COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 22, nº 3 (1970), p. 253-259
<http://www.numdam.org/item?id=CM_1970__22_3_253_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 3, 1970, pag. 253-259
Wolters-Noordhoff Publishing
Printed in the Netherlands

A THEOREM ON ENTIRE METHODS OF SUMMATION

by

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1. Introduction

Recently H. I. Brown [1] introduced the concept of entire methods of summation and proved a necessary and sufficient condition that an infinite matrix $A = (a_{n,k})$ be an entire method. In this paper we prove directly the necessity and sufficiency of a different condition which ensures that the matrix A is an entire method. We conclude the paper by considering applications of this theorem to the Sonnenschein matrix and to two recent generalizations of the Taylor matrix.

2. Entire methods

Let $x = \{x_k\}_0^\infty$ be a sequence of complex numbers. The sequence x is entire ($x \in \xi$) provided

$$\sum_{k=0}^{\infty} |x_k| q^k < \infty \text{ for every positive integer } q.$$

The infinite matrix $A = (a_{n,k})$ is an entire method provided the A-transform of $x \in \xi$ (written $A(x)$) is an entire sequence, i.e., the sequence $y = \{y_n\}_0^\infty \in \xi$ where

$$y_n = \sum_{k=0}^{\infty} a_{n,k} x_k.$$

In order to prove the main result (Theorem 3) we first state and prove two preparatory lemmas.

LEMMA 1. *If $A = (a_{n,k})$ is an entire method then*

(2.1) $\lim_{n\to\infty} a_{n,k} q^n = 0$ *for all integers $q > 0$ and each fixed $k = 0, 1, \cdots$.*

and

(2.2) *for each $n = 0, 1, \cdots$ there exists an integer $p_n > 0$ such that $|a_{n,k}| \leqq p_n^{k+1}$ for each $k = 0, 1, \cdots$.*

PROOF. (2.1) Let ν be a fixed non-negative integer and define the sequence $x = \{x_n\}_0^\infty$ by

$$x_n = \begin{cases} 1 & \text{if } n = \nu \\ 0 & \text{otherwise.} \end{cases}$$

Since $x \in \xi$ and A is an entire method we have that $A(x) \in \xi$. So, for $q > 0$ an integer, it follows that

$$\sum_{n=0}^{\infty} |y_n| q^n < \infty,$$

in particular, $0 = \lim_{n\to\infty} y_n q^n = \lim_{n\to\infty} a_{n,\nu} q^n$.

(2.2) Suppose there exists a non-negative integer N such that for each integer $p > 0$ there exists an integer $k_p \geqq 0$ where

$$|a_{N,k_p}| > p^{k_p+1}.$$

So, for $p_1 = 1$ there exists k_1 such that $|a_{N,k_1}| > 1^{k_1+1} = 1$. In general, choose $p_m > p_{m-1}$ $(m \geqq 2)$ such that

$$\max\{|a_{N,k}| : k \leqq k_{m-1}\} < p_m.$$

There exists $k_m > k_{m-1}$ such that $|a_{N,k_m}| > p_m^{k_m+1}$. Define the sequence $x = \{x_n\}_0^\infty$ by

$$x_n = \begin{cases} p_m^{-(k_m+1)} & \text{for } n = k_m \quad (m = 1, 2, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

The sequence $x \in \xi$ since, for $q > 0$ an integer, we have that

$$\sum_{n=0}^{\infty} |x_n| q^n = \sum_{m=1}^{\infty} p_m^{-(k_m+1)} q^{k_m} < \infty.$$

Since A is an entire method we have that $A(x) \in \xi$, however,

$$\sum_{k=0}^{\infty} a_{N,k} x_k = \sum_{m=1}^{\infty} a_{N,k_m} p_m^{-(k_m+1)}$$

diverges since $|a_{N,k_m} p_m^{-(k_m+1)}| \geqq 1$ for all $m \geqq 1$. Δ

LEMMA 2. *If $A = (a_{n,k})$ has properties* (2.1) *and* (2.2) *and, in addition,*

(2.3) *there exists an integer $q > 0$ such that for each integer $p > 0$ and each constant $M > 0$ there exist integers n, k where $|a_{n,k}| q^n > p^k M$*

then for a given integer $p > 0$, constant $M > 0$, and integers n_0, k_0 there exist integers $N > n_0$ $K > k_0$ such that

$$|a_{N,K}| q^N > p^K M.$$

PROOF. By (2.1) there exists $B_k > 0$ (for each $k = 0, 1, \cdots$) such that $|a_{n,k}q^n| < B_k$ for all $n = 0, 1, \cdots$. Let

$$B = \max\{1, B_0, \cdots, B_{k_0}\} \text{ and } P = \max\{1, p_0, \cdots, p_{n_0}\}$$

where p_n is given in (2.2). Therefore

$$|a_{n,k}q^n| < B \text{ for all } n = 0, 1, \cdots; \; k = 0, 1, \cdots, k_0.$$

Also

$$|a_{n,k}q^n| \leqq p_n^{k+1}q^n \leqq P^{k+1}q^{n_0} \text{ for all } n = 0, 1, \cdots, n_0; \; k = 0, 1, \cdots.$$

Therefore

$$|a_{n,k}q^n| \leqq q^{n_0}BP^{k+1} \text{ for all } n, k = 0, 1, \cdots \text{ with either } n \leqq n_0 \text{ or } k \leqq k_0. \tag{1}$$

By (2.3) there exist N, K such that

$$|a_{N,K}|q^N > P^K q^{n_0}BP.$$

So, by (1), $N > n_0$ and $K > k_0$. Δ

Before stating Theorem 3 we note that it can be proved using Brown's theorem [1]. Brown's theorem, however, used results of $l-l$ methods [3] (which depended upon the Uniform Boundedness Principle) and, thus, it is of interest that Theorem 3 can be proved directly. Such a proof is given.

THEOREM 3. *A matrix $A = (a_{n,k})$ is an entire method if and only if for each integer $q > 0$ there exists an integer $p = p(q) > 0$ and a constant $M = M(q) > 0$ such that*

$$|a_{n,k}|q^n \leqq p^k M \textit{ for all } n, k = 0, 1, \cdots.$$

PROOF. ($\Leftarrow$) Let $x \in \xi$, $y = A(x)$, and $q > 0$ be an arbitrary fixed integer. We have that

$$\sum_{n=0}^{\infty} |y_n|q^n \leqq \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{\infty} |a_{n,k}||x_k| \Big) q^n = \sum_{n=0}^{\infty} (\tfrac{1}{2})^n \sum_{k=0}^{\infty} |a_{n,k}|(2q)^n|x_k|.$$

There exists $p = p(2q) > 0$ and $M = M(2q) > 0$ such that

$$|a_{n,k}|(2q)^n \leqq p^k M \text{ for all } n, k = 0, 1, \cdots.$$

Therefore

$$\sum_{n=0}^{\infty} |y_n|q^n \leqq 2M \sum_{k=0}^{\infty} |x_k|p^k < \infty \text{ since } x \in \xi, \text{ i.e., } y = A(x) \in \xi.$$

($\Rightarrow$) Suppose there exists and integer $q > 0$ such that for each integer $p > 0$ and each constant $M > 0$ there exists integers n, k where $|a_{n,k}|q^n > p^k M$. By Lemma 1 (2.2) choose a sequence of positive integers $\{p_n\}_0^\infty$ such that

$$|a_{n,k}| \leqq p_n^{k+1} \text{ for all } k = 0, 1, \cdots. \tag{2}$$

Let $\bar{p}_n = \max\{p_i : i = 0, \cdots, n\}$. Construct the sequence $\{a_{n_j,k_j}\}_{j=1}^\infty$ as follows:

Choose n_1, k_1 such that $a_{n_1,k_1} \neq 0$. Suppose that $n_1, \cdots, n_j; k_1, \cdots, k_j$ ($j \geqq 1$) have been chosen. By Lemma 1 (2.1), given

$$\varepsilon = (4(j+1))^{-1} \min\{|a_{n_t,k_t}| : 1 \leqq t \leqq j\} > 0,$$

there exists $\bar{n}_j = \bar{n}_j(\varepsilon, k_j)$ such that

$$|a_{n,k}|q^n < \varepsilon \text{ for all } n \geqq \bar{n}_j,\, k \leqq k_j.$$

By Lemma 2 there exists $n_{j+1} > \max\{\bar{n}_j, n_j\}$ and $k_{j+1} > k_j$ such that

$$|a_{n_{j+1},k_{j+1}}|q^{n_{j+1}} > q^{n_j}[8(k_j+1)\bar{p}_{n_j}]^{k_{j+1}+1}.$$

So, we have a sequence $\{a_{n_j,k_j}\}_{j=1}^\infty$ such that

$$|a_{n,k_t}|q^n \leqq (4_j)^{-1}|a_{n_t,k_t}| \text{ for all } n > n_j,\, t < j \tag{3}$$

and

$$|a_{n_j,k_j}|q^{n_j} > q^{n_{j-1}}(8(k_{j-1}+1)\bar{p}_{n_{j-1}})^{k_j+1} \text{ for } j \geqq 2. \tag{4}$$

Define the sequence $x = \{x_m\}_1^\infty$ by

$$x_m = \begin{cases} [a_{n_j,k_j}q^{n_j}]^{-1} & \text{for } m = k_j \;\; (j = 2, 3, \cdots) \\ 0 & \text{otherwise.} \end{cases}$$

By (4) it follows that

$$|x_{k_{j+1}}| < [q^{n_j}(8(k_j+1)\bar{p}_{n_j})^{k_{j+1}+1}]^{-1} \leqq (k_j+1)^{-(k_j+1)},\, j \geqq 1.$$

Therefore $|x_n| \leqq (1/n)^n$ ($n \geqq 1$) and, hence, $x \in \xi$. So $y = A(x) \in \xi$. Now

$$\begin{aligned} |y_{n_j}| &= |\sum_{k=0}^{\infty} a_{n_j,k}x_k| \\ &\geqq |a_{n_j,k_j}||x_{k_j}| - \sum_{t=1}^{j-1}|a_{n_j,k_t}||x_{k_t}| - \sum_{t=j+1}^{\infty}|a_{n_j,k_t}||x_{k_t}|. \end{aligned} \tag{5}$$

From (3) we have that

$$\sum_{t=1}^{j-1}|a_{n_j,k_t}||x_{k_t}| \leqq (4jq^{n_j})^{-1}\sum_{t=1}^{j-1}|a_{n_t,k_t}||x_{k_t}| \leqq (4q^{n_j})^{-1} \tag{6}$$

and, from (2) and (4), we have that

$$(7)\quad \sum_{t=j+1}^{\infty} |a_{n_j,k_t}||x_{k_t}| \leqq \sum_{t=j+1}^{\infty} (\bar{p}_{n_j})^{k_t+1}|x_{k_t}|$$
$$\leqq \sum_{t=j+1}^{\infty} (\bar{p}_{n_j})^{k_t+1}[q^{n_{t-1}}(8\bar{p}_{n_{t-1}}(k_{t-1}+1))^{k_t+1}]^{-1}$$
$$\leqq \sum_{t=j+1}^{\infty} [q^{n_{t-1}}(8(k_{t-1}+1))^{k_t+1}]^{-1}$$
$$\leqq q^{-n_j} \sum_{t=j+1}^{\infty} 8^{-(k_t+1)} \leqq (4q^{n_j})^{-1}.$$

So, from (5), (6), and (7), it follows that

$$|y_{n_j}| \geqq \tfrac{1}{2}q^{-n_j}.$$

Therefore,

$$\sum_{n=0}^{\infty} |y_n|q^n \geqq \sum_{j=1}^{\infty} \tfrac{1}{2}$$

which diverges, i.e., $y = A(x) \notin \xi$. This contradicts the fact that A is an entire method. Δ

COROLLARY 4. *An upper triangular matrix $A = (a_{n,k})$ is an entire method if and only if there exists an integer $p > 0$ and a constant $M > 0$ such that*

$$|a_{n,k}| \leqq p^k M \text{ for all } n, k = 0, 1, \cdots.$$

PROOF. ($\Rightarrow$) This follows from Theorem 3 (let $q = 1$). ($\Leftarrow$) Let $q > 0$ be given. Thus

$$|a_{n,k}|q^n \leqq p^k M q^n = (pq)^k M q^{n-k} \leqq (pq)^k M \text{ for } n \leqq k.$$

Since $a_{n,k} = 0$ for $n > k$ we have, by Theorem 3, that A is an entire method. Δ

3. Applications

The Sonnenschein matrix $A(f) = (a_{n,k})$ has been studied by many people including Meyer-König [4]. It is defined by

$$[f(z)]^n = \sum_{k=0}^{\infty} a_{n,k} z^k \text{ (for } n \geqq 1)$$

where f is analytic at $z = 0$ and $a_{0,0} = 1$, $a_{0,k} = 0$ for $k \geqq 1$.

LEMMA 5. *The Sonnenschein matrix $A(f)$ is an entire method if and only if $f(0) = 0$.*

PROOF. ($\Rightarrow$) Suppose that $f(0) \neq 0$. Choose an integer $q > 0$ such that $|f(0)|\,q > 1$. By Theorem 3 there exists an integer $p = p(q) > 0$ and a constant $M = M(q) > 0$ such that

$$|a_{n,k}|q^n \leqq p^k M \quad \text{for all } n, k = 0, 1, \cdots.$$

However, there exists N such that if $n \geqq N$ then

$$|a_{n,0}|q^n = |f(0)|^n q^n > M.$$

Therefore $f(0) = 0$.

($\Leftarrow$) If $f(0) = 0$ then $A(f) = (a_{n,k})$ is upper triangular. Since $f(z)$ is analytic at $z = 0$ there exists $R > 0$ such that $f(z)$ is analytic on $\{z : |z| < 2R\}$ (hence $[f(z)]^n$ is analytic on $\{z : |z| < 2R\}$). Let $\Gamma = \{t : |t| = R\}$. There exists $M > 0$, such that $\sup\{|f(z)| : t \in \Gamma\} \leqq M$. So, by the Cauchy Integral Formula, we have that

$$|a_{n,k}| = \left| \frac{1}{2\pi i} \int_\Gamma \frac{[f(z)]^n}{t^{k+1}}\, dt \right| \leqq R^{-k} M^n \text{ for } k \geqq n > 0.$$

Choose an integer $p > R^{-1}(M+1)$ and $M^* = 1$. So

$$|a_{n,k}| \leqq p^k M^* \quad \text{for all } n, k = 0, 1, \cdots.$$

So, by Corollary 4, $A(f)$ is an entire method. Δ

In the special case of the Karamata matrix (see [4]) where

$$f(z) = \frac{\alpha + (1-\alpha-\beta)z}{1-\beta z}$$

we have

COROLLARY 6. *The Karamata matrix is an entire method if and only if* $\alpha = 0$.

In this case the Karamata matrix gives us the Taylor matrix $T(\beta)$ hence

COROLLARY 7. *The Taylor matrix* $T(\beta)$ *is an entire method for all complex numbers* β.

Brown proved this result in his examples [1].

There are two other generalizations of the Taylor matrix which are of interest, namely, the $T(r_n)$ matrix (see [2]) and the $\mathfrak{S}(r_n)$ matrix (see [5]). For $\{r_n\}_0^\infty$, a sequence of complex numbers, the $T(r_n) = (b_{n,k})$ matrix is defined by

$$\left\{\frac{(1-r_n)z}{1-r_n z}\right\}^{n+1} = \sum_{k=n}^{\infty} b_{n,k} z^{k+1},\ b_{n,k} = 0 \text{ for } k < n$$

and the $\mathfrak{S}(r_n) = (c_{n,k})$ matrix is defined by

$$\prod_{k=0}^{n} \frac{(1-r_k)z}{1-r_k z} = \sum_{k=n}^{\infty} c_{n,k} z^{k+1}, \; c_{n,k} = 0 \text{ for } k < n.$$

By applying Corollary 4 and the technique used in the proof of Lemma 5 we have

LEMMA 8. (i) *The $T(r_n)$ matrix is an entire method if and only if $\{r_n\}_0^\infty$ is bounded.*

(ii) *The $\mathfrak{S}(r_n)$ matrix is an entire method if and only if $\{r_n\}_0^\infty$ is bounded.*

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(Oblatum 23–V–69)

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