorphism  $\xi^a : S^a \rightarrow T$  (resp.  $\xi^w : S^w \rightarrow T$ ) such that  $\xi = \xi^a \circ \phi$  (resp.  $\xi = \xi^w \circ \psi$ ).*

**PROOF:** Unicity follows from the fact that  $T$  is a Hausdorff topological space and that  $\phi$  and  $\psi$  have dense images.

We now prove the existence of  $\xi^w$  in the case that  $T$  is a semitopological compact Hausdorff semigroup (the existence of  $\xi^a$  in the case that  $T$  is a topological compact Hausdorff semigroup can be proved in a similar way). First, we note that

$$(1) \quad \forall s, t \in S : \psi(s) = \psi(t) \Rightarrow \xi(s) = \xi(t)$$

Indeed:  $C(T)$  separates the points of  $T$ , so

$$\xi(s) \neq \xi(t) \Rightarrow \exists g \in C(T) : g(\xi(s)) \neq g(\xi(t)).$$

Since  $g \circ \xi \in W(S)$  it follows from 1.2 that in this case  $\psi(s) \neq \psi(t)$ .

Defining  $\psi' : S \rightarrow \text{Im } \psi$  by  $\psi = i \circ \psi'$  ( $i : \text{Im } \psi \rightarrow S^w$  is the inclusion map) it follows from (1) that there is a homomorphism  $\xi' : \text{Im } \psi \rightarrow T$  such that  $\xi = \xi' \circ \psi$ .

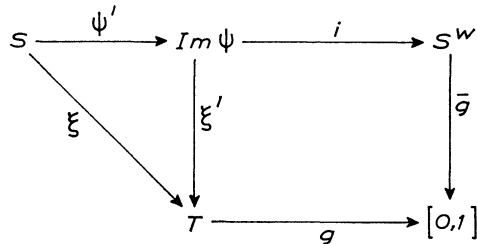
Now it is sufficient to show that  $\xi'$  is uniformly continuous ( $\text{Im } \psi$  provided with the relative uniformity of  $S^w$ ):  $T$  is complete as a uniform space ([3], Ch. 6, Theorem 32), so in this case  $\xi'$  has a uniformly con-

tinuous extension  $\xi^w : S^w \rightarrow T$  ([3], Ch. 6, Theorem 26). It is easy to see that this continuous extension  $\xi^w$  of the homomorphism  $\xi'$  is itself a homomorphism, using separate continuity of multiplication in  $S^w$  and  $T$ .

Uniform continuity of  $\xi'$  will follow from 2.1 if we succeed in proving the following assertion: if  $g : T \rightarrow [0, 1]$  is any continuous map, then  $g \circ \xi' : \text{Im } \psi \rightarrow [0, 1]$  is uniformly continuous.

By 1.1,  $g \circ \xi \in W(S)$ , so there is a  $\bar{g} \in C(S^w)$  such that  $g \circ \xi = \bar{g} \circ \psi$ , that is

$$g \circ \xi' \circ \psi' = \bar{g} \circ i \circ \psi'$$



Since  $S$  is mapped onto  $\text{Im } \psi$  by  $\psi'$ , it follows that

$$g \circ \xi' = \bar{g} \circ i.$$

Now uniform continuity of  $g \circ \xi'$  follows from uniform continuity of  $i$  and  $\bar{g}$ .

2.3. COROLLARY ([1], Corollary 5.6). *Two a.p. compactifications (weakly a.p. compactifications) of a semitopological semigroup are equivalent.*

PROOF: Let  $(\phi_1, T_1)$  and  $(\phi_2, T_2)$  be weakly a.p. compactifications of  $S$ . By 2.2 there are continuous homomorphisms  $\phi_1^w : T_2 \rightarrow T_1$  and  $\phi_2^w : T_1 \rightarrow T_2$  such that

$$\phi_2^w \circ \phi_1 = \phi_2 \quad \text{and} \quad \phi_1^w \circ \phi_2 = \phi_1.$$

We have to prove, that  $\phi_1$  (or  $\phi_2$ ) is a topological isomorphism. It is easy to see, that

$$(\phi_1^w \circ \phi_2^w) \circ \phi_1 = \phi_1 = I_1 \circ \phi_1$$

so by unicity it follows that  $\phi_1^w \circ \phi_2^w = I_1$ .

Similarly  $\phi_2^w \circ \phi_1^w = I_2$  ( $I_i$  denotes the identity mapping of  $T_i$  for  $i = 1, 2$ ). From this the result follows.

2.4. COROLLARY ([1], Theorem 5.5). *Let  $S$  and  $S_1$  be semitopological semigroups,  $(\phi, S')$  and  $(\phi_1, S'_1)$  a.p. compactifications (resp. w.a.p.*

compactifications) of  $S$  and  $S_1$  and  $\psi : S \rightarrow S_1$  a continuous homomorphism. Then there is a unique continuous homomorphism  $\psi' : S' \rightarrow S'_1$  for which  $\psi' \circ \phi = \phi_1 \circ \psi$ .

PROOF: Existence: take  $\psi' = (\phi_1 \circ \psi)^a$  (respectively  $\psi' = (\phi_1 \circ \psi)^w$ ; notation as in 2.2).

Unicity follows from unicity in 2.2: if  $\psi'' : S' \rightarrow S'_1$  satisfies  $\psi'' \circ \phi = \phi_1 \circ \psi$ , then  $\psi'' = (\phi_1 \circ \psi)^a$  (respectively  $\psi'' = (\phi_1 \circ \psi)^w$ ).

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