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A RESULT CONCERNING MEROMORPHIC SOLUTIONS IN THE UNIT DISK OF ALGEBRAIC DIFFERENTIAL EQUATIONS (*)

by

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1. Introduction

In [4], Valiron proved that an analytic function $g(z)$ in the unit disk, which is a solution of a first order equation $P(z, y, y') = 0$, where P is a polynomial, must be of finite order of growth (i.e. $\limsup_{r\to 1} [\log (\log M(r; g))/\log((1-r)^{-1})] < +\infty$, where $M(r; g)$ is the maximum modulus of g). We treat here a broader class of equations, namely those first order equations of the form $\Omega(z, y, y') = \sum Q_{kj}(z)y^k(y')^j = 0$, where Ω is a polynomial in y and y', whose coefficients are subject to the following restriction: If $p = \max\{k+j : Q_{kj} \not\equiv 0\}$, then we allow $Q_{kj}(z)$ to be an analytic function of finite order in the unit disk if $k+j < p$, while if $k + j = p$, we require $Q_{kj}(z)$ to be a polynomial.

In this paper we investigate solutions of $\Omega(z, y, dy/dz) = 0$ which are meromorphic in the unit disk (i.e. meromorphic functions $h(z)$ in $|z| < 1$ such that $\Omega(z, h(z), h'(z)) = 0$ at each point z where h is analytic). More specifically, we deal with the following problem: If a meromorphic solution in $|z| < 1$ is represented as the quotient of two analytic functions in $|z| < 1$, what can be said about the orders of growth of these analytic functions? In particular, we would like to show that a meromorphic solution $h(z)$ in $|z| < 1$, cannot be written as the quotient of two analytic functions f/g, where g is of finite order in $|z| < 1$ and f is of infinite order in $|z| < 1$. However, we cannot prove this result without an additional assumption on $h(z)$ because of the following difficulty which arises: Our technique consists in viewing a first order equation which has $h = f/g$ for a solution, as a relation between f, g and their logarithmic derivatives. Now if f is of infinite order in $|z| < 1$, then the Valiron–Wiman theory for the unit disk [4; p. 299], provides a useful relation between the values of $f(z)$ and $f'(z)/f(z)$ at certain points on some

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sequence of 'remarkable' circles $|z| = r_n$ where $\lim_{n\to\infty} r_n = 1$. These certain points on $|z| = r_n$ are those z where $|f(z)|$ is sufficiently close to $M(r_n;f)$ in the sense that $|f(z)| \geqq M(r_n;f)(\log M(r_n;f))^{-\lambda}$ for some non-negative constant $\lambda < 1/2$. On the other hand, if $g(z)$ is of finite order in $|z| < 1$, then of course, as $|z| \to 1$, $|g(z)|$ is easily estimated, but obviously as $|z| \to 1$, there is difficulty in estimating the growth of $|g'(z)/g(z)|$ whenever z is in the immediate neighbourhood of a zero of g. We prove (see § 4 below) that there exist small disks around the zeros of g such that if D is the union of these disks, then as $|z| \to 1$ in the complement of D, $|g'(z)/g(z)| = 0((1-|z|)^{-A})$ for some $A \geqq 0$. Thus the information on $f(z)$, $f'(z)/f(z)$, $g(z)$ and $g'(z)/g(z)$ can be used together with the differential equation (and will yield a contradiction) provided an additional assumption is imposed to insure that as $r \to 1$, there is at least one point on $|z| = r$, not in D, at which $|f(z)|$ is sufficiently close to $M(r;f)$ in the sense that $|f(z)| \geqq M(r;f)(\log M(r;f))^{-\lambda}$. Thus if $W(r,f;g)$ denotes $\max\{|f(z)| : |z| = r, z \notin D\}$, then our main result states that a meromorphic function $h(z)$ in $|z| < 1$, which is a solution of the first order equation $\Omega(z, y, y') = 0$, cannot be written as the quotient of two analytic functions f/g where g is of finite order in $|z| < 1$ while f is of infinite order in $|z| < 1$, and where

$$\text{(A)} \qquad \limsup_{r\to 1} \frac{M(r;f)(\log M(r;f))^{-\lambda}}{W(r,f;g)} < +\infty \text{ for some } \lambda < \tfrac{1}{2}.$$

The condition (A) is of course automatically satisfied for the broad class of functions f/g, where g is of finite order in $|z| < 1$, f is of infinite order in $|z| < 1$, and where for all r in some interval $[r_0, 1)$, $|f(z)|$ assumes its maximum value $M(r;f)$ at some point on $|z| = r$ not lying in D. (In this case, $M(r;f) = W(r,f;g)$ on $[r_0, 1)$.) In the other case (i.e. where $W(r,f;g) < M(r;f)$ for a sequence of r tending to 1), we show in § 6 that there are examples where condition (A) is satisfied and examples where it is not satisfied.

In view of our main result, all analytic solutions of equations in the class treated here must be of finite order in the unit disk. However, we remark that such equations can possess meromorphic solutions of infinite order in the disk (e.g. $(\sin \exp (1/(1-z)))^{-1}$).

2. Definition

Let $g(z)$ be an analytic function in $|z| < 1$ which is not identically zero and which is of finite order in $|z| < 1$. Let $a_1, a_2, \cdots$ be the sequence of zeros of $g(z)$ in $0 < |z| < 1$, arranged so that $|a_1| \leqq |a_2| \leqq \cdots$. Since g

is of finite order, there exists $\sigma \geqq 0$ such that $\sum_{n\geqq 1}(1-|a_n|)^\sigma < +\infty$ (see [1; p. 140]). Following Tsuji [3], let $\mu \geqq 0$ be the convergence exponent of the sequence $\{a_n\}$ (i.e. $\mu = 0$ if $\sum_{n\geqq 1}(1-|a_n|) < +\infty$, while if $\sum_{n\geqq 1}(1-|a_n|) = +\infty$, then let $\mu \geqq 0$ be such that $\sum_{n\geqq 1}(1-|a_n|)^{\mu+1-\varepsilon} = +\infty$ and $\sum_{n\geqq 1}(1-|a_n|)^{\mu+1+\varepsilon} < +\infty$ for every $\varepsilon > 0$). For each n, let D_n be the disk $|z-a_n| < (1-|a_n|)^{\mu+4}$, and let $D = \bigcup_{n\geqq 1} D_n$. Then clearly the closure of each D_n lies in $|z| < 1$, and it is proved in [3; p. 14] that the radii of the D_n satisfy the condition

$$(1) \qquad \sum_{r\leqq |a_n|<1}(1-|a_n|)^{\mu+4} = 0((1-r)^2) \quad \text{as} \quad r \to 1.$$

(It easily follows that for any $r \in [0, 1)$, there is an $r' \in (r, 1)$ such that the circle $|z| = r'$ is disjoint from D).

Let $f(z)$ be any analytic function in $|z| < 1$ which is not identically zero. We define the *maximum modulus of f relative to the zeros of g* to be the function,

$$(2) \qquad W(r,f;g) = \max\{|f(z)| : |z| = r, z \notin D\}.$$

It follows easily from (1) that there exists an interval $[r^*, 1)$ on which $W(r,f;g)$ is defined and is nowhere zero.

3. Main theorem

Let $\Omega(z, y, y') = \sum Q_{kj}(z)y^k(y')^j$ be a polynomial in y and y' whose total degree in y and y' is p. If $k+j < p$, let $Q_{kj}(z)$ be analytic and of finite order in $|z| < 1$, while if $k+j = p$, let $Q_{kj}(z)$ be a polynomial. Then a meromorphic function $h(z)$ in $|z| < 1$, which is a solution of $\Omega(z, y, y') = 0$, cannot be written as the quotient of two analytic functions f/g in $|z| < 1$, where $g(z)$ is of finite order in $|z| < 1$, while $f(z)$ is of infinite order in $|z| < 1$, and where for some real number $\lambda < 1/2$,

$$(3) \qquad \limsup_{r\to 1} \frac{M(r;f)(\log M(r;f))^{-\lambda}}{W(r,f;g)} < +\infty.$$

4.

Before proving the main result, we prove a lemma which shows the significance of the disks D_n of § 2.

LEMMA A: *Let $g(z)$ be an analytic function in $|z| < 1$ which is not identically zero and which is of finite order in $|z| < 1$. Let $\{a_n\}$ be the sequence of zeros of g in $0 < |z| < 1$, and let $\mu \geqq 0$ be the convergence exponent of $\{a_n\}$. For each n, let D_n be the disk $|z-a_n| < (1-|a_n|)^{\mu+4}$,* *and let D be the union of the D_n. Then there exist real numbers $r_0 \in [0, 1)$, $L_0 > 0$ and $A > 0$ such that for $r \in [r_0, 1)$,*

(4) $$|g'(z)/g(z)| \leqq L_0(1-r)^{-A} \quad \text{on} \quad |z| = r \quad \text{if} \quad z \notin D.$$

PROOF: Let q be a non-negative integer such that

(5) $$\sum_{n \geqq 1} (1-|a_n|)^{q+1} \text{ converges, say} \sum_{n \geqq 1} (1-|a_n|)^{q+1} = b.$$

Let $E_{q+1}(u) = (1-u) \exp [u+u^2/2+ \cdots +u^{q+1}/(q+1)]$, and let

$$S(z) = \prod_{n \geqq 1} E_{q+1} \left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right).$$

It is easily verified that for any $\varepsilon > 0$, there exists n_0 such that for $n \geqq n_0$, $1-|a_n|^2 \leqq (1/2)|1-\bar{a}_n z|$ on $|z| \leqq 1-\varepsilon$. Hence in view of the inequality, $|E_{q+1}(u)-1| \leqq 3|u|^{q+2}$ for $|u| \leqq 1/2$ (see [2; p. 297]), and the fact that $\sum_{n \geqq 1} (1-|a_n|)^{q+2}$ converges (in view of (5)), it is easy to see that the product defining $S(z)$ converges uniformly in each compact subset of $|z| < 1$, and hence $S(z)$ represents an analytic function in $|z| < 1$ whose sequence of zeros is $\{a_n\}$. We note here that $S(z)$ differs slightly from the canonical product of Tsuji [3; p. 8], which is defined as

$$\prod_{n \geqq 1} E_t \left(\frac{1-|a_n|^2}{1-\bar{a}_n z}\right),$$

where t is the smallest non-negative integer such that $\sum_{n \geqq 1} (1-|a_n|)^{t+1}$ converges. However, it is easy to verify that the proof and conclusion of [3; Theorem 1] on the growth of the canonical product, hold for our function $S(z)$ with t replaced by $q+1$. That is,

(6) $$\log^+ |S(z)| \leqq 2^{q+2} \sum_{n \geqq 1} \left\{\frac{1-|a_n|^2}{|1-\bar{a}_n z|}\right\}^{q+2}.$$

Since $|1-\bar{a}_n z| \geqq 1-|z|$, and since $\sum_{n \geqq 1} (1-|a_n|^2)^{q+2}$ converges (in view of (5)), it follows from (6) that

(7) $S(z)$ is of finite order $\leqq q+2$ in $|z| < 1$.

Let $g(z)$ have a k-fold root at $z = 0$ ($k \geqq 0$). Then $z^{-k}g(z)$ and $S(z)$ are both analytic and have the same sequence of zeros in $|z| < 1$. Hence there is an analytic function $\varphi(z)$ in $|z| < 1$ which is nowhere zero, such that $z^{-k}g(z) = \varphi(z)S(z)$. We may write $\varphi(z) = e^{\psi(z)}$ where ψ is analytic in $|z| < 1$, and hence,

(8) $$g(z) = z^k e^{\psi(z)} S(z).$$

Thus,

$$g'(z)/g(z) = kz^{-1}+\psi'(z)+S'(z)/S(z). \tag{9}$$

Now an easy computation (see [2; p. 292]) yields,

$$S'(z)/S(z) = \sum_{n\geqq 1}\Gamma(z, a_n) \text{ for } z\notin\{a_n : n\geqq 1\}, \tag{10}$$

where,

$$\Gamma(z, a_n) = (z-a_n)^{-1}((1-|a_n|^2)/(1-\bar{a}_n z))^{q+2}. \tag{11}$$

Let $z\notin D$. Then there are only finitely many n such that $(1-|a_n|^2)\geqq (1/2)|1-\bar{a}_n z|$ since for such n, $|a_n|\leqq ((1+|z|)/2)^{\frac{1}{2}}$. We write (10) as,

$$S'(z)/S(z) = \sum_1 \Gamma(z, a_n)+\sum_2 \Gamma(z, a_n), \tag{12}$$

where $\sum_1$ corresponds to all n such that $(1-|a_n|^2)\geqq (1/2)|1-\bar{a}_n z|$, while $\sum_2$ corresponds to all n such that $(1-|a_n|^2) < (1/2)|1-\bar{a}_n z|$.

We first consider $\sum_1$. Since $z\notin D$, $|z-a_n|\geqq (1-|a_n|)^{\mu+4}$ for each n in $\sum_1$. Thus $|z-a_n|\geqq 2^{-(\mu+4)}(1-|a_n|^2)^{\mu+4}$, and hence by assumption about $\sum_1$, $|z-a_n|\geqq 2^{-(2\mu+8)}|1-\bar{a}_n z|^{\mu+4}$. Thus,

$$\sum_1 |\Gamma(z, a_n)| \leqq 2^{2\mu+8}\sum_1 (1-|a_n|^2)^{q+2}|1-\bar{a}_n z|^{-(q+\mu+6)}. \tag{13}$$

Since $|1-\bar{a}_n z|\geqq 1-|z|$ and since $\sum_{n\geqq 1}(1-|a_n|^2)^{q+2}\leqq 2^{q+1}b$ (by (5)), we have

$$\sum_1 |\Gamma(z, a_n)| \leqq 2^{2\mu+q+9}b(1-|z|)^{-(q+\mu+6)} \tag{14}$$

Now consider $\sum_2$. For such an n, $(1-|a_n|^2) < (1/2)|1-\bar{a}_n z|$. It easily follows that $|z-(1/\bar{a}_n)| > 2(1-|a_n|^2)/|a_n|$. Since $|z-(1/\bar{a}_n|)\leqq |z-a_n|+|a_n-(1/\bar{a}_n)|$, we easily obtain, $|z-a_n|\geqq (1-|a_n|^2)/|a_n|$. Thus,

$$\sum_2 |\Gamma(z, a_n)| \leqq \sum_2 |a_n|(1-|a_n|^2)^{q+1}|1-\bar{a}_n z|^{-(q+2)}. \tag{15}$$

Since $|a_n| < 1$ and $|1-\bar{a}_n z|\geqq 1-|z|$, we have (in view of (5)),

$$\sum_2 |\Gamma(z, a_n)| \leqq 2^{q+1}b(1-|z|)^{-(q+2)}. \tag{16}$$

In view of (12), (14) and (16), we obtain for $z\notin D$,

$$|S'(z)/S(z)| \leqq K_1(1-|z|)^{-(q+\mu+6)}, \tag{17}$$

where $K_1 = (2^{2\mu+q+9}+2^{q+1})b$.

We now consider the function $\varphi(z) = e^{\psi(z)}$, which by (8) satisfie- $\varphi(z) = z^{-k}g(z)/S(z)$. Letting $T(r, -)$ represent the Nevanlinna characs teristic [1; p. 12], we have by [1; pp. 11, 14] that

$$(18) \qquad T(r, \varphi) \leqq T(r, g) + T(r, z^k) + T(r, S) + K_2,$$

where K_2 is a constant. By hypothesis, g is of finite order in $|z| < 1$, and by (7), S is of finite order in $|z| < 1$. Since z^k is also of finite order in $|z| < 1$ (being bounded), it follows easily from (18) that φ is of finite order in $|z| < 1$. Thus there exist $r^* \in [0, 1)$ and $\sigma > 0$ such that for $r \in [r^*, 1)$,

$$(19) \qquad |\varphi(z)| \leqq \exp((1-r)^{-\sigma}) \quad \text{on} \quad |z| = r.$$

Since $\varphi = e^{\psi}$, we thus have,

$$(20) \qquad \operatorname{Re} \psi(z) \leqq (1-r)^{-\sigma} \quad \text{on} \quad |z| = r \quad \text{when} \quad r \in [r^*, 1).$$

But by an inequality of Caratheodory [2; p. 338], if $r \leqq R < 1$, $M(r; \psi) \leqq (R-r)^{-1}(R+r)(A(R) + |\psi(0)|)$ where $A(R) = \max_{|z|=R} \operatorname{Re} \psi(z)$. Applying this with $R = (1+r)/2$, where $r \in [r^*, 1)$, we obtain (in view of (20)),

$$(21) \qquad M(r, \psi) \leqq K_3(1-r)^{-(\sigma+1)} \quad \text{where} \quad K_3 > 0.$$

Let $r_0 = \max\{r^*, 1/2\}$. Then if $r \in [r_0, 1)$ and z is on $|z| = r$, we have by the Cauchy formula for derivatives that

$$\psi'(z) = (2\pi i)^{-1} \int_J (\psi(\zeta)/(\zeta - z)^2) d\zeta$$

where $J : |\zeta - z| = (1-r)/2$. In view of (21) we obtain,

$$(22) \quad |\psi'(z)| \leqq 2^{\sigma+2} K_3(1-r)^{-(\sigma+2)} \quad \text{on} \quad |z| = r \quad \text{if} \quad r \in [r_0, 1).$$

Finally, since $r_0 \geqq 1/2$, we have

$$(23) \qquad |kz^{-1}| \leqq 2k \quad \text{on} \quad |z| = r \quad \text{if} \quad r \in [r_0, 1).$$

In view of (9), (17), (22) and (23), if we set $A = \max\{q+\mu+6, \sigma+2\}$ and $L_0 = K_1 + 2^{\sigma+2}K_3 + 2k$, then for $r \in [r_0, 1)$, we have

$$(24) \qquad |g'(z)/g(z)| \leqq L_0(1-r)^{-A} \quad \text{if} \quad |z| = r \quad \text{and} \quad z \notin D.$$

This proves Lemma A.

5. Proof of the theorem of § 3

We assume the contrary and suppose that there exists a meromorphic function $h(z)$ in $|z| < 1$ which is a solution of the first order algebraic differential equation $\Omega(z, y, y') = 0$, and which can be written as the quotient of two analytic functions f/g, where g is of finite order in $|z| < 1$, f is of infinite order in $|z| < 1$ and condition (3) holds.

Thus,

$$h = f/g. \tag{26}$$

By hypothesis concerning the coefficients $Q_{kj}(z)$, there exist constants $L > 0$, $\Delta > 0$ and r^* $(0 \leqq r^* < 1)$ such that,

$$(27)\quad \begin{aligned} &|Q_{kj}(z)| \leqq L \quad \text{on} \quad |z| < 1 \quad \text{if} \quad k+j = p, \quad \text{and} \\ &|Q_{kj}(z)| \leqq \exp((1-r)^{-\Delta}) \quad \text{on} \quad |z| = r > r^*, \quad \text{if} \quad k+j < p. \end{aligned}$$

Let m be defined by,

$$m = \max\{j : Q_{p-j,j}(z) \not\equiv 0\}, \tag{28}$$

and consider the coefficient $Q_{p-m,m}$. We now prove,

LEMMA B: *There exist real numbers $r_1 \in [r^*, 1)$, $c \geqq 0$ and $L_1 > 0$ such that when $r \in [r_1, 1)$,*

$$|Q_{p-m,m}(z)| \geqq L_1(1-r)^c \quad \text{on} \quad |z| = r. \tag{29}$$

PROOF: We may write $Q_{p-m,m}(z) = K(z-b_1)\cdots(z-b_d)$, where $K \neq 0$ and where the roots $b_1, \cdots, b_d$ are so arranged that $b_1, \cdots, b_s$ lie in $|z| < 1$, $b_{s+1}, \cdots, b_t$ lie on $|z| = 1$ and $b_{t+1}, \cdots, b_d$ lie in $|z| > 1$. Let $\gamma = \max\{|b_j| : 1 \leqq j \leqq s\}$ and let $r_1 = (1+\gamma)/2$. Then for $r \in [r_1, 1)$ and $j = 1, \cdots, s$ we clearly have $|z-b_j| \geqq (1-\gamma)/2$ on $|z| = r$. For $j = s+1, \cdots, t$, we have $|b_j| = 1$ so clearly $|z-b_j| \geqq 1-r$ on $|z| = r$. Finally if $\gamma_1 = \min\{|b_j| : t+1 \leqq j \leqq d\}$, then $\gamma_1 > 1$ and clearly, $|z-b_j| \geqq \gamma_1 - 1$ on $|z| < 1$ for $j = t+1, \cdots, d$. Setting $c = t-s$ and $L_1 = |K|((1-\gamma)/2)^s(\gamma_1-1)^{d-t}$, we clearly obtain (29), proving Lemma B.

Since $g(z)$ is of finite order in $|z| < 1$, there exist real numbers $r_2 \in [0, 1)$ and $B > \Delta$ such that

$$|g(z)| \leqq \exp((1-r)^{-B}) \quad \text{on} \quad |z| = r \quad \text{when} \quad r \in [r_2, 1). \tag{30}$$

Letting $\{a_n\}$ be the sequence of zeros of g in $0 < |z| < 1$ with convergence exponent $\mu \geqq 0$ and letting D be the union of the disks $|z-a_n| < (1-|a_n|)^{\mu+4}$, then by Lemma A there exist $r_0 \in [0, 1)$, $L_0 > 0$ and $A > 0$ such that for $r \in [r_0, 1)$,

$$|g'(z)/g(z)| \leqq L_0(1-r)^{-A} \quad \text{on} \quad |z| = r \quad \text{if} \quad z \notin D. \tag{31}$$

We now consider $f(z)$. Let $\sum_{j=0}^{\infty} H_j z^j$ be the power series expansion of $f(z)$. For each $r \in [0, 1)$ let $N(r) = \max_{j \geqq 0} |H_j|r^j$ and $n(r) = \max\{j : |H_j|r^j = N(r)\}$. For convenience let $M(r) = M(r; f)$ and $W(r) = W(r, f; g)$ (where W is as in § 2). Then since $f(z)$ is of infinite order in $|z| < 1$, the following is proved in [5; pp. 209, 210, 212]: If α is any real number in $(0, 1)$ and t is a positive integer such that the corresponding number $\beta = (t+1)^{-1}(t+2)(1-(\alpha/2))$ is less than 1, then there exists in (0, 1) a sequence of values of r (called *remarkable*) tending to one, such that,

$$\log M(r) > \gamma'(n(r))^{\alpha}; \; n(r) > \gamma''(1-r)^{-\delta} \tag{32}$$

where $\delta = (1-\alpha)^{-1}$ and γ', γ'' are strictly positive constants independent of r,

$$M(r) < N(r)n(r), \tag{33}$$

and such that at every point of $|z| = r$,

$$|zf'(z)-n(r)f(z)| < K'(n(r))^{\beta}M(r), \tag{34}$$

where K' is a positive constant independent of r.

We now construct the value of α for which we will apply the Valiron theory, (32)–(34) above. By condition (3), there exist constants $r_3 \in [0, 1)$ and $K_4 > 0$, such that

$$(\log M(r))^{-\lambda}M(r)/W(r) \leqq K_4 \quad \text{for} \quad r \in [r_3, 1). \tag{35}$$

Since $\lambda < 1/2$, let $\theta \in (0, 1/2)$ be such that $\lambda+\theta < 1/2$. Since the function $t \to (t+1)^{-1}((t+2)/2)$ tends to 1/2 as $t \to +\infty$, there exists a positive integer t_0 such that $(t_0+1)^{-1}((t_0+2)/2) < ((1+\theta)/2)$. Now each of the four numbers, $1-(t_0+2)^{-1}\theta(t_0+1)$, $c/(1+c)$, $B/(1+B)$, $A/(1+A)$, is less than 1 (where c, B, A are as in (29), (30), (31)). Let α be a real number such that,

$$1 > \alpha > \max\{1-(t_0+2)^{-1}\theta(t_0+1), \; c/(1+c), \; B/(1+B), \; A/(1+A)\}. \tag{36}$$

Then $\alpha \in (0, 1)$, and using $t = t_0$, the corresponding

$$\beta = (t_0+1)^{-1}(t_0+2)(1-(\alpha/2)),$$

clearly satisfies,

$$\beta < (\tfrac{1}{2})+\theta. \tag{37}$$

Thus $\beta < 1$, and hence we can apply (32)–(34) for the value of α given by (36).

Now define $\varepsilon(z)$ at points z where $f(z) \neq 0$, by the relation,

$$f'(z)/f(z) = (1+\varepsilon(z))n(r)/z. \tag{38}$$

We note by the definition of $W(r)$ in § 2, that for each r in some interval $[r^*, 1)$, there is at least one point $z \notin D$ lying on $|z| = r$ at which $|f(z)| = W(r)$. We now prove:

LEMMA C: *There exists* $r_4 \in [r^*, 1)$ *such that for any remarkable value* $r \in [r_4, 1)$, *we have*

$$(39) \qquad |\varepsilon(z)| < \tfrac{1}{2}, \text{ at each point } z \notin D \quad \text{on} \quad |z| = r$$

at which $|f(z)| = W(r)$.

PROOF: By (34), (35) and (38), it follows that for any remarkable $r \geqq \max\{r_3, r^*\}$, we have,

$$(40) \qquad |\varepsilon(z)| \leqq K'K_4(n(r))^{-1+\beta}(\log M(r))^{\lambda},$$

at each point of $|z| = r$ at which $|f(z)| = W(r)$. Now by [5; p. 196], for any $R_0 \in (0, 1)$,

$$(41) \qquad \log N(r) = \log N(R_0) + \int_{R_0}^{r} (n(x,/x)dx \quad \text{if} \quad R_0 < r < 1.$$

We apply (41) with $R_0 = 3/4$. Then noting [5; p. 195] that since f is of infinite order,

(42) $n(r)$ is a non-decreasing function of r, with $\lim_{r\to 1} n(r) = +\infty$,

we have by (41) that $N(r) \leqq N(3/4)e^{n(r)/3}$ if $r \in (3/4, 1)$. Thus by (33), for any remarkable $r > 3/4$, we have

$$(43) \qquad M(r) < N(\tfrac{3}{4})n(r)e^{n(r)/3}$$

Since $\lim_{r\to 1} n(r) = +\infty$, it follows from (43), that there exists $r' \in (3/4, 1)$ such that $M(r) < e^{n(r)}$ for all remarkable $r > r'$. Hence, since $\beta < 1$, we obtain from (40), that

$$(44) \qquad |\varepsilon(z)| \leqq K'K_4(\log M(r))^{-1+\beta+\lambda},$$

at each point of $|z| = r$ at which $|f(z)| = W(r)$, if r is a remarkable value $\geqq \max\{r_3, r^*, r'\}$. But $-1+\beta+\lambda < 0$ by (37) and the definition of θ. Hence since $\log M(r) \to +\infty$ as $r \to 1$, the right side of (44) tends to zero as $r \to 1$, and so is less than 1/2 for all r greater than some number $r_4 \in [r^*, 1)$. This proves Lemma C.

Now $h = f/g$ satisfies the relation,

$$(45) \qquad \sum Q_{kj}(z)(h(z))^k(h'(z))^j = 0,$$

at all points z where h is analytic. Let $r \in [r_4, 1)$ and let $z \notin D$ be a point on $|z| = r$ at which $|f(z)| = W(r)$. Then $f(z) \neq 0$ and $g(z) \neq 0$, and so by dividing equation (45) through by $(h(z))^p$ (where p is as given), and noting that $h'/h = (f'/f)-(g'/g)$, we can write (45) in the form,

$$(46) \qquad \Lambda(z) = \Phi(z),$$

where

$$(47) \qquad \Lambda(z) = \sum_{j=0}^{m} Q_{p-j,j}(z) \left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \right)^j,$$

and

$$(48) \qquad \Phi(z) = -\sum_{k+j<p} Q_{kj}(z) \left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \right)^j \left(\frac{f(z)}{g(z)} \right)^{k+j-p}.$$

Let $R_1 = \max_{1 \leqq j \leqq 4} r_j$.

We now assert that there exists $r_5 \in [R_1, 1)$, such that for any remarkable $r \in [r_5, 1)$,

$$(49) \qquad |\Phi(z)| \leqq (M(r))^{-\frac{1}{2}} \exp(p'(1-r)^{-B}), \quad \text{where} \quad p' = p+1,$$

at each point $z \notin D$ on $|z| = r$ at which $|f(z)| = W(r)$.

To prove (49), we note first that since $M(r) \to +\infty$ as $r \to 1$, it follows from (35) that $W(r) \to +\infty$ as $r \to 1$. Since also $n(r) \to +\infty$ as $r \to 1$ (by (42)), there exists $R_2 \in [R_1, 1)$ with $R_2 > 1/2$, such that

$$(50) \qquad W(r) > 1 \quad \text{and} \quad n(r) > 1 \quad \text{for} \quad r \in [R_2, 1).$$

Let r be a remarkable value in $[R_2, 1)$, and let $z \notin D$ be a point on $|z| = r$ at which $|f(z)| = W(r)$. We refer to the right side of (48). If $k+j < p$, then $p-(k+j) \geqq 1$, so

$$|f(z)|^{k+j-p} = (W(r))^{k+j-p} \leqq (W(r))^{-1}.$$

Hence by (35),

$$|f(z)|^{k+j-p} \leqq K_4(\log M(r))^{\lambda}(M(r))^{-1}.$$

Also $|g(z)| \leqq \exp((1-r)^{-B})$ by (30), so

$$|g(z)|^{p-(k+j)} \leqq \exp(p(1-r)^{-B}).$$

Thus,

$$|f(z)/g(z)|^{k+j-p} \leqq K_4(\log M(r))^{\lambda}(M(r))^{-1} \exp(p(1-r)^{-B}).$$

Also by (27) and $B > \Delta$, $|Q_{kj}(z)| \leqq \exp((1-r)^{-B})$ and $|g'(z)/g(z)| \leqq L_0(1-r)^{-\Delta}$. Since $|\varepsilon(z)| < 1/2$ by (39), we have by (38) that

$$|f'(z)/f(z)| \leqq 4n(r) \quad (\text{since } r \geqq R_2 > 1/2).$$

Thus noting that $j < p$ if $k+j < p$, we have from (48) and the above estimates that,

$$(51) \qquad |\Phi(z)| \leqq u(r)(M(r))^{-\frac{1}{2}} \exp(p'(1-r)^{-B}),$$

where

$$u(r) = K_5(4n(r)+L_0(1-r)^{-\Delta})^p(\log M(r))^{\lambda}(M(r))^{-\frac{1}{2}}$$

for some constant $K_5 > 0$. Then $u(r) \geqq 0$, and we will show that $u(r) \to 0$ as $r \to 1$ through remarkable values. Now it follows from (32) that

$$n(r) < ((1/\gamma') \log M(r))^{1/\alpha} \quad \text{and} \quad (1-r)^{-A} < (n(r)/\gamma'')^{A/\delta}.$$

Thus clearly,

$$(52) \quad u(r) \leqq K_5[\sigma_1(\log M(r))^{1/\alpha}+\sigma_2(\log M(r))^{A/\alpha\delta}]^p(\log M(r))^{\lambda}(M(r))^{-\frac{1}{2}},$$

where σ_1, σ_2 are constants. Since $M(r) \to +\infty$ as $r \to 1$, clearly the right side of (52) tends to zero as $r \to 1$. Hence there exists $r_5 \in [R_2, 1)$ such that $u(r) < 1$ for $r \in [r_5, 1)$. In view of (51), this proves (49).

We now consider $\Lambda(z)$ given by (48). We distinguish two cases:

CASE I: $m = 0$. Then $\Lambda(z) = Q_{p,0}(z)$. Since $\Lambda(z) = \Phi(z)$ by (46), we have by (29) and (49) that $L_1(1-r)^c \leqq (M(r))^{-\frac{1}{2}} \exp(p'(1-r)^{-B})$ for all remarkable $r \in [r_5, 1)$. But by (32), $M(r) > \exp(\gamma'(\gamma'')^{\alpha}(1-r)^{-\alpha\delta})$. Hence we obtain,

$$(53) \qquad (1-r)^{2c} \exp(\gamma'(\gamma'')^{\alpha}(1-r)^{-\alpha\delta}-2p'(1-r)^{-B}) \leqq (1/L_1)^2,$$

for all remarkable $r \in [r_5, 1)$. But $\gamma'(\gamma'')^{\alpha} > 0$ and since $\alpha > B/(1+B)$ by (36), we have $\alpha\delta > B > 0$. Hence the left side of (53) clearly tends to $+\infty$ as $r \to 1$, and so (53) is impossible (since there exist remarkable values tending to 1). This contradiction proves the theorem in the case $m = 0$.

CASE II: $m > 0$. We first note that $\Lambda(z)$ may be written in the form,

$$(54) \quad \Lambda(z) = Q_{p-m,m}(z)\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right)^m (1+\sum_{j=0}^{m-1} \Psi_j(z)),$$

where

$$(55) \quad \Psi_j(z) = \frac{Q_{p-j,j}(z)}{Q_{p-m,m}(z)}\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right)^{j-m} \quad \text{for} \quad j = 0, 1, \ldots, m-1.$$

We consider $\Psi_j(z)$ at points $z \notin D$ on $|z| = r$ at which $|f(z)| = W(r)$, where r is a remarkable value in $[r_5, 1)$. By (38),

$$|f'(z)/f(z)| \geqq (1-|\varepsilon(z)|)n(r)/r.$$

Since $|\varepsilon(z)| < 1/2$ by (39) and since $r < 1$, $|f'(z)/f(z)| \geqq n(r)/2$. Thus by (32), $|f'(z)/f(z)| \geqq (\gamma''/2)(1-r)^{-\delta}$. By (31), $|g'(z)/g(z)| \leqq L_0(1-r)^{-A}$. Hence,

$$(56) \quad \begin{aligned} \left|\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right| &\geqq \left|\frac{f'(z)}{f(z)}\right| - \left|\frac{g'(z)}{g(z)}\right| \\ &\geqq (1-r)^{-\delta}\left(\frac{\gamma''}{2} - L_0(1-r)^{\delta-A}\right). \end{aligned}$$

By (36), $\alpha > A/(1+A)$ and so $\delta > A+1$. Thus $(1-r)^{\delta-A} \to 0$ as $r \to 1$, so from (56) there exists $r_6 \in [r_5, 1)$ such that

$$\left| \frac{f'(z)}{(fz)} - \frac{g'(z)}{g(z)} \right| \geqq (\gamma''/4)(1-r)^{-\delta} \quad \text{for} \quad r \in [r_6, 1). \tag{57}$$

Hence in view of (27), (29) and (57), we have for $j = 0, 1, \cdots, m-1$,

$$|\Psi_j(z)| \leqq (L/L_1)(4/\gamma'')^{m-j}(1-r)^{(m-j)\delta-c}; \tag{58}$$

when r is a remarkable value in $[r_6, 1)$. But since $m-j \geqq 1$, $(m-j)\delta \geqq \delta$, and since $\alpha > c/(1+c)$ by (36), we have $\delta > c$. Thus the right side of (58) tends to zero as $r \to 1$, so there exists $r_7 \in [r_6, 1)$ such that for remarkable r in $[r_7, 1)$, we have,

$$|\Psi_j(z)| \leqq (m+1)^{-1} \quad \text{for} \quad j = 0, 1, \cdots, m-1. \tag{59}$$

By (54),

$$|A(z)| \geqq |Q_{p-m,m}(z)| \left| \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} \right|^m \left(1 - \sum_{j=0}^{m-1} |\Psi_j(z)|\right).$$

Thus in view of (29), (57) and (59), we have that when r is a remarkable value in $[r_7, 1)$, then,

$$|A(z)| \geqq (L_1/(m+1))(\gamma''/4)^m(1-r)^{c-m\delta}, \tag{60}$$

at each point $z \notin D$ on $|z| = r$ at which $|f(z)| = W(r)$.

Since $A(z) = \Phi(z)$ by (46), we have by (49) and (60) that when r is a remarkable value in $[r_7, 1)$, then

$$K_6(1-r)^{c-m\delta} \leqq (M(r))^{-\frac{1}{2}} \exp(p'(1-r)^{-B})$$

where K_6 is a strictly positive constant. Since

$$M(r) > \exp(\gamma'(\gamma'')^\alpha(1-r)^{-\alpha\delta})$$

by (32), we thus obtain,

$$(1-r)^{2(c-m\delta)} \exp(\gamma'(\gamma'')^\alpha(1-r)^{-\alpha\delta} - 2p'(1-r)^{-B}) \leqq (1/K_6)^2, \tag{61}$$

for all remarkable $r \in [r_7, 1)$. But since $\alpha > B/(1+B)$, we have $\alpha\delta > B > 0$. Since also $\gamma'(\gamma'')^\alpha > 0$, clearly the left side of (61) tends to $+\infty$ as $r \to 1$ and hence (61) is impossible (since there exist remarkable values tending to 1). This contradiction proves the theorem in Case II and thus the proof of the theorem is complete.

6. Remarks and examples concerning condition (3)

We discuss here the condition (3):

$$\limsup_{r\to 1} \frac{M(r;f)(\log M(r;f))^{-\lambda}}{W(r,f;g)} < +\infty \text{ for some } \lambda < \tfrac{1}{2},$$

which is a relation between the two functions f and g, each analytic in $|z| < 1$, g being of finite order. The condition is of course automatically satisfied for the broad class of pairs (f, g) where for all r in some interval $[r_0, 1)$, the maximum value of $|f(z)|$ on $|z| = r$ is assumed at some point outside the union of the small disks D_n around the zeros of g, which are described in § 2. (In this case, $M(r;f) = W(r,f;g)$ for all $r \in [r_0, 1)$.)

In the other case (i.e. where $M(r;f) > W(r,f;g)$ for a sequence of r tending to 1), we now show that there are examples where (3) is satisfied and examples where it is not satisfied.

First we show that if $f(z) = \exp(\exp((1-z)^{-1}))$ and $g(z) = \sin(\pi(1-z)^{-1})$, then (3) is not satisfied. Here, $M(r;f) > W(r,f;g)$ for a sequence of $r \to 1$, since the maximum of $|f(z)|$ on $|z| = r$ is achieved only on the positive real axis, while in $|z| < 1$ the non-zero roots of g are $a_n = 1-(1/n)$ for $n = 2, 3, \cdots$, which are positive real numbers. The exponent of convergence of the sequence $\{a_n\}$ is $\mu = 0$. We now consider $W(r,f;g)$ for $r = a_n = 1-(1/n)$. The disk D_n of § 2 is $|z-r| < (1-r)^4$. An easy calculation then shows that

$$\log W(r,f;g) \leqq \exp\left(\frac{1-x(r)}{1-2x(r)+r^2}\right),$$

where

$$x(r) = r - \left(\frac{(1-r)^8}{2r}\right).$$

Writing

$$\frac{1-x(r)}{1-2x(r)+r^2} = \frac{1-\varepsilon(r)}{1-r},$$

it easily follows that $\varepsilon(r) = (1-r)^6\varphi(r)$ where $\lim_{r\to 1}\varphi(r) = 1$. Since $\log M(r;f) = \exp((1-r)^{-1})$, we see that,

$$\log M(r;f) - \log W(r,f;g) \geqq \exp((1-r)^{-1})[1-\exp(-\varepsilon(r)/(1-r))],$$

from which it easily follows that for all sufficiently large n,

$$\log M(r;f) - \log W(r,f;g) \geqq (1/4)(1-r)^5 \exp((1-r)^{-1}).$$

It is now easy to see that for any λ,

$$\frac{M(r;f)(\log M(r;f))^{-\lambda}}{W(r,f;g)}$$

tends to $+\infty$ as r tends to 1 through the sequence $r = 1-(1/n)$. Hence in this example, condition (3) is not satisfied.

To construct examples where $M(r;f) > W(r,f;g)$ for a sequence of

$r \to 1$, but where condition (3) is satisfied, we modify the above example. Again let $f(z) = \exp(\exp((1-z)^{-1}))$, and let $\delta(r)$ be a positive function on some interval $(r', 1)$ such that $\delta(r) < 1/2$. Let r_0 be such that $1 > r_0 > \max\{r', 1/2\}$, and for each $r \in [r_0, 1)$, define $u = u(r)$ by the relation

$$\frac{1-u}{1-2u+r^2} = \frac{1-\delta(r)}{1-r}.$$

Then it is easy to verify that $-r < u(r) < r$, and so there exists $v(r) > 0$ such that $z(r) = u(r)+iv(r)$ lies on $|z| = r$. Since each $z(r)$ lies above the real axis, we can choose, for each integer $n \geqq (1-r_0)^{-1}$, a point a_n on $|z| = 1-(1/n)$, such that the disk $D_n : |z-a_n| < (1-|a_n|)^4$ intersects the positive real axis, but does not contain any point $z(r)$ (where $r \in [r_0, 1)$). The exponent of convergence of $\{a_n\}$ is zero, and so by [3; p. 8], we can form the canonical product $g(z)$ with zeros at the points a_n for $n \geqq (1-r_0)^{-1}$. g is of finite order in $|z| < 1$ by [3; Theorem 1]. By construction, $M(r;f) > W(r,f;g)$ when $r = 1-(1/n)$ for each $n \geqq (1-r_0)^{-1}$. We now derive a condition on $\delta(r)$ to guarantee that condition (3) will be satisfied. Since for each $r \in [r_0, 1)$, $z(r)$ does not lie in the union of the disks $D_n(n \geqq (1-r_0)^{-1})$, clearly $W(r,f;g) \geqq |f(z(r))|$ for $r \in [r_0, 1)$. Hence,

$$\log W(r,f;g) \geqq \cos b(r)\left(\exp\left(\frac{1-u(r)}{1-2u(r)+r^2}\right)\right), \tag{62}$$

where $b(r) = v(r)/(1-2u(r)+r^2)$. A routine calculation shows that $(b(r))^2 \leqq 2\delta(r)(1-r)^{-2}$. Hence if $\delta(r)$ is chosen so that $\delta(r)(1-r)^{-2} \to 0$ as $r \to 1$, then on some interval $[r_1, 1)$,

$$1-\cos b(r) \leqq 2\delta(r)(1-r)^{-2}. \tag{63}$$

Since $\log M(r;f) = \exp((1-r)^{-1})$, it follows from (62) that

$$\log M(r;f)-\log W(r,f;g) \leqq \exp((1-r)^{-1}) - \cos b(r)\exp\left(\frac{1-\delta(r)}{1-r}\right). \tag{64}$$

Subtracting and adding the term $\cos b(r)\exp((1-r)^{-1})$ to the right side of (64), and using (63), it follows that on some interval $[r_2, 1)$,

$$\log M(r;f)-\log W(r,f;g) \leqq 4\delta(r)(1-r)^{-2}\exp((1-r)^{-1}). \tag{65}$$

Hence if $\lambda < 1/2$ and $\delta(r)$ is chosen to be any positive function which satisfies,

$$\delta(r) \leqq (\lambda/4)(1-r)\exp(-(1-r)^{-1}), \tag{66}$$

on some interval $[r_3, 1)$, then in view of (65), condition (3) will be satisfied for the corresponding pair (f, g).

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