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# SOME PROPERTIES OF SIMPLE $I$-REGULAR SEMIGROUPS

by

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Let $S$ be a regular semigroup and let $E_S$ denote its set of idempotents. As usual, $E_S$ is partially ordered in the following manner: if $e, f \in E_S$, $e \leqq f$ if and only if $ef = fe = e$. We then say that $E_S$ is under or assumes its natural order. Let $I$ denote the integers. If $E_S$, under the natural order, is order isomorphic to $I$ under the reverse of the usual order, we call $S$ an $I$-regular semigroup. We determined the structure of $I$-regular semigroups mod groups in [10].

In section 1, we develop the ideal extension theory of simple $I$-regular semigroups. In section 2, we obtain the maximal group homomorphic image of a simple $I$-regular semigroup including the defining homomorphism. In section 3, we determine the nature of the congruences admitted by a simple $I$-regular semigroup, and we describe the idempotents separating congruences.

In the special case $S$ is bisimple, the results of this paper reduce to the corresponding results for $I$-bisimple semigroups (bisimple semigroups $S$ such that $E_S$ is order isomorphic to $I$ under the reverse of the usual order) [6, 7].

Unless otherwise specified, we utilize the definitions, terminology, and notation of [1].

## 1. Ideal extension theory

In this section, we determine the translational hull $\bar{S}$ of a simple $I$-regular semigroup $S$. All ideal extensions of $S$ by a semigroup $T$ with zero, $o$, can then be described if one knows the structure of $T$ and the partial homomorphisms $\theta$ of $T^* = T \setminus 0$ into $\bar{S}$ such that $AB = 0$ in $T$ implies that $A\theta B\theta \in S$ [1]. This determination is carried out if $T$ is a completely 0-simple (Brandt) semigroup. We also completely determine the extensions of a Brandt semigroup with finite index set by a simple $I$-regular semigroup (with zero appended) by specializing our general determination of the extensions of a Brandt semigroup by an arbitrary semigroup [5, theorem 1].

Before commencing, let us state the structure theorem for simple $I$-regular semigroups.

Let $C_1^* = IxI$ under the multiplication $(a, b)(c, d) = (a+c-\min(b, c), b+d-\min(b, c))$. We called $C_1^*$ the extended bicyclic semigroup in [6].

THEOREM 1.1 (Warne, [10]). *$S$ is a simple $I$-regular semigroup if and only if $S = (U(G_j : j = 0, 1, \cdots, d-1)) \times C_1^*$, where $d$ is a positive integer, $\{G_j : 0 \leqq j \leqq d-1\}$ is a collection of pairwise disjoint groups, and $C_1^*$ is the extended bicyclic semigroup, under the multiplication*

$$(g_s, (m, n))(h_r, (p, q)) = (t, (m, n)(p, q)) \qquad (*)$$

*where*

$$g_s \in G_s, g_r \in G_r \quad (0 \leqq r,\ s \leqq d-1) \quad and \quad t =$$

$$g_s(f_{n-p,p}^{-1} \prod_{j=0}^{s-1} \gamma_j)(h_r \prod_{j=pd+r}^{nd+s-1} \gamma_j)(f_{n-p,q} \prod_{j=0}^{s-1} \gamma_j),$$

$$(f_{p-n,m}^{-1} \prod_{j=0}^{r-1} \gamma_j)(g_s \prod_{j=nd+s}^{pd+r-1} \gamma_j)(f_{p-n,n} \prod_{j=0}^{r-1} \gamma_j)h_r, \qquad or$$

$$(g_s \prod_{j=s}^{v-1} \gamma_j)(h_r \prod_{j=r}^{v-1} \gamma_j) \quad (v = \max(r, s))$$

*according to whether $n > p$, $p > n$, or $p = n$ where $\gamma_j = \gamma_{j(\mathrm{mod}\, d)}(j \in I, j \geqq 0)$ is a homomorphism of $G_{j(\mathrm{mod}\, d)}$ into $G_{(j+1)\mathrm{mod}\, d}$. Juxtaposition denotes multiplication in $C_1^*$ and in the appropriate $G_j$. For $m \in I^0$, the non-negative integers, $n \in I, f_{0,n} = k_0$, the identity of $G_0$, while, for $m > 0$,*

$$f_{m,n} = u_{(n+1)d}(\prod_{j=0}^{d-1} \gamma_j)^{m-1} u_{(n+2)d}(\prod_{j=0}^{d-1} \gamma_j)^{m-2} \cdots u_{(n+(m-1))d}(\prod_{j=0}^{d-1} \gamma_j) u_{(n+m)d}$$

*where $\{u_{kd} : k \in I\}$ is a collection of elements of $G_0$ with $u_{kd} = k_0$ for $k > 0$. In $(*)$ $\prod_{j=a}^{a-1} \gamma_j$ will denote the identity automorphism of $G_{a(\mathrm{mod}\, d)}$.*

Let $S$ be a simple $I$-regular semigroup. In connection with theorem 1.1, we write $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$.

For convenience, we write $\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$ if $m < n$ and let $\alpha_{n,n}$ denote the identity automorphism of $G_{n(\mathrm{mod}\, d)}$.

LEMMA 1.1. *A simple $I$-regular semigroup is left and right reductive.*

PROOF. This lemma is an immediate consequence of theorem 1.1. We will utilize the multiplication of theorem 1.1 without explicit mention.

THEOREM 1.2 *Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ be a simple $I$-regular semigroup. Let $W = \{(\theta, p) : \theta : I \to G_0, p \in I$, and $(i+1)\theta = m_{(i+1)d}^{-1}(i\theta \prod_{j=0}^{d-1} \gamma_j)\, m_{(i+p+1)d}$ for all $i \in I\}$. Let $\rho_i\ (i \in I)$ denote the inner right translation of $(I, +)$ determined by $i \cdot W$, under the multiplication*

$$^*(\theta, w)(\eta, p) = (\theta \circ \rho_w \eta, w+p),$$

*where $\circ$ denotes pointwise multiplication of mappings and juxtaposition denotes iteration of mappings is a group. Let $\bar{S}$ be the translational hull of $S$. Then, $\bar{S} = W \cup S$ ($W \cap S = \square$), under the multiplication*

$$(\theta, a) \cdot (\eta, p) = (\theta, a)(\eta, p)$$

$$(g_s, a, b) \cdot (h_r, c, d) = (g_s, a, b)(h_r, c, d)$$

*where juxtaposition denotes multiplication in $W$ and $S$ and*

$$(\theta, p) \cdot (g_r, a, b) = ((a-p)\theta \prod_{j=0}^{r-1} \gamma_j g_r, a-p, b)$$

$$(g_r, a, b) \cdot (\theta, p) = (g_r(b\theta \prod_{j=0}^{r-1} \gamma_j), a, b+p).$$

PROOF. Let $\lambda$ be a left translation of $S$. Then, if $e_0$ is the identity of $G_0$,

$$(e_0, i, i)\lambda = (i\delta, i\delta_1, i+i\rho_1)$$

where $\delta : I \to U(G_j : 0 \leqq j \leqq d-1)$; $\delta_1 : I \to I$; and $\rho_1 : I \to I^0$ the non-negative integers. Since $(e_0, i, i)(e_0, i+1, i+1) = (e_0, i+1, i+1)$, we have the following two possibilities: If $i\rho_1 = 0$,

$$\begin{aligned} (i+1)\delta &= m^{-1}_{(i\delta_1+1)d}(i\delta\alpha_{r,d})m_{(i+1)d} \quad \text{where} \quad i\delta \in G_r, \\ (i+1)\rho_1 &= 0, \quad \text{and} \\ (i+1)\delta_1 &= i\delta_1+1 \end{aligned} \tag{1.1}$$

while, if $i\rho_1 \geqq 1$,

$$\begin{aligned} (i+1)\delta &= i\delta \\ (i+1)\rho_1 &= i\rho_1-1 \\ (i+1)\delta_1 &= i\delta_1 . \end{aligned} \tag{1.2}$$

Let us first consider the case $i\rho_1 = 0$ for all $i \in I$. In this case it is easily seen that $\lambda | D_0$, where $D_0 = \{(g_0, m, n) : g_0 \in G_0, m, n \in I\}$, is a left translation of $D_0$. Hence, since $D_0$ is the $I$-bisimple semigroup ($G_0$, $C_1^*$, $\alpha_{0,d}$, $m_{id}$) [6, theorem 1.2] (notation of [6]),

$$(e_0, i, i)\lambda = (i\delta, i+p, i)$$

where $p \in I$ and $\delta$ is a mapping of $I$ into $G_0$ such that

$$(i+1)\delta = m^{-1}_{(i+p+1)d} i\delta\alpha_{0,d} m_{(i+1)d} \tag{1.3}$$

by virtue of [7, 8] or by [9, proof of theorem 1]. Hence, since $(g_r, i, j) = (e_0, i, i)(g_r, i, j)$,

$$(g_r, i, j)\lambda_{(\delta, p)} = ((i\delta)\alpha_{0,r} g_r, i+p, j) \tag{1.4}$$

where $\lambda = \lambda_{(\delta, p)}$, $p \in I$ and $\delta$ is a mapping of $I$ into $G_0$ satisfying (1.3).

Conversely, (1.3) and (1.4) define a left translation of $D_0$ by [7] or by [9, proof of theorem 1]. By (1.3),

$$(g_r, a, b)\lambda_{(\delta, p)} = (e_0, a, a)\lambda_{(\delta, p)}(g_r, a, b).$$

Thus,

$$\begin{aligned}((g_r, a, b)(h_s, c, d))\lambda_{(\delta, p)} &\\ &= (e_0, a+c-\min(b, c), a+c-\min(b, c))\lambda_{(\delta, p)}(g_r, a, b)(h_s, c, d)\\ &= (e_0, a, a)\lambda_{(\delta, p)}(g_r, a, b)(h_s, c, d) = (g_r, a, b)\lambda_{(\delta, p)}(h_s, c, d).\end{aligned}$$

Hence, $\lambda_{(\delta, p)}$ is a left translation of $S$.

Next, suppose that there exists $u \in I$ such that $u\rho_1 \neq 0$. Utilizing (1.1) and (1.2), we obtain: $(t+i)\rho_1 = 0$, where $t$ is a unique element in $I$, for $i \geqq 0$, and $(t+i)\rho_1 = -i$ for $i < 0$; $(t+i)\delta_1 = a+i$, where $a \in I$, for $i \geqq 0$, and $(t+i)\delta_1 = a$ for $i < 0$; and $(t+i)\delta = f_{i,a}^{-1} g_s \alpha_{s,d} \alpha_{0,d}^{i-1} f_{i,t}$ for $i > 0$, and $(t+i)\delta = g_s \in G_s$ for $i \leqq 0$. Since $(e_0, i, i)(e_0, i+n, i) = (e_0, i+n, i)$ for all $n \geqq 0$, we are able to determine $(e_0, i+n, i)\lambda$. Next, since $(g_r, i+n, i+m) = (e_0, i+n, i)(g_r, i, i+m)$ for $i \in I$, $m, n \in I^0$, we are able to determine $(g_r, i+n, i+m)\lambda$. By [3] and theorem 1.1, every element of $S$ may be written in the form $(g_r, i+n, i+m)$ where $g_r \in G_r$, $i \in I$, and $m, n \in I^0$. We let $i = t+q$ and determine $(g_r, t+q+n, t+q+m)\lambda$ in terms of the values of $\delta$, $\rho_1$, and $\delta_1$ given above. In this calculation, we utilize the identity $f_{m+c,n} f_{c,m+n}^{-1} = f_{m,n} \alpha_{0,d}^{c}$ for $m, c \in I^0$ and $n \in I$ [10]. (This identity may be developed by a routine calculation.) Finally, if $a_1 = t+q+n$ and $b_1 = t+q+m$, we show that $(g_r, a_1, b_1)\lambda = (g_s, a, t)(g_r, a_1, b_1)$, i.e. $\lambda$ is an inner left translation. (We omit the details of these calculations as they parallel calculations given in [7] and [9]).

In a similar manner, it may be shown that the semigroup of right translations of $S$ consists of the inner right translations of $S$ and the transformations of $S$ defined by

$$(g_r, i, j)\rho_{(\theta, w)} = (g_r(j\theta\alpha_{0,r}), i, j+w) \tag{1.5}$$

where $w \in I$ and $\theta$ is a mapping of $I$ into $G_0$ such that

$$(i+1)\theta = m_{(i+1)d}^{-1}(i\theta\alpha_{0,d})m_{(i+w+1)d} \quad \text{for all } i \in I. \tag{1.6}$$

It is easily seen that $\rho_{(\theta, w)}$ as defined by (1.5) and (1.6) is not an inner right translation of $S$, and $\lambda_{(\delta, p)}$ as defined by (1.4) and (1.3) is not an inner left translation of $S$. Hence, by lemma 1.1 and [7, lemma 1], $\lambda_{(\delta, p)}$ and $\rho_{(g_r, a, b)}$ are not linked and $\lambda_{(g_r, a, b)}$ and $\rho_{(\theta, w)}$ are not linked. Similarly, $\lambda_{(h_r, c, d)}$ and $\rho_{(g_s, a, b)}$ are linked if and only if $(h_r, c, d) = (g_s, a, b)$. Next, suppose that $\rho_{(\theta, w)}$ and $\lambda_{(\delta, p)}$ are linked. Then $\rho_{(\theta, w)}|D_0$ and $\lambda_{(\delta, p)}|D_0$

are linked. Thus, by the proof of [9, theorem 1] or [7], $w = -p$ and $\delta = \rho_{-w}\theta$. By the proof of [9, theorem 1] or [7], $\rho_{(\theta, w)}|D_0$ and $\lambda_{(p_{-w}\theta, -w)}$ $|D_0$ are linked. Thus, $(g_s, a, b)\rho_{(\theta,w)}(h_r, c, d) = ((g_s, a, b)(e_0, b, b))$ $\rho_{(\theta,w)}(h_r, c, d) = (g_s, a, b)((e_0, b, b)\rho_{(\theta,w)}(e_0, c, c))(h_r, c, d) = (g_s, a, b)$ $((e_0, c, c)\lambda_{(\rho_{-w}\theta, -w)}(h_r, c, d)) = (g_s, a, b)((h_r, c, d)\lambda_{(\rho_{-w}\theta, -w)})$. Thus, $\rho_{(\theta, w)}$ and $\lambda_{(\rho_{-w}\theta, -w)}$ are linked. The mapping $\rho \to (\lambda, \rho)$, where $\rho$ is a right translation of $S$ and $\lambda$ is the left translation of $S$ linked with $\rho$, is an isomorphism of the semigroup of right translations of $S$ onto $\bar{S}$. If $\rho_{(\theta, q)}, \rho_{(\eta, p)} \in \bar{S} \setminus S$, $\rho_{(\theta, q)}\rho_{(\eta, p)} = \rho_{(\theta \circ p_q \eta, q+p)}$ by (1.5) and (1.6). Hence $\bar{S} \setminus S$ is a semigroup. The mapping $(\theta, p) \to \rho_{(\theta, p)}$ is an isomorphism of $W$, under the multiplication $*$, onto $\bar{S} \setminus S$. Clearly, $W$ is a group. The remainder of the theorem is a consequence of [1, p. 12, lemma 1.2], (1.4), and (1.5).

REMARK 1.1. In the case $d = 1$, we obtain [7, theorem 1] (see also [8]).

COROLLARY 1.1. *Let $S$ be a weakly reductive semigroup and let $\bar{S}$ be its translational hull. Let $T$ be a* 0*-simple semigroup having proper divisors of zero. If $S = \bar{S}$ or $\bar{S} \setminus S$ is a subsemigroup of $S$, then every extension of $S$ by $T$ is given by a partial homomorphism* [4].

PROOF. Replace $\mathfrak{D}$ by $\mathfrak{J}$ in the proof of [7, theorem 3].

REMARK 1.2. Let $S = (d, G_0, G_1, \cdots, G_{d-1}, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ be a simple $I$-regular semigroup. $S$ has $d$ $\mathfrak{D}$-classes, $D_0, D_1, \cdots, D_{d-1}$. $D_r = \{(g_r, a, b) : g_r \in G_r : a, b \in I\}$ is the $I$-bisimple semigroup $(G_r, C_1^*,$ $\alpha_{r, r+d}, m_{id}\alpha_{0,r})$. (Notation of [6]). Let $T$ be a 0-bisimple semigroup. To determine the partial homomorphisms of $T \setminus 0$ into $S$ one must just determine the partial homomorphisms of $T \setminus 0$ into $D_r$ for each $r \in \{0, 1, 2, \cdots, d-1\}$. In the case $T$ is a completely 0-simple semigroup, (a Brandt semigroup), these determinations are given mod groups by [7, theorem 2] ([7, corollary 1]). By lemma 1.1, theorem 1.2, and Corollary 1.1, if $T$ is a 0-simple semigroup with proper divisors of zero, every extension of $S$ by $T$ is given by a partial homomorphism. In particular, this is valid if $T$ is a completely 0-simple semigroup (Brandt Semigroup) with proper divisors of zero.

COROLLARY 1.2. *Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \delta_0, \cdots, \delta_{d-1},$ $m_{id})$ be a simple $I$-regular semigroup and let $T = M^0(R; K; \Lambda; P)$ be a completely* 0*-simple semigroup (with zero,* 0'*) without proper divisors of zero. Let $V$ be an extension of $S$ by $T$. Then, either $V$ is given by a partial homomorphism and an explicit multiplication is thus given by employing remark* 1.2. *(Conversely, every partial homomorphism of $T \setminus 0'$ into $S$ determines an extension of $S$ by $T$), or $V = (T \setminus 0') \cup S$ under the multiplication*

A) $(a; s, \lambda)^*(g_r, m, n)$

$$= ((m-k_s-i_a-t_\lambda)(\beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_\lambda)\prod_{j=0}^{r-1}\delta_j g_r, m-k_s-i_a-t_\lambda, n)$$

B) $(g_r, m, n)^*(a; s, \lambda)$

$$= (g_r((n\beta_s \circ \rho_{k_s}\theta_a \circ \rho_{k_s+i_a}\gamma_\lambda)\prod_{j=0}^{r-1}\delta_j), m, k_s+i_a+t_\lambda+n)$$

*where* $(g_s, m, n) \in S$ *and* $(a; s, \lambda) \in T\setminus 0'$, $\circ$ *denotes pointwise multiplication of mappings,* $a \to i_a$ *is a homomorphism of* $R$ *into* $(I, +)$, $a \to \theta_a$ *is a mapping of* $R$ *into* $H = \{\beta : (\beta, a) \in W$ *for some* $a \in I\}$ *(see statement of theorem* 1.2) *such that* $\theta_{ab} = \theta_a \circ \rho_{i_a}\theta_b$ *for all* $a, b \in R$, $s \to \beta_s$ *is a mapping of* $K$ *into* $H$, $s \to k_s$ *is a mapping of* $K$ *into* $I$, $\lambda \to \gamma_\lambda$ *is a mapping of* $\Lambda$ *into* $H$, *and* $\lambda \to t_\lambda$ *is a mapping of* $\Lambda$ *into* $I$ *such that* $i_{p_{\lambda s}} = t_\lambda + k_s$ *and* $\theta_{p_{\lambda s}} = \gamma_\lambda \circ \rho_{t_\lambda}\beta_s$. *Conversely,* (A) *and* (B) *define an extension of* $S$ *by* $T$.

PROOF. The proof utilizes theorem 1.1, theorem 1.2, corollary 1.1, and [1, theorem 4.20 and theorem 4.22]. It is similar in nature to the proof of [7, theorem 4] (see also [8]) and [9, theorem 4] and it will be omitted.

REMARK 1.3. In the case $d = 1$, we obtain [7, theorem 4][see also [8]).

REMARK 1.4. In the special case that $T\setminus 0'$ is a group $R$, $V$ is either given by a partial homomorphism or (A) and (B) become

$$a^*(g_r, m, n) = ((m-i_a)\theta_a\prod_{j=0}^{r-1}\gamma_j g_r, m-i_a, n)$$

$$(g_r, m, n)^*a = (g_r((n\theta_a)\prod_{j=0}^{r-1}\gamma_j), m, n+i_a).$$

REMARK 1.5. If $T$ is a 0-simple semigroup without proper divisors of zero, an extension of $S$ by $T$ is either given by a partial homomorphism or by the equations in the above remark with $a \to \theta_a$ a mapping of $T\setminus 0'$ into $H$ and with $a \to i_a$ a homomorphism of $T\setminus 0'$ into $(I, +)$.

We close this section by giving a specialization of [5, theorem 1]. The theorem is obtained by combining theorem 3.1 (below), [5, theorem 1], and [5, lemma 1]. The theorem is quite similar to [9, theorem 7].

In the theorem below, capital roman letters will denote elements of $T^*$.

THEOREM 1.3. *Let* $S = M^0(G; J; J; \Delta)$, *where* $J$ *is a finite set, be a Brandt semigroup; let* $T^* = (d, U_0, U_1, \cdots, U_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ *be a simple I-regular semigroup; and let* $V$ *be an extension of* $S$ *by* $T$. *Then, there exists a homomorphism* $w : A \to w_A$ *of* $T^*$ *into* $H_r$, *the full symmetric group on some* $r$ *element subset* $Q$ *of* $J$. *This homomorphism is explicitly given by theorem* 2.4. *For each* $A \in T^*$, *there exists a mapping* $\psi_A$ *of* $Q$ *into the group* $G$ *such that*

$$(i\psi_A)(iw_A\psi_B) = i\psi_{AB} \quad \textit{for all} \quad i \in Q.$$

*The products in $V$ are given by*

$$A \circ B = AB \tag{1.7}$$

$$\begin{aligned}(a; i, j) \circ A &= (a(j\psi_A), i, jw_A) \quad \textit{if} \quad j \in Q \\ &= 0', \textit{ the zero of } S, \textit{ if } j \,\bar{\in}\, Q \\ 0' \circ A &= 0'\end{aligned} \tag{1.8}$$

$$\begin{aligned}A \circ (a; i, j) &= ((iw_A^{-1}\psi_A)a, iw_A^{-1}, j) \textit{ if } i \in Q \\ &= 0' \quad \textit{if} \quad i \,\bar{\in}\, Q \\ A \circ 0' &= 0'.\end{aligned} \tag{1.9}$$

*Conversely, let $S$ be a Brandt semigroup and let $T^*$ be a simple I-regular semigroup. If we are given the mappings $w$ and $\psi_A$ described above and define product $\circ$ in the class sum of $S$ and $T^*$ by* (1.7)–(1.9), *then $V$ is an extension of $S$ by $T$.*

## 2. The maximal group homomorphic image

The major purpose of this section is to determine the maximal group homomorphic image of a simple $I$-regular semigroup including the defining homomorphism.

To do this, we first determine the homomorphisms of a simple regular $\omega$-semigroup (a simple regular semigroup $S$ such that $E_S$ is order isomorphic to $I^0$, the non-negative integers, under the reverse of the usual order) into a group (theorem 2.1). Utilizing this result and our determination of the maximal group homomorphic image of an $\omega$-bisimple semigroup (a bisimple semigroup $S$ such that $E_S$ is order isomorphic to $I^0$ under the reverse of the usual order) [6, theorem 3.4], we determine the maximal group homomorphic image a simple regular $\omega$-semigroup including the defining homomorphism (theorem 2.2). Finally, utilizing theorem 2.1, theorem 2.2, and 'an inverse limit process' and 'an inductive process' (introduced in [6]), we determine the maximal group homomorphic image of a simple $I$-regular semigroup. We also completely determine the homomorphisms of a simple $I$-regular semigroup into a group. This result was used in section 1.

The multiplication for a simple regular $\omega$-semigroup $S$ (due to Munn [2]) may be obtained from theorem 1.1 by considering the triples $\{(g_r, m, n) : g_r \in G_r\ (0 \leqq r \leqq d-1), m, n \in I^0\}$. Thus, we may write $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ where $C_1$ is the bicyclic semigroup.

THEOREM 2.1. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ *be a simple regular* $\omega$*-semigroup and let* $H$ *be a group. For each* $i \in \{0, 1, \cdots, d-1\}$, *let* $f_i$ *be a homomorphism of* $G_i$ *into* $H$ *and let* $z \in H$ *such that*

$$f_{d-1} C_z = \gamma_{d-1} f_0, \quad \textit{where } xC_z = zxz^{-1} \quad \textit{for} \quad x \in H, \tag{2.1}$$

*and*

$$f_r = \gamma_r f_{r+1} \quad \textit{for} \quad 0 \leqq r \leqq d-2. \tag{2.2}$$

*Then,*

$$(g_r, m, n)\phi = z^{-m}(g_r f_r)z^n \tag{2.3}$$

*is a homomorphism of* $S$ *into* $H$ *and, conversely every such homomorphism is obtained in this fashion.*

PROOF. Let $\phi$ be a homomorphism of $S$ into $H$. Define $(g_r, 0, 0)\phi = g_r f_r$. Clearly, $f_r$ is a homomorphism of $G_r$ into $H$. Let $(e_0, 0, 1)\phi = z$, where $e_0$ is the identity of $G_0$. Hence $(g_r, m, n)\phi = z^{-m} g_r f_r z^n$ and (2.3) is valid. Since $(g_{d-1}\gamma_{d-1}, 0, 0)(e_0, 0, 1) = (e_0, 0, 1)(g_{d-1}, 0, 0)$, (2.1) is valid. Since, for $0 \leqq r \leqq d-2$, $(g_r, 0, 0)(e_{r+1}, 0, 0) = (g_r\gamma_r, 0, 0)(e_{r+1}, 0, 0)$, (2.2) is valid.

Conversely, let us show that (2.3) subject to the conditions (2.1) and (2.2) defines a homomorphism of $S$ into $H$. Clearly, $\phi$ is a well defined mapping of $S$ into $H$. Form (2.1) and (2.2), we obtain

$$\alpha_{j,d} f_0 = f_j C_z \tag{2.4}$$

By induction, we obtain

$$t^r b_j f_j = b_j \alpha_{j,rd} f_0 z^r \tag{2.5}$$

for each positive integer $r$ and each $b_j \in G_j$ $(0 \leqq j \leqq d-1)$.

Utilizing (2.5) and (2.2), it is easy to show that (2.3) defines a homomorphism of $S$ into $H$.

REMARK 2.1 In the case $d = 1$, we obtain [6, theorem 3.5].

THEOREM 2.2. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ *be a simple regular* $\omega$*-semigroup. Let* $N = \{g \in G_0 | g(\gamma_0\gamma_1 \cdots \gamma_{d-1})^n = k_0$, *the identity of* $G_0$, *for some* $n \in I^0\}$. *Then,* $N$ *is a normal subgroup of* $G_0$. *Let* $g \to \bar{g}$ *be the natural homomorphism of* $G_0$ *onto* $G_0/N$. *Define* $\bar{x}\theta = \overline{x\gamma_0\gamma_1 \cdots \gamma_{d-1}}$ *for* $x \in G_0$. *Then,* $\theta$ *is an endomorphism of* $G_0/N$. *Define a relation* $\sigma$ *on* $G_0/N \times (I^0)^2$ *by the rule* $((\bar{g}_0, a, b), (\bar{h}_0, c, d)) \in \sigma$ *if and only if there exist* $x, y \in I^0$ *such that* $x+a = y+c$, $x+b = y+d$ *and* $\bar{g}_0\theta^x = \bar{h}_0\theta^y$. *Define a binary operation on* $V = G_0/N x(I^0)^2/\sigma$ *by the rule*

$$(\bar{g}_0, a, b)_\sigma(\bar{h}_0, c, d)_\sigma = (\bar{g}_0\theta^c\bar{h}_0\theta^b, a+c, b+d)_\sigma.$$

*Then, V is a group which is the maximal group homomorphic image of S. The canonical homomorphism of S onto V is given by*

$$(g_r, m, n)\zeta = (\overline{g_r \prod_{j=r}^{d-1} \gamma_j}, m+1, n+1)_\sigma \quad \textit{where} \quad g_r \in G_r$$

PROOF. For simplicity, let $\alpha_{n,m} = \prod_{j=n}^{m-1} \gamma_j$ if $m > n$ and let $\alpha_{n,n}$ denote the identity automorphism of $G_{n (\mathrm{mod}\, d)}$. Let $T = \{(g_0, a, b) : g_0 \in G_0; a, b \in I^0\}$. Then, $T$ is the $\omega$-bisimple semigroup $(G_0, C_1, \alpha_{0,d})$ by theorem 1.1 and [6, theorem 1.1] (notation of [6]). Thus, by [6, theorem 3.4], $\sigma$ is an equivalence relation and $V$ is a group. By a routine calculation $(\bar{k}_0, 0, 0)_\sigma$ is the identity of $V$ and $(\bar{g}_0^{-1}, b, a)_\sigma$ is the inverse of $(\bar{g}_0, a, b)_\sigma$. We first employ theorem 2.1 to show that $\xi$ is a homomorphism of $S$ into $V$. Let $z = (\bar{k}_0, 0, 1)_\sigma$ and $g_r f_r = (\overline{g_r \alpha_{r,d}}, 1, 1)_\sigma$ for $0 \leqq r \leqq d-1$. By a straight forward calculation, (2.1) and (2.2) of theorem 2.1 are valid, and $(g_r, m, n)\xi = z^{-m} g_r f_r z^n$.

Since

$$\begin{aligned}(g_0, m, n)\zeta &= (\overline{g_0 \alpha_{0,d}}, m+1, n+1)_\sigma \\ &= (\overline{g_0 \theta}, m+1, n+1)_\sigma \\ &= (\bar{g}_0, m, n)_\sigma,\end{aligned}$$

$\xi$ maps $S$ onto $V$.

Let $\delta$ be a homomorphism of $S$ onto a group $X$. We show that $\delta | T$ is a homomorphism of $T$ onto $X$. By theorem 2.1, for each $r \in \{0, \cdots, d-1\}$, there exists a homomorphism $\delta_r$ of $G_r$ into $X$ and $ap \in X$ such that (2.1) and (2.2) of theorem 2.1 are valid and

$$(g_r, m, n)\delta = p^{-m} g_r \delta_r p^n$$

where $g_r \in G_r$. Thus, if $x \in X$, there exists $g_r \in G_r$, $a, b \in I^0$, such that

$$x = p^{-a} g_r \delta_r p^b.$$

Hence, utilizing (2.1) and (2.2) of theorem 2.1,

$$\begin{aligned}x &= p^{-a} g_r \gamma_r \delta_{r+1} p^b \\ &= p^{-a} g_r \gamma_r \cdots \gamma_{d-2} \delta_{d-1} p^b \\ &= p^{-(a+1)} g_r \alpha_{r,d} \delta_0 p^{(b+1)} \\ &= (g_r \alpha_{r,d}, a+1, b+1)\delta.\end{aligned}$$

By [6, theorem 3.4], $V$ is the maximal group homomorphic image of $T$ under the homomorphism

$$(g_0, a, b)\phi = (g_0, a, b)_\sigma.$$

Hence, there exists a homomorphism $\eta$ of $V$ onto $X$ such that $\phi\eta = \delta | T$.

We will show that $V$ is the maximal group homomorphic image of $S$ under the homomorphism $\xi$. We note that

$$(\bar{g}_0, m, n)_\sigma \eta = (g_0, m, n)\phi\eta = (g_0, m, n)\delta. \tag{2.6}$$

Hence, by (2.6), (2.2), and (2.1),

$$\begin{aligned}(g_r, m, n)\zeta\eta &= (\overline{g_r \alpha_{r,d}}, m+1, n+1)_\sigma \eta \\ &= p^{-(m+1)}(g_r \alpha_{r,d})\delta_0 p^{n+1} \\ &= p^{-m} g_r \delta_r p^n \\ &= (g_r, m, n)\delta.\end{aligned}$$

REMARK 2.2. In the case $d = 1$, we obtain [6, theorem 3.4].

The following remarks will be utilized in giving the canonical homomorphism in theorem 2.3 (below) a convenient form.

Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$ be a simple $I$-regular semigroup. Let $\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$ for $m < n$ let $\alpha_{m,m}$ denote the identity automorphism of $G_{m (\mathrm{mod}\, d)}$. Let $a_1$ denote a non-negative integer. Define

$$t_{id, a_1} = \begin{cases} f_{a_1-1, i+1}^{-1} u_{(i+1)d} \alpha_{0,d}^{a_1-2} \cdots u_{(i+1)d} \alpha_{0,d} u_{(i+1)d} & \text{if } a_1 \geqq 2 \\ k_0, \text{ the identity of } G_0, \text{ otherwise.} \end{cases} \tag{2.7}$$

By the proof of [10, theorem 1] *, $S \cong (U(S_{id} : i \in I, i \leqq 0))\lambda$ where $S_{id}$ is the simple regular $\omega$-semigroup $S_{id} = (d, G_0, \cdots, G_{d-1}, C_1, \gamma_{id,0}, \gamma_{id,1} \cdots, \gamma_{id,d-1})_{id}$ the congruence $\lambda$ defined in [10],

$$\gamma_{id, d-1} = \gamma_{d-1} C_{u_{(i+1)d}^{-1}}, \tag{2.8}$$

and

$$\gamma_{id, s} = \gamma_s \quad \text{for} \quad 0 \leqq s \leqq d-2 \tag{2.9}$$

under an isomorphism $\Psi$ (defined in [10]).

For $g_r \in G_r$ for $0 \leqq r \leqq d-1$,

$$\begin{aligned}(g_r, m, n)_{(i+1)d}\lambda = ((&s_{id}^{-1} \alpha_{id,0,d}^{m-1} \cdots s_{id}^{-1} \alpha_{id,0,d} s_{id}^{-1}) \alpha_{id,0,r} g_r \\ &((s_{id} \cdot s_{id} \alpha_{id,0,d} \cdots s_{id} \alpha_{id,0,d}^{n-1}) \alpha_{id,0,r}), m+1, n+1)_{id}\lambda\end{aligned} \tag{2.10}$$

where if $m = 0$ $(n = 0)$ the right (left) multiplier of $g_r$ is $k_r$, the identity of $G_r$ and

$$s_{id} = u_{(i+2)d}^{-1} u_{(i+1)d}, \tag{2.11}$$

$$g_{d-1} \gamma_{id, d-1} = s_{id}^{-1} (g_{d-1} \gamma_{(i+1)d, d-1}) s_{id} \tag{2.12}$$

By the proof of [10, theorem 1], if $\Psi_{id}$ is as in [10],

* In [10], $S_{id}$ is denoted by $X_{id}$.

$$(g_r, a_1, b_1)_{id} = ((t_{id,a_1}\alpha_{0,r})g_r(t_{id,b_1}^{-1}\alpha_{0,r}), a_1+i, b_1+i)\Psi_{id} \tag{2.13}$$

and

$$(g_r, a_1+i, b_1+i)\Psi_{id} = ((t_{id,a_1}^{-1}\alpha_{0,r})g_r t_{id,b_1}\alpha_{0,r}, a_1, b_1)_{id}. \tag{2.14}$$

THEOREM 2.3. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$ *be a simple I-regular semigroup. Let* $N = \{g \in G_0 | g(\gamma_0\gamma_1\cdots\gamma_{d-1})^n = k_0$, *the identity of* $G_0$, *for some* $n \in I^0\}$. *Then,* $N$ *is a normal subgroup of* $G_0$. *Let* $g \to \bar{g}$ *be the natural homomorphism of* $G_0$ *onto* $G_0/N$. *Define* $\bar{x}\theta = \overline{x\gamma_0\gamma_1\cdots\gamma_{d-1}}$ *for* $x \in G_0$. *Then,* $\theta$ *is an endomorphism of* $G_0/N$. *Define a relation* $\sigma$ *on* $G_0/N x(I^0)^2$ *by the rule* $((\bar{g}_0, a, b), (\bar{h}_0, c, d)) \in \sigma$ *if and only if there exist* $x, y \in I^0$ *such that* $x+a = y+c$, $x+b = y+d$, *and* $\bar{g}_0\theta^x = \bar{h}_0\theta^y$. *Define a binary operation on* $H = G_0/N x(I^0)^2/\sigma$ *by the rule* $(\bar{g}_0, a, b)_\sigma(\bar{h}_0, c, d)_\sigma = (\bar{g}_0\theta^c\bar{h}_0\theta^b, a+c, b+d)_\sigma$. *Then,* $H$ *is a group which is the maximal group homorphic image of* $S$. *The canonical homorphism of* $S$ *onto* $V$ *is given by*

$$(g_r, a, b)\bar{\Phi} = \begin{cases} (x_{id}^{-1}\theta^{a_1-i-1}\cdots x_{id}^{-1}\theta x_{id}^{-1}((t_{id,a-i}^{-1}\alpha_{0,r})g_r(t_{id,b-i}\alpha_{0,r}))\delta_{id} \\ \quad \cdot x_{id}\cdot x_{id}\theta\cdots x_{id}\theta^{b_1-i-1}, a_1-i, b_1-i)_\sigma \text{ for } i \leqq -1. \\ \text{If } a = i\ (b = i), \text{ the corresponding factor is } \bar{k}_0; \\ (\overline{g_r\alpha_{r,d}}, a-i+1, b-i+1)_\sigma \text{ for } i = 0, \end{cases} \tag{2.15}$$

*where* $(g_r, a, b) \in (k_0, i, i)S(k_0, i, i)$ *and where*

$x_0 = \bar{k}_0$,

$x_{-d} = \bar{u}_0^{-1}$ *while for* $i \leqq -2$,

$x_{id} = \bar{u}_0^{-1}(\bar{u}_{-d}^{-1}\theta)\cdots\bar{u}_{(i+1)d}^{-1}\theta^{-(i+1)}\bar{u}_{(i+2)d}\theta^{-(i+1)}\bar{u}_{(i+3)d}\theta^{-(i+2)}\cdots\bar{u}_0\theta$

$g_r\delta_0 = \overline{g_r\alpha_{r,d}}$ *while for* $i \leqq -1$

$g_r\delta_{id} = \bar{u}_0^{-1}\bar{u}_{-d}^{-1}\theta\cdots\bar{u}_{(i+1)d}^{-1}\theta^{-(i+1)}\overline{g_r\alpha_{r,d}}\theta^{-i-1}\bar{u}_{(i+1)d}\theta^{-(i+1)}\cdots\bar{u}_{-d}\theta\bar{u}_0.$

PROOF. As our proof parallels that of [6, theorem 3.6], we will just give a sketch of the proof. We first use theorem 2.1 to determine a homomorphism $\phi_{id}$ of $S_{id}$ into $H$ for each $i \in I$ with $i \leqq 0$. Let $x_{id}$ and $\delta_{id}$ be defined as in the statement of the theorem. In the notation of theorem 2.1, let $z_{id} = (x_{id}, 0, 1)_\sigma$, $g_r f_{r,0} = (g_r\delta_0, 1, 1)_\sigma$ and $g_r f_{r,id} = (g_r\delta_{id}, 0, 0)_\sigma$ for $i \leqq -1$ where $g_r \in G_r$ $(0 \leqq r \leqq d-1)$. Utilizing (2.8), we show that (2.1) and (2.2) are valid.

Hence, by (2.3),

$$(g_r, m, n)_{id}\phi_{id} = \begin{cases} (x_{id}^{-1}\theta^{m-1}\cdots x_{id}^{-1}\theta x_{id}^{-1})(g_r\delta_{id})(x_{id}\cdot x_{id}\theta\cdots x_{id}\theta^{n-1}), m, n)_\sigma \text{ if } i \leqq -1. \\ \text{If } m = 0\ (n = 0) \text{ the corresponding factor is } \bar{k}_0; \\ (\overline{g_r\alpha_{r,d}}, m+1, n+1)_\sigma \text{ if } i = 0, \end{cases} \tag{2.16}$$

defines a homomorphism of $S_{id}$ into $H$.

We note that $(g_r, m, n)_0\phi_0 = (g_r\alpha_{r,d}, m+1, n+1)_\sigma$. Hence, by theorem 2.2, $\phi_0$ is a homomorphism of $S_0$ onto $H$.

Let us define $x\lambda\phi = x\phi_{id}$ if $x \in S_{id}$. We will show that $\phi$ is a homomorphism of $S\Psi$ onto $H$. We note that $(g_r, 1, 1)_{id}\phi_{id} = (g_r, 0, 0)_{(i+1)d}\phi_{(i+1)d}$. Utilizing (2.11), we obtain $(s_{id}, 1, 2)_{id}\phi_{id} = (k_0, 0, 1)_{(i+1)d}\phi_{(i+1)d}$. The desired result is then a consequence of (2.10).

Let $G^*$ be an arbitrary group and let $\rho$ be a homomorphism of $S\Psi$ onto $G^*$. We denote $\lambda\rho|S_{id}$ by $\rho_{id}$. Thus, $\rho_{id}$ is a homomorphism of $S_{id}$ into $G^*$. Since $H$ is the maximal group homomorphic image of $S_0$ under the homomorphism $\phi_0$ by virtue of theorem 2.2, there exists a homomorphism $\gamma$ of $H$ onto the subgroup $S_0\rho_0$ of $G^*$ such that $(g_r, m, n)_0\phi_0\gamma = (g_r, m, n)_0\rho_0$ for all $(g_r, m, n)_0 \in S_0$.

Next suppose that $(g_r, m, n)_{(i+1)d}\phi_{(i+1)d}\gamma = (g_r, m, n)_{(i+1)d}\rho_{(i+1)d}$ where $\gamma$ is a homomorphism of $H$ onto $S_{(i+1)d}\rho_{(i+1)d}$.

By virtue of theorem 2.1, there exists $v_{id}$ in $G^*$ and a homomorphism $\eta_{r,id}$ of $G_r$ into $G^*$ for each $r \in \{0, 1, 2, \cdots, d-1\}$ such that $v_{id}(g_{d-1}\eta_{d-1,id})v_{id}^{-1} = g_{d-1}\gamma_{id,d-1}\eta_{0,id}$ and $g_r\eta_{r,id} = g_r\gamma_{id,r}\eta_{r+1,id}$ for $0 \leqq r \leqq d-2$. Furthermore $(g_r, m, n)_{id}\rho_{id} = v_{id}^{-m}(g_r\eta_{r,id})v_{id}^n$ for $(g_r, m, n)_{id} \in S_{id}$. Since $(g_r, 0, 0)_{(i+1)d}\lambda = (g_r, 1, 1)_{id}\lambda$, when $g_r \in G_r$, by (2.10), $(g_r, 0, 0)_{(i+1)d}\rho_{(i+1)d} = (g_r, 1, 1)_{id}\rho_{id}$. Thus, $g_r\eta_{r,id} = v_{id}(g_r\eta_{r,(i+1)d})v_{id}^{-1}$. Hence, since $(k_0, 0, 1)_{(i+1)d}\rho_{(i+1)d} = (s_{id}, 1, 2)_{id}\rho_{id}$ by (2.10), $v_{id} = (s_{id}^{-1}\eta_{0,(i+1)d})v_{(i+1)d}$. Thus, $g_r\eta_{r,id} = (s_{id}^{-1}(g_r\alpha_{(i+1)d,r,d})s_{id})\eta_{0,(i+1)d}$. Utilizing (2.8), (2.9), and (2.2), we obtain $\overline{s_{id}^{-1}(g_r\alpha_{(i+1)d,r,d})s_{id}} = \bar{u}_{(i+1)d}^{-1}(g_r\alpha_{r,d})\bar{u}_{(i+1)d}$. Hence, $(s_{id}^{-1}(g_r\alpha_{(i+1)d,r,d})\, s_{id}, 0, 0)_{(i+1)d}\phi_{(i+1)d} = (g_r, 0, 0)_{id}\phi_{id}$ and $(g_r, 0, 0)_{id}\rho_{id} = (g_r, 0, 0)_{id}\phi_{id}\gamma$. We also note that $(s_{id}^{-1}, 0, 1)_{(i+1)d}\phi_{(i+1)d} = (k_0, 0, 1)_{id}\phi_{id}$ by (2.16). Hence, $(k_0, 0, 1)_{id}\rho_{id} = (k_0, 0, 1)_{id}\phi_{id}\gamma$. Thus, $(g_r, m, n)_{id}\phi_{id}\gamma = (g_r, m, n)_{id}\rho_{id}$ for all $(g_r, m, n) \in S_{id}$. Hence, $H$ is the maximal group homomorphic image of $S\Psi$ under the homomorphism $\phi$. We put $\phi$ in the form (2.15) by combining (2.16) and (2.14).

REMARK 2.3. In the case $d = 1$, we obtain [6, theorem 3.6].

The following result is needed to give an explicit determination of the extensions of a Brandt semigroup by a simple I-regular semigroup (theorem 1.3).

THEOREM 2.4. *Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id})$ be a simple I-regular semigroup and let $X$ be a group. Let $\{z_{id} : i \in I, i \leqq 0\}$ be a sequence of elements of $X$ and for each $r \in \{0, 1, \cdots, d-1\}$ let $\{f_{id,r} : i \in I, i \leqq 0\}$ be a sequence of homomorphisms of $G_r$ into $X$ such that*

$$
\begin{aligned}
f_{id,d-1} C_{z_{id}} &= \gamma_{d-1} C_{u^{-1}_{(i+1)d}} f_{id,0}, \\
f_{id,r} &= \gamma_r f_{id,r+1} \text{ for } 0 \leqq r \leqq d-2, \\
z_{(i+1)d} &= z_{id}^{-1}((u^{-1}_{(i+2)d} u_{(i+1)d}) f_{0,id}) z_{id}^2, \quad \text{and} \\
f_{r,(i+1)d} &= f_{r,id} C z_{id}
\end{aligned}
$$

*For each $(g_r, a, b) \in (k_0, i, i) S(k_0, i, i)$, define $(g_r, a, b)\phi = z_{id}^{-(a-i)}((t_{id,a-i}^{-1}\alpha_{0,r}) g_r (t_{id,b-i}\alpha_{0,r})) f_{r,id} z_{id}^{b-i}$. Then, $\phi$ defines a homomorphism of $S$ into $X$ and conversely every such homomorphism is defined in this fashion.*

PROOF. We utilize theorem 2.1, (2.14), and the 'inverse limit' process (see [10]).

## 3. The congruences

In this section, we show that each congruence $\rho$ on a simple *I*-regular semigroup $S$ is a group congruence ($S/\rho$ is a group), an idempotent separating congruence (each $\rho$-class contains at most one idempotent) or that $S/\rho$ is a simple *I*-regular semigroup with fewer $\mathfrak{D}$-classes than $S$. We determine the idempotent separating congruences in terms of certain normal subgroups of the structure groups of $S$. The group congruences of $S$ are in a $1-1$ correspondence with the normal subgroups of the maximal group homomorphic image of $S$.

THEOREM 3.1. *Let $S$ be a simple I-regular semigroup. Let $\rho$ be a congruence on $S$. Then $\rho$ is a group congruence, $\rho$ is an idempotent separating congruence, or $S/\rho$ is a simple I-regular semigroup with $t$ $\mathfrak{D}$-classes where $t < d$, the number of $\mathfrak{D}$-classes of $S$.*

PROOF. Let $\{(f_i, n, n) : 0 \leqq i \leqq d-1, n \in I\}$ denote the set of idempotents of $S$. Each $D_i = \{(g_i, m, n); g_i \in G_i, m, n \in I\}$ is an *I*-bisimple semigroup for $0 \leqq i \leqq d-1$. Thus, by [6, theorem 4.2], $\rho | D_i$ is a group congruence or an idempotent separating congruence for $0 \leqq i \leqq d-1$. Suppose that $\rho$ is not an idempotent separating congruence. First suppose that $\rho | D_i$ is a group congruence for some $i$. Hence, $(f_i, 0, 0)\rho = (f_i, k, k)\rho$ for all $k \in I$. Let $(f_j, n, n) \in E_{D_j}$ and $(f_k, p, p) \in E_{D_k}$ and suppose that $(f_j, n, n) < (f_k, p, p)$. Thus, $(f_i, n+1, n+1) < (f_j, n, n) < (f_k, p, p) < (f_i, p-1, p-1)$. Hence, $(f_i, n, n)\rho = (f_k, p, p)\rho$ and $\rho$ is a group congruence. Next, suppose that $\rho | D_i$ is an idempotent separating congruence for each $0 \leqq i \leqq d-1$. Then, there exist $(f_i, n, n)$, $(f_r, q, q) \in E_S$ such that $(f_i, n, n)\rho = (f_k, q, q)\rho$. Thus, $D_i\rho$ and $D_k\rho$ lie in the same $\mathfrak{D}$-class of $S/\rho$. Hence, $S/\rho$ is a simple *I*-regular semigroup with $t$ $\mathfrak{D}$-classes with $t < d$.

REMARK 3.1. In the case $d = 1$, we obtain [6, theorem 4.2].

REMARK 3.2. We may replace 'simple $I$-regular' by 'simple $\omega$-regular' in theorem 3.1. The proof is analogous.

We next determine the idempotent separating congruences of a simple $I$-regular semigroup.

Let $G_0, G_1, \cdots, G_{d-1}$ be a collection of disjoint groups and let $\gamma_i$ be a homomorphism of $G_i$ into $G_{i+1}$ for $0 \leqq i \leqq d-2$ and let $\gamma_{d-1}$ be a homomorphism of $G_{d-1}$ into $G_0$. Let $V_i$ be a normal subgroup of $G_i$ for $0 \leqq i \leqq d-1$ such that $V_i\gamma_i \subseteq V_{i+1}$ for $c \leqq i \leqq d-2$ and $V_{d-1}\gamma_{d-1} \subseteq V_0$. Then, $(V_0, V_1, \cdots, V_{d-1})$ will be called a $\gamma_0-\gamma_1 \cdots -\gamma_{d-1}$ invariant $d$-tuple of $(G_0, G_1, \cdots, G_{d-1})$. Let $(V_0, V_1, \cdots, V_{d-1})$ and $(U_0, U_1, \cdots, U_{d-1})$ be $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ invariant $d$-tuples of $(G_0, G_1, \cdots, G_{d-1})$. Then, we say $(V_0, V_1, \cdots, V_{d-1}) \subseteq (U_0, U_1, \cdots, U_{d-1})$ if and only if $V_i \subseteq U_i$ for $0 \leqq i \leqq d-1$.

In the proof of the following theorem, we will utilize a theorem of Preston [6, theorem 4.3]. We also utilize the notation of this theorem. We will sketch the following proof where it parallels the proof of [6, theorem 4.4].

THEOREM 3.2. *Let* $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ *be a simple I-regular semigroup. There exists a* $1-1$ *correspondence between the idempotent separating congruences on S and the* $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ *invariant d-tuples of* $(G_0, G_1, \cdots, G_{d-1})$. *If* $\rho^{(V_0, V_1, \cdots, V_{d-1})}$ *is the idempotent separating congruence corresponding to the* $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ *invariant d-tuple* $(V_0, V_1, \cdots, V_{d-1})$, $(g_r, a, b)\rho^{(V_0, \cdots, V_{d-1})}(h_s, c, d)$ *if and only if* $r = s$, $a = c$, $b = d$ *and* $V_rg_r = V_rh_r$. *If* $(V_0, V_1, \cdots, V_{d-1})$ *and* $(U_0, U_1, \cdots, U_{d-1})$ *are two* $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ *invariant d-tuples* $(V_0, V_1, \cdots, V_{d-1}) \subseteq (U_0, U_1, \cdots, U_{d-1})$ *if and only if* $\rho^{(V_0, V_1, \cdots, V_{d-1})} \subseteq \rho^{(U_0, U_1, \cdots, U_{d-1})}$.

PROOF. Let $(V_0, V_1, \cdots, V_{d-1})$ be a $\gamma_0-\gamma_1- \cdots -\gamma_{d-1}$ invariant $d$-tuple of $(G_0, G_1, \cdots, G_{d-1})$. Let $N_{(k_r, a, a)} = \{(v_r, a, a) : v_r \in V_r\}$ and let $N = U(N_{(k_r, a, a)} : 0 \leqq r \leqq d-1, a \in I)$. By a routine calculation, $N_{(k_r, a, a)}$ is a subgroup of $S$ isomorphic to $V_r$. By [6, theorem 4.3] $\rho_N$ is an idempotent separating congruences of $S$. We denote $\rho_N$ by $\rho^{(V_0, V_1, \cdots, V_{d-1})}$.

Let $\rho$ be an idempotent separating congruence of $S$. Then, by [6, theorem 4.3] $\rho = \rho_N$ where $N$ is given in the statement of [6, theorem 4.3]. $N_{(e_r, a, a)} = \{(v_r, a, a) : v_r \in V_r\}$ where $V_r$ is an invariant subgroup of $G_r$. Since $(e_{r+1}, 0, 0)(e_r, 0, 0) = (e_{r+1}, 0, 0)$, $(e_{r+1}, 0, 0)(v_r, 0, 0) \in N_{(e_{r+1}, 0, 0)}$. Thus $v_r\gamma_r \subseteq V_{r+1}$ for $0 \leqq r \leqq d-2$. Since $(e_0, 0, 1)(e_{d-1}, 0, 0)(e_0, 1, 0) = (e_0, 0, 0)$, $(e_0, 0, 1)(v_{d-1}, 0, 0)(e_0, 1, 0) \in N_{(e_0, 0, 0)}$. Thus, $v_{d-1}\gamma_{d-1} \in V_0$. Hence, $\rho = \rho^{(V_0, V_1, \cdots, V_{d-1})}$ and we have the desired correspondence.

REMARK. In the case $d = 1$, we obtain [6, theorem 4.4].

REMARK. We may replace 'simple $I$-regular semigroup' by 'simple regular $\omega$-semigroup' in theorem 3.2. The proof is analogous.

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