R. J. Warne

Some properties of simple I-regular semigroups


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Let $S$ be a regular semigroup and let $E_S$ denote its set of idempotents. As usual, $E_S$ is partially ordered in the following manner: if $e, f \in E_S$, $e \leq f$ if and only if $ef = fe = e$. We then say that $E_S$ is under or assumes its natural order. Let $I$ denote the integers. If $E_S$, under the natural order, is order isomorphic to $I$ under the reverse of the usual order, we call $S$ an I-regular semigroup. We determined the structure of I-regular semigroups mod groups in [10].

In section 1, we develop the ideal extension theory of simple I-regular semigroups. In section 2, we obtain the maximal group homomorphic image of a simple I-regular semigroup including the defining homomorphism. In section 3, we determine the nature of the congruences admitted by a simple I-regular semigroup, and we describe the idempotents separating congruences.

In the special case $S$ is bisimple, the results of this paper reduce to the corresponding results for I-bisimple semigroups (bisimple semigroups $S$ such that $E_S$ is order isomorphic to $I$ under the reverse of the usual order) [6, 7].

Unless otherwise specified, we utilize the definitions, terminology, and notation of [1].

1. Ideal extension theory

In this section, we determine the translational hull $\bar{S}$ of a simple I-regular semigroup $S$. All ideal extensions of $S$ by a semigroup $T$ with zero, $o$, can then be described if one knows the structure of $T$ and the partial homomorphisms $\theta$ of $T^* = T \setminus 0$ into $\bar{S}$ such that $AB = 0$ in $T$ implies that $A\theta B\theta \in S$ [1]. This determination is carried out if $T$ is a completely 0-simple (Brandt) semigroup. We also completely determine the extensions of a Brandt semigroup with finite index set by a simple I-regular semigroup (with zero appended) by specializing our general determination of the extensions of a Brandt semigroup by an arbitrary semigroup [5, theorem 1].

Before commencing, let us state the structure theorem for simple I-regular semigroups.
Let \( C^*_1 = IxI \) under the multiplication \((a, b)(c, d) = (a + c - \min(b, c), b + d - \min(b, c))\). We called \( C^*_1 \) the extended bicyclic semigroup in [6].

**Theorem 1.1** (Warne, [10]). \( S \) is a simple \( I \)-regular semigroup if and only if \( S = (U(G_j : j = 0, 1, \cdots, d-1)) \times C^*_1 \), where \( d \) is a positive integer, \( \{G_j : 0 \leq j \leq d-1\} \) is a collection of pairwise disjoint groups, and \( C^*_1 \) is the extended bicyclic semigroup, under the multiplication

\[
(g_s, (m, n))(h_r, (p, q)) = (t, (m, n)(p, q))
\]

(*)

where

\[
g_s \in G_s, g_r \in G_r \quad (0 \leq r, s \leq d-1) \quad \text{and} \quad t = \frac{d}{p+d+r-1} \sum_{j=0}^{d-1} \gamma_j (f^n_{p-n, p} \prod_{j=0}^{d-1} \gamma_j) (f^n_{p-n, p} \prod_{j=0}^{d-1} \gamma_j) (f^n_{p-n, p} \prod_{j=0}^{d-1} \gamma_j) h_r,
\]

\(::j=n\sum_{j=n}^{d-1}-1 \prod_{j=0}^{d-1} \gamma_j) (f^n_{p-n, p} \prod_{j=0}^{d-1} \gamma_j) h_r,
\]

\(:v=-1 \prod_{j=0}^{d-1} \gamma_j) (v = \text{max}(r, s))
\]

according to whether \( n > p, p > n, \) or \( p = n \) where \( \gamma_j = \gamma_{j \text{mod } d}(j \in I, j \geq 0) \) is a homomorphism of \( G_{j \text{mod } d} \) into \( G_{(j+1) \text{mod } d} \). Juxtaposition denotes multiplication in \( C^*_1 \) and in the appropriate \( G_j \). For \( m \in I^0 \), the non-negative integers, \( n \in I, f_0, n = k_0, \) the identity of \( G_0 \), while, for \( m > 0, \)

\[
f_{m, n} = u_{(n+1)d} \prod_{j=0}^{d-1} \gamma_j u_{(n+2)d} \prod_{j=0}^{d-1} \gamma_j \prod_{j=0}^{d-1} \gamma_j u_{(n+m)d}
\]

where \( \{u_{kd} : k \in I\} \) is a collection of elements of \( G_0 \) with \( u_{kd} = k_0 \) for \( k > 0 \). In (*) \( \prod_{j=0}^{d-1} \gamma_j \) will denote the identity automorphism of \( G_{a \text{mod } d} \).

Let \( S \) be a simple \( I \)-regular semigroup. In connection with theorem 1.1, we write \( S = (d, G_0, G_1, \cdots, G_{d-1}, C^*_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id}) \).

For convenience, we write \( \alpha_{m, n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1} \) if \( m < n \) and let \( \alpha_{m, n} \) denote the identity automorphism of \( G_{n \text{mod } d} \).

**Lemma 1.1.** A simple \( I \)-regular semigroup is left and right reductive.

**Proof.** This lemma is an immediate consequence of theorem 1.1. We will utilize the multiplication of theorem 1.1 without explicit mention.

**Theorem 1.2** Let \( S = (d, G_0, G_1, \cdots, G_{d-1}, C^*_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id}) \) be a simple \( I \)-regular semigroup. Let \( W = \{(\theta, p) : \theta : I \to G_0, p \in I, \) and \( (i+1)\theta = m_{(i+1)d} (\theta \prod_{j=0}^{d-1} \gamma_j) m_{(i+p+1)d} \) for all \( i \in I\). Let \( \rho_i (i \in I) \) denote the inner right translation of \((I, +)\) determined by \( i \cdot W \), under the multiplication

\[
*(\theta, w)(\eta, p) = (\theta \circ \rho_w \eta, w + p),
\]
where \( \circ \) denotes pointwise multiplication of mappings and juxtaposition denotes iteration of mappings is a group. Let \( \mathcal{S} \) be the translational hull of \( S \). Then, \( \mathcal{S} = W \cup S \) (\( W \cap S = \emptyset \)), under the multiplication

\[
(\theta, a) \cdot (\eta, p) = (\theta, a)(\eta, p)
\]

\[
(g_s, a, b) \cdot (h_r, c, d) = (g_s, a, b)(h_r, c, d)
\]

where juxtaposition denotes multiplication in \( W \) and \( S \) and

\[
(\theta, p) \cdot (g_r, a, b) = ((a-p)\theta \prod_{j=0}^{r-1} g_j g_r, a-p, b)
\]

\[
(g_r, a, b) \cdot (\theta, p) = (g_r(b\theta \prod_{j=0}^{r-1} g_j), a, b+p).
\]

**Proof.** Let \( \lambda \) be a left translation of \( S \). Then, if \( e_0 \) is the identity of \( G_0 \),

\[
(e_0, i, i)\lambda = (i\delta, i\delta_1, i+ip_1)
\]

where \( \delta : I \to U(G_j : 0 \leq j \leq d-1) \); \( \delta_1 : I \to I \); and \( p_1 : I \to I^0 \) the non-negative integers. Since \( (e_0, i, i)(e_0, i+1, i+1) = (e_0, i+1, i+1) \), we have the following two possibilities: If \( ip_1 = 0 \),

\[
(i+1)\delta = m_{(i+1)\delta}^{i\delta \alpha_{r, d}} m_{(i+1)d} \quad \text{where} \quad \delta \in G_r,
\]

\[
(i+1)p_1 = 0, \quad \text{and}
\]

\[
(i+1)\delta_1 = i\delta_1 + 1
\]

while, if \( ip_1 \geq 1 \),

\[
(i+1)\delta = i\delta
\]

\[
(i+1)p_1 = ip_1 - 1
\]

\[
(i+1)\delta_1 = i\delta_1.
\]

Let us first consider the case \( ip_1 = 0 \) for all \( i \in I \). In this case it is easily seen that \( \lambda|D_0 \), where \( D_0 = \{(g_0, m, n) : g_0 \in G_0, m, n \in I \} \), is a left translation of \( D_0 \). Hence, since \( D_0 \) is the I-bisimple semigroup \( (G_0, C_1^*, \alpha_{0, d}, m_d) \) [6, theorem 1.2] (notation of [6]),

\[
(e_0, i, i)\lambda = (i\delta, i+p, i)
\]

where \( p \in I \) and \( \delta \) is a mapping of \( I \) into \( G_0 \) such that

\[
(i+1)\delta = m_{(i+p+1)d}^{-1} i\delta \alpha_{0, d} m_{(i+1)d}
\]

by virtue of [7, 8] or by [9, proof of theorem 1]. Hence, since \( (g_r, i, j) = (e_0, i, i)(g_r, i, j) \),

\[
(g_r, i, j)\lambda_{(\delta, p)} = ((i\delta)\alpha_{r, g_r, i+p, j})
\]
where \( \lambda = \lambda(\delta, \rho) \), \( p \in I \) and \( \delta \) is a mapping of \( I \) into \( G_0 \) satisfying (1.3).

Conversely, (1.3) and (1.4) define a left translation of \( D_0 \) by [7] or by [9, proof of theorem 1]. By (1.3),

\[
(g_r, a, b)\lambda(\delta, \rho)(g_r, a, b) = (e_0, a, a)\lambda(\delta, \rho)(g_r, a, b).
\]

Thus,

\[
((g_r, a, b)(h_s, c, d))\lambda(\delta, \rho) = (e_0, a, a)\lambda(\delta, \rho)(g_r, a, b)(h_s, c, d)
\]

Hence, \( \lambda(\delta, \rho) \) is a left translation of \( S' \).

Next, suppose that there exists \( u \in I \) such that \( u \rho_1 \neq 0 \). Utilizing (1.1) and (1.2), we obtain: \((t+i)\rho_1 = 0\), where \( t \) is a unique element in \( I \), for \( i \geq 0 \), and \((t+i)\rho_1 = -i\) for \( i < 0 \); \((t+i)\delta_1 = a+i\), where \( a \in I \), for \( i \geq 0 \), and \((t+i)\delta_1 = a\) for \( i < 0 \); and \((t+i)\delta = f_i^{-1}g_s a x_i a^{-1} f_i\), for \( i > 0 \), and \((t+i)\delta = g_s \in G_s \) for \( i \leq 0 \). Since \((e_0, i, i)(e_0, i+n, i) = (e_0, i+n, i)\) for all \( n \geq 0 \), we are able to determine \((e_0, i+n, i)\). Next, since \((g_r, i+n, i+m) = (e_0, i+n, i)(g_r, i, i+m)\) for \( i \in I, m, n \in I^0 \), we are able to determine \((g_r, i+n, i+m)\) by [3] and theorem 1.1, every element of \( S \) may be written in the form \((g_r, i+n, i+m)\) where \( g_s \in G_r, i \in I, m, n \in I^0 \). We let \( i = t+q \) and determine \((g_r, t+q+n, t+q+m)\) in terms of the values of \( \delta, \rho_1 \), and \( \delta_1 \) given above. In this calculation, we utilize the identity \( f_m+c,n f_c^{-1} f_{m+n} = f_{m+n} a x_c a^{-1} f_{m+n}\) for \( m, c \in I^0 \) and \( n \in I \) [10]. (This identity may be developed by a routine calculation.) Finally, if \( a_1 = t+q+n \) and \( b_1 = t+q+m \), we show that \((g_r, a_1, b_1)\lambda = (g_s, a, t)(g_r, a_1, b_1)\), i.e. \( \lambda \) is an inner left translation. (We omit the details of these calculations as they parallel calculations given in [7] and [9]).

In a similar manner, it may be shown that the semigroup of right translations of \( S \) consists of the inner right translations of \( S \) and the transformations of \( S \) defined by

\[
(g_r, i, j)\rho(\theta, w) = (g_r(j \theta x_0, r), i, j+w)
\]

where \( w \in I \) and \( \theta \) is a mapping of \( I \) into \( G_0 \) such that

\[
(i+1)\theta = m_{(i+1)d}(i \theta x_0, d) m_{(i+w+1)d} \quad \text{for all} \quad i \in I.
\]
are linked. Thus, by the proof of [9, theorem 1] or [7], $w = -p$ and $\delta = \rho_{-w,0}$. By the proof of [9, theorem 1] or [7], $\rho_{(\theta, w)}D_0$ and $\lambda_{(p_{-w}, -w)} |D_0$ are linked. Thus, $(g_s, a, b)_{D_0}((e_0, b, b))$

$$\rho_{(\theta, w)}(h_r, c, d) = (g_s, a, b)((e_0, b, b)\rho_{(\theta, w)}(e_0, c, c))(h_r, c, d) = (g_s, a, b)_{D_0}(e_0, c, c)\lambda_{(p_{-w}, -w)}(h_r, c, d) = (g_s, a, b)((h_r, c, d)\lambda_{(p_{-w}, -w)}).$$

Thus, $\rho_{(\theta, w)}$ and $\lambda_{(p_{-w}, -w)}$ are linked. The mapping $\rho \to (\lambda, \rho)$, where $\rho$ is a right translation of $S$ and $\lambda$ is the left translation of $S$ linked with $\rho$, is an isomorphism of the semigroup of right translations of $S$ onto $\mathcal{S}$. If $\rho_{(\theta, q)}, \rho_{(\eta, p)} \in S \setminus \mathcal{S}$, $\rho_{(\theta, q)}\rho_{(\eta, p)} = \rho_{(\theta \circ \rho_{(\eta, q+p)}}$ by (1.5) and (1.6). Hence $S \setminus \mathcal{S}$ is a semigroup. The mapping $(\theta, p) \to \rho_{(\theta, p)}$ is an isomorphism of $W$, under the multiplication $\ast$, onto $S \setminus \mathcal{S}$. Clearly, $W$ is a group. The remainder of the theorem is a consequence of [1, p. 12, lemma 1.2], (1.4), and (1.5).

**Remark 1.1.** In the case $d = 1$, we obtain [7, theorem 1] (see also [8]).

**Corollary 1.1.** Let $S$ be a weakly reductive semigroup and let $S$ be its translational hull. Let $T$ be a 0-simple semigroup having proper divisors of zero. If $S = S$ or $S \setminus S$ is a subsemigroup of $S$, then every extension of $S$ by $T$ is given by a partial homomorphism [4].

**Proof.** Replace $\mathcal{S}$ by $\mathcal{S}$ in the proof of [7, theorem 3].

**Remark 1.2.** Let $S = (d, G_0, G_1, \cdots, G_{d-1}, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, m_{id})$ be a simple $I$-regular semigroup. $S$ has $d$ $\mathcal{S}$-classes, $D_0, D_1, \cdots, D_{d-1}$. $D_r = \{(g_r, a, b) : g_r \in G_r, a, b \in I\}$ is the $I$-bisimple semigroup $(G_r, C_r^1, \alpha_r, \beta_r, m_{id}^0)$. (Notation of [6]). Let $T$ be a 0-bisimple semigroup. To determine the partial homomorphisms of $T \setminus 0$ into $S$ one must just determine the partial homomorphisms of $T \setminus 0$ into $D_r$ for each $r \in \{0, 1, 2, \cdots, d-1\}$. In the case $T$ is a completely 0-simple semigroup, (a Brandt semigroup), these determinations are given mod groups by [7, theorem 2] ([7, corollary 1]). By lemma 1.1, theorem 1.2, and Corollary 1.1, if $T$ is a 0-simple semigroup with proper divisors of zero, every extension of $S$ by $T$ is given by a partial homomorphism. In particular, this is valid if $T$ is a completely 0-simple semigroup (Brandt Semigroup) with proper divisors of zero.

**Corollary 1.2.** Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C^*, \delta_0, \cdots, \delta_{d-1}, m_{id})$ be a simple $I$-regular semigroup and let $T = M^0(R; K; A; P)$ be a completely 0-simple semigroup (with zero, $0'$) without proper divisors of zero. Let $V$ be an extension of $S$ by $T$. Then, either $V$ is given by a partial homomorphism and an explicit multiplication is thus given by employing remark 1.2. (Conversely, every partial homomorphism of $T \setminus 0'$ into $S$ determines an extension of $S$ by $T$), or $V = (T \setminus 0') \cup S$ under the multiplication
A) \((a; s, \lambda)^*(g_r, m, n)\) 
\begin{align*}
&= ((m - k_s - i_a - t_\lambda)(\beta_s \circ \rho_{k_s} \theta_a \circ \rho_{k_s + i_a} \gamma_\lambda) \prod_{j=0}^{r-1} \delta_j g_r, m - k_s - i_a - t_\lambda, n) \\
B) \((g_r, m, n)^*(a; s, \lambda)\) 
&= (g_r((n \beta_s \circ \rho_{k_s} \theta_a \circ \rho_{k_s + i_a} \gamma_\lambda) \prod_{j=0}^{r-1} \delta_j), m, k_s + i_a + t_\lambda + n) 
\end{align*}

where \((g_s, m, n) \in S\) and \((a; s, \lambda) \in T'\), \(\circ\) denotes pointwise multiplication of mappings, \(a \mapsto i_a\) is a homomorphism of \(R\) into \((I, +)\), \(a \mapsto \theta_a\) is a mapping of \(R\) into \(H = \{\beta : (\beta, a) \in W\} \) (see statement of theorem 1.2) such that \(\theta_{ab} = \theta_a \circ \rho_{ia} \theta_b\) for all \(a, b \in R\), \(s \mapsto \beta_s\) is a mapping of \(K\) into \(H\), \(s \mapsto k_s\) is a mapping of \(K\) into \(I\), \(\lambda \mapsto t_\lambda\) is a mapping of \(\Lambda\) into \(I\), and \(\lambda \mapsto t_\lambda\) is a mapping of \(\Lambda\) into \(I\) such that \(i_{p, a_e} = t_\lambda + k_s\) and \(\theta_{p, a_e} = \gamma_\lambda \circ \rho_{t_\lambda} \beta_s\). Conversely, \((A)\) and \((B)\) define an extension of \(S\) by \(T\).

**Proof.** The proof utilizes theorem 1.1, theorem 1.2, corollary 1.1, and [1, theorem 4.20 and theorem 4.22]. It is similar in nature to the proof of [7, theorem 4] (see also [8]) and [9, theorem 4] and it will be omitted.

**Remark 1.3.** In the case \(d = 1\), we obtain [7, theorem 4] [see also [8]].

**Remark 1.4.** In the special case that \(T'\) is a group \(R\), \(V\) is either given by a partial homomorphism or \((A)\) and \((B)\) become

\[ a^*(g_r, m, n) = ((m - i_a) \theta_a \prod_{j=0}^{r-1} \gamma_j g_r, m - i_a, n) \]
\[ (g_r, m, n)^*a = (g_r((n \theta_a) \prod_{j=0}^{r-1} \gamma_j), m, n + i_a). \]

**Remark 1.5.** If \(T\) is a 0-simple semigroup without proper divisors of zero, an extension of \(S\) by \(T\) is either given by a partial homomorphism or by the equations in the above remark with \(a \mapsto \theta_a\) a mapping of \(T'\) into \(H\) and with \(a \mapsto i_a\) a homomorphism of \(T'\) into \((I, +)\).

We close this section by giving a specialization of [5, theorem 1]. The theorem is obtained by combining theorem 3.1 (below), [5, theorem 1], and [5, lemma 1]. The theorem is quite similar to [9, theorem 7].

In the theorem below, capital roman letters will denote elements of \(T^*\).

**Theorem 1.3.** Let \(S = \mathcal{M}^0(G; J; J; \Delta)\), where \(J\) is a finite set, be a Brandt semigroup; let \(T^* = (d, U_0, U_1, \ldots, U_{d-1}, C_1, \gamma_0, \gamma_1, \ldots, \gamma_{d-1}, m_a)\) be a simple \(I\)-regular semigroup; and let \(V\) be an extension of \(S\) by \(T\). Then, there exists a homomorphism \(w : A \to w_A\) of \(T^*\) into \(H_r\), the full symmetric group on some \(r\) element subset \(Q\) of \(J\). This homomorphism is explicitly given by theorem 2.4. For each \(A \in T^*\), there exists a mapping \(\psi_A\) of \(Q\) into the group \(G\) such that
The products in $V$ are given by

\[(i\psi_A)(iw_A\psi_B) = i\psi_{AB} \text{ for all } i \in Q.\]

The major purpose of this section is to determine the maximal group homomorphic image of a simple $I$-regular semigroup including the defining homomorphism.

To do this, we first determine the homomorphisms of a simple regular $\omega$-semigroup (a simple regular semigroup $S$ such that $E_S$ is order isomorphic to $\mathbb{Z}^0$, the non-negative integers, under the reverse of the usual order) into a group (theorem 2.1). Utilizing this result and our determination of the maximal group homomorphic image of an $\omega$-bisimple semigroup (a bisimple semigroup $S$ such that $E_S$ is order isomorphic to $\mathbb{Z}^0$ under the reverse of the usual order) [6, theorem 3.4], we determine the maximal group homomorphic image a simple regular $\omega$-semigroup including the defining homomorphism (theorem 2.2). Finally, utilizing theorem 2.1, theorem 2.2, and ‘an inverse limit process’ and ‘an inductive process’ (introduced in [6]), we determine the maximal group homomorphic image of a simple $I$-regular semigroup. We also completely determine the homomorphisms of a simple $I$-regular semigroup into a group. This result was used in section 1.

The multiplication for a simple regular $\omega$-semigroup $S$ (due to Munn [2]) may be obtained from theorem 1.1 by considering the triples \[\{(g_r, m, n) : g_r \in G_r (0 \leq r \leq d-1), m, n \in \mathbb{Z}^0\}.\] Thus, we may write $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ where $C_1$ is the bicyclic semigroup.
THEOREM 2.1. Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ be a simple regular $\omega$-semigroup and let $H$ be a group. For each $i \in \{0, 1, \cdots, d-1\}$, let $f_i$ be a homomorphism of $G_i$ into $H$ and let $z \in H$ such that

$$f_{d-1}C_x = \gamma_{d-1}f_0,$$

where $xC_x = xz^{-1}$ for $x \in H$, \hspace{1cm} (2.1)

and

$$f_r = \gamma_r f_{r+1} \hspace{1cm} \text{for} \hspace{0.5cm} 0 \leq r \leq d-2.$$ \hspace{1cm} (2.2)

Then,

$$(g_r, m, n)\phi = z^{-m}(g_r f_r)z^n \hspace{1cm} \text{(2.3)}$$

is a homomorphism of $S$ into $H$ and, conversely every such homomorphism is obtained in this fashion.

PROOF. Let $\phi$ be a homomorphism of $S$ into $H$. Define $(g_r, 0, 0)\phi = g_r f_r$. Clearly, $f_r$ is a homomorphism of $G_r$ into $H$. Let $(e_0, 0, 1)\phi = z$, where $e_0$ is the identity of $G_0$. Hence $(g_r, m, n)\phi = z^{-m}g_r f_r z^n$ and (2.3) is valid. Since $(g_{d-1} \gamma_{d-1}, 0, 0)(e_0, 0, 1) = (e_0, 0, 1)(g_{d-1}, 0, 0)$, (2.1) is valid. Since, for $0 \leq r \leq d-2$, $(g_r, 0, 0)(e_{r+1}, 0, 0) = (g_r \gamma_r, 0, 0)(e_{r+1}, 0, 0)$, (2.2) is valid.

Conversely, let us show that (2.3) subject to the conditions (2.1) and (2.2) defines a homomorphism of $S$ into $H$. Clearly, $\phi$ is a well defined mapping of $S$ into $H$. Form (2.1) and (2.2), we obtain

$$\alpha_{j, d} f_0 = f_j C_x \hspace{1cm} \text{(2.4)}$$

By induction, we obtain

$$r'b_j f_j = b_j \alpha_{j, r} f_0 z^r \hspace{1cm} \text{(2.5)}$$

for each positive integer $r$ and each $b_j \in G_j \ (0 \leq j \leq d-1)$.

Utilizing (2.5) and (2.2), it is easy to show that (2.3) defines a homomorphism of $S$ into $H$.

REMARK 2.1 In the case $d = 1$, we obtain [6, theorem 3.5].

THEOREM 2.2. Let $S = (d, G_0, G_1, \cdots, G_{d-1}, C_1, \gamma_0, \gamma_1, \cdots, \gamma_{d-1})$ be a simple regular $\omega$-semigroup. Let $N = \{g \in G_0 \mid g(g_0 \gamma_1 \cdots \gamma_{d-1})^n = k_0$, the identity of $G_0$, for some $n \in I^0\}$. Then, $N$ is a normal subgroup of $G_0$. Let $g \rightarrow \tilde{g}$ be the natural homomorphism of $G_0$ onto $G_0/N$. Define $\bar{x} = x\gamma_0 \gamma_1 \cdots \gamma_{d-1}$ for $x \in G_0$. Then, $\theta$ is an endomorphism of $G_0/N$. Define a relation $\sigma$ on $G_0/N \times (I^0)^2$ by the rule $((\bar{g}_0, a, b), (\bar{h}_0, c, d)) \in \sigma$ if and only if there exist $x, y \in I^0$ such that $x + a = y + c$, $x + b = y + d$ and $\bar{g}_0 \theta x = \bar{h}_0 \theta y$. Define a binary operation on $V = G_0/N \times (I^0)^2/\sigma$ by the rule

$$(\bar{g}_0, a, b) \sigma (\bar{h}_0, c, d)_\sigma = (\bar{g}_0 \theta^c \bar{h}_0 \theta^d, a + c, b + d)_\sigma.$$
Then, $V$ is a group which is the maximal group homomorphic image of $S$. The canonical homomorphism of $S$ onto $V$ is given by

$$(g_r, m, n)x = (g_r, \prod_{j=r}^{m-1} y_j, m+1, n+1)$$

where $g_r \in G_r$.

**Proof.** For simplicity, let $\alpha_{n,m} = \prod_{j=n}^{m-1} y_j$ if $m > n$ and let $\alpha_{n,n}$ denote the identity automorphism of $G_{n(\text{mod } d)}$. Let $T = \{(g_0, a, b) : g_0 \in G_0; a, b \in I^0\}$. Then, $T$ is the $\omega$-bisimple semigroup $(G_0, C_1, \alpha_{0,d})$ by theorem 1.1 and [6, theorem 1.1] (notation of [6]). Thus, by [6, theorem 3.4], $\sigma$ is an equivalence relation and $V$ is a group. By a routine calculation $(k_0, 0, 0)_\sigma$ is the identity of $V$ and $(\bar{g}_0^{-1}, b, a)_\sigma$ is the inverse of $(\bar{g}_0, a, b)_\sigma$. We first employ theorem 2.1 to show that $\xi$ is a homomorphism of $S$ into $V$. Let $z = (k_0, 0, 1)_\sigma$ and $g_rf_r = (g_r, \alpha_{r,d}, a, b)_\sigma$ for $0 \leq r \leq d-1$. By a straightforward calculation, (2.1) and (2.2) of theorem 2.1 are valid, and $(g_r, m, n)z = z^{-m}g_rf_rz^n$.

Since

$$(g_0, m, n)z = (g_0, \alpha_{0,d}, m+1, n+1)\sigma = (g_0, b, m+1, n+1)\sigma = (\bar{g}_0, m, n)\sigma,$$

$\xi$ maps $S$ onto $V$.

Let $\delta$ be a homomorphism of $S$ onto a group $X$. We show that $\delta|T$ is a homomorphism of $T$ onto $X$. By theorem 2.1, for each $r \in \{0, \ldots, d-1\}$, there exists a homomorphism $\delta_r$ of $G_r$ into $X$ and $a_p \in X$ such that (2.1) and (2.2) of theorem 2.1 are valid and

$$(g_r, m, n)\delta = p^{-m}g_r\delta_r p^n$$

where $g_r \in G_r$. Thus, if $x \in X$, there exists $g_r \in G_r$, $a, b \in I^0$, such that $x = p^{-a}g_r\delta_r p^{b}$. Hence, utilizing (2.1) and (2.2) of theorem 2.1,

$$x = p^{-a}g_r\gamma_r\delta_{r+1} p^{b} = p^{-a}g_r\gamma_r \cdots \gamma_{d-2}\delta_{d-1} p^{b} = p^{-(a+1)}g_r\alpha_{r,d} \delta_0 p^{(b+1)} = (g_r\alpha_{r,d}, a+1, b+1)\delta.$$

By [6, theorem 3.4], $V$ is the maximal group homomorphic image of $T$ under the homomorphism

$$(g_0, a, b)\phi = (g_0, a, b)_\sigma.$$

Hence, there exists a homomorphism $\eta$ of $V$ onto $X$ such that $\phi\eta = \delta|T$. 

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We will show that \( V \) is the maximal group homomorphic image of \( S \) under the homomorphism \( \xi \). We note that
\[
(g_0, m, n)\eta = (g_0, m, n)\phi \eta = (g_0, m, n)\delta.
\]
(2.6)

Hence, by (2.6), (2.2), and (2.1),
\[
(g_r, m, n)\zeta \eta = (g_r\alpha_{r,d}, m+1, n+1)\eta
= p^{-(m+1)}(g_r\alpha_{r,d})\delta_0 p^{n+1}
= p^{-m}g_r\delta_r p^n
= (g_r, m, n)\delta.
\]

**REMARK 2.2.** In the case \( d = 1 \), we obtain [6, theorem 3.4].

The following remarks will be utilized in giving the canonical homomorphism in theorem 2.3 (below) a convenient form.

Let \( S = (d, G_0, G_1, \cdots, G_{d-1}, C_1^*, \gamma_0, \gamma_1, \cdots, \gamma_{d-1}, u_{id}) \) be a simple \( I \)-regular semigroup. Let \( \alpha_{m,n} = \gamma_m\gamma_{m+1} \cdots \gamma_{n-1} \) for \( m < n \) let \( \alpha_{m,m} \) denote the identity automorphism of \( G_m(\mod d) \). Let \( a_k \) denote a non-negative integer. Define
\[
t_{id,a_k} = \begin{cases} f_{a_1-1,i+1}^{-1}u_{(i+1)d}^{-2}u_{(i+1)d}^{-1}a_0u_{(i+1)d} & \text{if } a_1 \geq 2 \\
k_0, \text{ the identity of } G_0, \text{ otherwise.} \end{cases}
\]
(2.7)

By the proof of [10, theorem 1] *, \( S \cong (U(S_{id} : i \in I, i \leq 0))\lambda \) where \( S_{id} \) is the simple regular \( \omega \)-semigroup \( S_{id} = (d, G_0, \cdots, G_{d-1}, C_1, \gamma_{id}, 0, \gamma_{id}, 1 \cdots, \gamma_{id,d-1}) \) the congruence \( \lambda \) defined in [10],
\[
\gamma_{id,d-1} = \gamma_{d-1}C_{u_{(i+1)d}},
\]
(2.8)
and
\[
\gamma_{id,s} = \gamma_s \text{ for } 0 \leq s \leq d-2
\]
(2.9)
under an isomorphism \( \Psi \) (defined in [10]).

For \( g_r \in G_r \) for \( 0 \leq r \leq d-1 \),
\[
(g_r, m, n)_{(i+1)d} = ((s_{id}^{-1}\alpha_{id,0,d}^{-1} \cdots s_{id}^{-1}\alpha_{id,0,d}^{-1})_{(i+1)d} x_{id,0,r} g_r
= ((s_{id} \cdot s_{id} \alpha_{id,0,d} \cdots s_{id} \alpha_{id,0,d})_{(i+1)d} x_{id,0,r}, m+1, n+1)_{id} \lambda
\]
(2.10)
where if \( m = 0 \) \( (n = 0) \) the right (left) multiplier of \( g_r \) is \( k_r \), the identity of \( G_r \) and
\[
s_{id} = u_{(i+2)d}^{-1}u_{(i+1)d},
\]
(2.11)
\[
g_{d-1} \gamma_{id,d-1} = s_{id}^{-1}(g_{d-1} \gamma_{(i+1)d,d-1} s_{id}
\]
(2.12)

By the proof of [10, theorem 1], if \( \Psi_{id} \) is as in [10],

* In [10], \( S_{id} \) is denoted by \( X_{id} \).
THEOREM 2.3. Let $S = (d, G_0, G_1, \ldots, G_{d-1}, C^*, \gamma_0, \gamma_1, \ldots, \gamma_{d-1}, u_{id})$ be a simple $I$-regular semigroup. Let $N = \{g \in G_0 | g(\gamma_0, \gamma_1, \ldots, \gamma_{d-1})^n = k_0, \text{the identity of } G_0, \text{for some } n \in I^0\}$. Then, $N$ is a normal subgroup of $G_0$. Let $\varphi$ be the natural homomorphism of $G_0$ onto $G_0/N$. Define $0 \delta = x\gamma_0, \gamma_1, \ldots, \gamma_{d-1}$ for $x \in G_0$. Then, $\delta$ is an endomorphism of $G_0/N$. Define $\sigma$ on $G_0/N \times (I^0)^2$ by the rule $((\bar{g}_0, a, b), (\bar{h}_0, c, d)) \in \sigma$ if and only if there exist $x, y \in I^0$ such that $x + a = y + c, x + b = y + d,$ and $\bar{g}_0 \theta^2 = \bar{h}_0 \theta^2$. Define a binary operation on $H = G_0/N \times (I^0)^2/\sigma$ by the rule $(\bar{g}_0, a, b)(\bar{h}_0, c, d) = (\bar{g}_0 \theta^2 \bar{h}_0, a + c, b + d)$. Then, $H$ is a group which is the maximal group homomorphic image of $S$. The canonical homomorphism of $S$ onto $V$ is given by

\[
(g_r, a, b) \bar{\phi} = \begin{cases} (x_{id}^{-1} \theta a_{i-1}^{-1} \cdots x_{id}^{-1} \theta x_{id}^{-1} (t_{id, a_{i-r}} g_r (t_{id, b_{i-r}} \alpha_0, r)), a_{i-r}, b_{i-r}) & \text{for } i \leq -1, \\
(g_r, a_{i+1}, b_{i+1}) & \text{for } i = 0,
\end{cases}
\]
where $(g_r, a, b) \in (k_0, i, i)S(k_0, i, i)$ and where

1. $x_0 = k_0$,
2. $x_{-d} = \bar{u}_0^{-1}$ while for $i \leq -2$,
3. $x_{id} = \bar{u}_0^{-1}(\bar{u}_{i-1}^{-1} \theta) \cdots \bar{u}_{(i+1)d}^{-1} \theta \bar{(i+1d)} \bar{u}_{(i+2)d} \theta \bar{(i+2d)} \cdots \bar{u}_0 \theta$,
4. $g_r \delta_{id} = g_r x_{r, d}$ while for $i \leq -1$,
5. $g_r \delta_{id} = \bar{u}_0^{-1} \bar{u}_{-d}^{-1} \theta \cdots \bar{u}_{(i+1)d}^{-1} \theta \bar{(i+1d)} \bar{g_r x_{r, d}} \theta \bar{(i+1d)} \cdots \bar{u}_{-d} \theta \bar{u}_0$.

**Proof.** As our proof parallels that of [6, theorem 3.6], we will just give a sketch of the proof. We first use theorem 2.1 to determine a homomorphism $\phi_{id}$ of $S_{id}$ into $H$ for each $i \in I$ with $i \leq 0$. Let $x_{id}$ and $\delta_{id}$ be defined as in the statement of the theorem. In the notation of theorem 2.1, let $z_{id} = (x_{id}, 0, 1), g_r f_r = (g_r \delta_{id}, 1, 1)$ and $g_r f_{r, id} = (g_r \delta_{id}, 0, 0)$ for $i \leq -1$ where $g_r \in G_r (0 \leq r \leq d - 1)$. Utilizing (2.8), we show that (2.1) and (2.2) are valid.

Hence, by (2.3),

\[
(g_r, a, b) \phi_{id} = \begin{cases} (x_{id}^{-1} \theta a_{i-1}^{-1} \cdots x_{id}^{-1} \theta x_{id}^{-1} (g_r \delta_{id})(x_{id} \cdot x_{id} \theta \cdots x_{id} \theta a_{i-1}^{-1}), a_{i-r}, b_{i-r}) & \text{if } i \leq -1, \\
(g_r x_{r, d}, m+1, n+1) & \text{if } i = 0,
\end{cases}
\]
defines a homomorphism of $S_{id}$ into $H$.

We note that $(g, m, n)_{0} \phi_{0} = (g_{r}, m + 1, n + 1)_{o}$. Hence, by theorem 2.2, $\phi_{0}$ is a homomorphism of $S_{0}$ onto $H$.

Let us define $x \lambda \phi = x \phi_{id}$ if $x \in S_{id}$. We will show that $\phi$ is a homomorphism of $S_{\Psi}$ onto $H$. We note that $(g_{r}, 1, 1)_{id} \phi_{id} = (g_{r}, 0, 0)_{(i+1)d} \phi_{(i+1)d}$. Utilizing (2.11), we obtain $(s_{id}, 1, 2)_{id} \phi_{id} = (k_{0}, 0, 1)_{(i+1)d} \phi_{(i+1)d}$. The desired result is then a consequence of (2.10).

Let $G^{*}$ be an arbitrary group and let $\rho$ be a homomorphism of $S_{\Psi}$ onto $G^{*}$. We denote $\lambda \rho | S_{id}$ by $\rho_{id}$. Thus, $\rho_{id}$ is a homomorphism of $S_{id}$ into $G^{*}$. Since $H$ is the maximal group homomorphic image of $S_{0}$ under the homomorphism $\phi_{0}$ by virtue of theorem 2.2, there exists a homomorphism $\gamma$ of $H$ onto the subgroup $S_{0} \rho_{0}$ of $G^{*}$ such that $(g, m, n)_{0} \phi_{0} \gamma = (g, m, n)_{0} \rho_{0}$ for all $(g, m, n)_{0} \in S_{0}$.

Next suppose that $(g, m, n)_{(i+1)d} \phi_{(i+1)d} \gamma = (g, m, n)_{(i+1)d} \rho_{(i+1)d}$ where $\gamma$ is a homomorphism of $H$ onto $S_{(i+1)d} \rho_{(i+1)d}$.

By virtue of theorem 2.1, there exists $v_{id}$ in $G^{*}$ and a homomorphism $\eta_{r}, id$ of $G_{r}$ into $G^{*}$ for each $r \in \{0, 1, 2, \cdots, d-1\}$ such that $v_{id}(g_{r-1}) \eta_{r-1, id}^{-1} = g_{r-1} \eta_{r-1, id}^{-1} \eta_{0, id}$ and $g_{r} \eta_{r, id} = g_{r} \eta_{r, id} \eta_{r+1, id}$ for $0 \leq r \leq d-2$. Furthermore $(g_{r}, m, n)_{id} \rho_{id} = v_{id}^{-1}(g_{r}, m, n)_{id} \eta_{r, id}$ for $(g_{r}, m, n)_{id} \in S_{id}$. Since $(g_{r}, 0, 0)_{(i+1)d} \lambda = (g_{r}, 1, 1)_{id} \lambda$, when $g_{r} \in G_{r}$, by (2.10), $(g_{r}, 0, 0)_{(i+1)d} \rho_{(i+1)d} = (g_{r}, 1, 1)_{id} \rho_{id}$. Thus, $g_{r} \eta_{r, id} = v_{id}(g_{r}, m, n)_{id} \eta_{r, id}^{-1}$.

Next suppose that $(g_{r}, m, n)_{(i+1)d} \phi_{(i+1)d} \gamma = (g_{r}, m, n)_{(i+1)d} \rho_{(i+1)d}$ where $\gamma$ is a homomorphism of $H$ onto $S_{(i+1)d} \rho_{(i+1)d}$.

By virtue of theorem 2.1, there exists $v_{id}$ in $G^{*}$ and a homomorphism $\eta_{r, id}$ of $G_{r}$ into $G^{*}$ for each $r \in \{0, 1, 2, \cdots, d-1\}$ such that $v_{id}(g_{r-1}) \eta_{r-1, id}^{-1} = g_{r-1} \eta_{r-1, id}^{-1} \eta_{0, id}$ and $g_{r} \eta_{r, id} = g_{r} \eta_{r, id} \eta_{r+1, id}$ for $0 \leq r \leq d-2$. Furthermore $(g_{r}, m, n)_{id} \rho_{id} = v_{id}^{-1}(g_{r}, m, n)_{id} \eta_{r, id}$ for $(g_{r}, m, n)_{id} \in S_{id}$. Since $(g_{r}, 0, 0)_{(i+1)d} \lambda = (g_{r}, 1, 1)_{id} \lambda$, when $g_{r} \in G_{r}$, by (2.10), $(g_{r}, 0, 0)_{(i+1)d} \rho_{(i+1)d} = (g_{r}, 1, 1)_{id} \rho_{id}$. Thus, $g_{r} \eta_{r, id} = v_{id}(g_{r}, m, n)_{id} \eta_{r, id}^{-1}$.

Next suppose that $(g_{r}, m, n)_{(i+1)d} \phi_{(i+1)d} \gamma = (g_{r}, m, n)_{(i+1)d} \rho_{(i+1)d}$ where $\gamma$ is a homomorphism of $H$ onto $S_{(i+1)d} \rho_{(i+1)d}$.

By virtue of theorem 2.1, there exists $v_{id}$ in $G^{*}$ and a homomorphism $\eta_{r, id}$ of $G_{r}$ into $G^{*}$ for each $r \in \{0, 1, 2, \cdots, d-1\}$ such that $v_{id}(g_{r-1}) \eta_{r-1, id}^{-1} = g_{r-1} \eta_{r-1, id}^{-1} \eta_{0, id}$ and $g_{r} \eta_{r, id} = g_{r} \eta_{r, id} \eta_{r+1, id}$ for $0 \leq r \leq d-2$. Furthermore $(g_{r}, m, n)_{id} \rho_{id} = v_{id}^{-1}(g_{r}, m, n)_{id} \eta_{r, id}$ for $(g_{r}, m, n)_{id} \in S_{id}$. Since $(g_{r}, 0, 0)_{(i+1)d} \lambda = (g_{r}, 1, 1)_{id} \lambda$, when $g_{r} \in G_{r}$, by (2.10), $(g_{r}, 0, 0)_{(i+1)d} \rho_{(i+1)d} = (g_{r}, 1, 1)_{id} \rho_{id}$. Thus, $g_{r} \eta_{r, id} = v_{id}(g_{r}, m, n)_{id} \eta_{r, id}^{-1}$.

Remark 2.3. In the case $d = 1$, we obtain [6, theorem 3.6].

The following result is needed to give an explicit determination of the extensions of a Brandt semigroup by a simple I-regular semigroup (theorem 1.3).

Theorem 2.4. Let $S = (d, G_{0}, G_{1}, \cdots, G_{d-1}, C_{1}^{*}, \cdots, C_{d-1}^{*}, \gamma_{0}, \gamma_{1}, \cdots, \gamma_{d-1}, u_{id})$ be a simple I-regular semigroup and let $X$ be a group. Let $\{z_{id} : i \in I, i \leq 0\}$ be a sequence of elements of $X$ and for each $r \in \{0, 1, \cdots, d-1\}$ let $\{s_{id,r} : i \in I, i \leq 0\}$ be a sequence of homomorphisms of $G_{r}$ into $X$ such that
For each \((g_r, a, b) \in (k_0, i, i)S(k_0, i, i)\), define \((g_r, a, b) \sim = z_{id}^{-1}(a-i)\) \((t_{id}^{-1}(a-i) \in \sigma_{id}(t_{id}, b-i))g_{id}(t_{id}, b-i)\)) \(fr, id z_{id}^2\), and let \(f_{id, (i+1)d} = f_{id} id Cz_{id}\)

For each \((g_r, a, b) \in (k_0, i, i)S(k_0, i, i)\), define \((g_r, a, b) \sim = z_{id}^{-1}(a-i)\) \((t_{id}^{-1}(a-i) \in \sigma_{id}(t_{id}, b-i))g_{id}(t_{id}, b-i)\)) \(f_{id} id z_{id}^2\). Then, \(\phi\) defines a homomorphism of \(S\) into \(X\) and conversely every such homomorphism is defined in this fashion.

**Proof.** We utilize theorem 2.1, (2.14), and the ‘inverse limit’ process (see [10]).

### 3. The congruences

In this section, we show that each congruence \(\rho\) on a simple I-regular semigroup \(S\) is a group congruence (\(S/\rho\) is a group), an idempotent separating congruence (each \(\rho\)-class contains at most one idempotent) or that \(S/\rho\) is a simple I-regular semigroup with fewer \(\mathcal{D}\)-classes than \(S\). We determine the idempotent separating congruences in terms of certain normal subgroups of the structure groups of \(S\). The group congruences of \(S\) are in a 1-1 correspondence with the normal subgroups of the maximal group homomorphic image of \(S\).

**Theorem 3.1.** Let \(S\) be a simple I-regular semigroup. Let \(\rho\) be a congruence on \(S\). Then \(\rho\) is a group congruence, \(\rho\) is an idempotent separating congruence, or \(S/\rho\) is a simple I-regular semigroup with \(t\) \(\mathcal{D}\)-classes where \(t < d\), the number of \(\mathcal{D}\)-classes of \(S\).

**Proof.** Let \(\{(f_i, n, n) : 0 \leq i \leq d-1, n \in I\}\) denote the set of idempotents of \(S\). Each \(D_i = \{(g_i, m, n) : g_i \in G_i, m, n \in I\}\) is an \(I\)-bisimple semigroup for \(0 \leq i \leq d-1\). Thus, by [6, theorem 4.2], \(\rho|D_i\) is a group congruence or an idempotent separating congruence for \(0 \leq i \leq d-1\).

Suppose that \(\rho\) is not an idempotent separating congruence. First suppose that \(\rho|D_i\) is a group congruence for some \(i\). Hence, \((f_i, 0, 0)\rho = (f_i, k, k)\rho\) for all \(k \in I\). Let \((f_j, n, n) \in E_{D_j}\) and \((f_k, p, p) \in E_{D_k}\) and suppose that \((f_j, n, n) < (f_k, p, p)\). Thus, \((f_i, n+1, n+1) < (f_j, n, n) < (f_k, p, p)\). Hence, \((f_i, n, n)\rho = (f_k, p, p)\rho\) and \(\rho\) is a group congruence. Next, suppose that \(\rho|D_i\) is an idempotent separating congruence for each \(0 \leq i \leq d-1\). Then, there exist \((f_i, n, n), (f_r, q, q)\) \(\in E_\mathcal{D}\) such that \((f_i, n, n)\rho = (f_k, q, q)\rho\). Thus, \(D_i\rho\) and \(D_k\rho\) lie in the same \(\mathcal{D}\)-class of \(S/\rho\). Hence, \(S/\rho\) is a simple I-regular semigroup with \(t\) \(\mathcal{D}\)-classes with \(t < d\).

**Remark 3.1.** In the case \(d = 1\), we obtain [6, theorem 4.2].
REMARK 3.2. We may replace ‘simple I-regular’ by ‘simple \(\omega\)-regular’ in theorem 3.1. The proof is analogous.

We next determine the idempotent separating congruences of a simple I-regular semigroup.

Let \(G_0, G_1, \ldots, G_{d-1}\) be a collection of disjoint groups and let \(\gamma_i\) be a homomorphism of \(G_i\) into \(G_{i+1}\) for \(0 \leq i \leq d-2\) and let \(\gamma_{d-1}\) be a homomorphism of \(G_{d-1}\) into \(G_0\). Let \(V_i\) be a normal subgroup of \(G_i\) for \(0 \leq i \leq d-1\) such that \(V_i \gamma_i \subseteq V_{i+1}\) for \(c \leq i \leq d-2\) and \(V_{d-1} \gamma_{d-1} \subseteq V_0\). Then, \((V_0, V_1, \ldots, V_{d-1})\) will be called a \(\gamma_0-\gamma_1-\cdots-\gamma_{d-1}\) invariant \(d\)-tuple of \((G_0, G_1, \ldots, G_{d-1})\). Let \((V_0, V_1, \ldots, V_{d-1})\) and \((U_0, U_1, \ldots, U_{d-1})\) be \(\gamma_0-\gamma_1-\cdots-\gamma_{d-1}\) invariant \(d\)-tuples of \((G_0, G_1, \ldots, G_{d-1})\). Then, we say \((V_0, V_1, \ldots, V_{d-1}) \subseteq (U_0, U_1, \ldots, U_{d-1})\) if and only if \(V_i \subseteq U_i\) for \(0 \leq i \leq d-1\).

In the proof of the following theorem, we will utilize a theorem of Preston [6, theorem 4.3]. We also utilize the notation of this theorem. We will sketch the following proof where it parallels the proof of [6, theorem 4.4].

**THEOREM 3.2.** Let \(S = (d, G_0, G_1, \ldots, G_{d-1}, C^*, \gamma_0, \gamma_1, \ldots, \gamma_{d-1},\ m_1d)\) be a simple I-regular semigroup. There exists a 1-1 correspondence between the idempotent separating congruences on \(S\) and the \(\gamma_0-\gamma_1-\cdots-\gamma_{d-1}\) invariant \(d\)-tuples of \((G_0, G_1, \ldots, G_{d-1})\). If \(\rho(V_0, V_1, \ldots, V_{d-1})\) is the idempotent separating congruence corresponding to the \(\gamma_0-\gamma_1-\cdots-\gamma_{d-1}\) invariant \(d\)-tuple \((V_0, V_1, \ldots, V_{d-1})\), \((g_r, a, b)\rho(V_0, V_1, \ldots, V_{d-1}) (h_s, c, d)\) if and only if \(r = s, a = c, b = d\) and \(V_r g_r = V_r h_s\). If \((V_0, V_1, \ldots, V_{d-1})\) and \((U_0, U_1, \ldots, U_{d-1})\) are two \(\gamma_0-\gamma_1-\cdots-\gamma_{d-1}\) invariant \(d\)-tuples \((V_0, V_1, \ldots, V_{d-1}) \subseteq (U_0, U_1, \ldots, U_{d-1})\) if and only if \(\rho(V_0, V_1, \ldots, V_{d-1}) \subseteq \rho(U_0, U_1, \ldots, U_{d-1})\).

**PROOF.** Let \((V_0, V_1, \ldots, V_{d-1})\) be a \(\gamma_0-\gamma_1-\cdots-\gamma_{d-1}\) invariant \(d\)-tuple of \((G_0, G_1, \ldots, G_{d-1})\). Let \(N_{(\nu_r, a, a)} = \{v_r, a, a : v_r \in V_r\}\) and let \(N = U(N_{(\nu_r, a, a)}) : 0 \leq r \leq d-1, a \in I\). By a routine calculation, \(N_{(\nu_r, a, a)}\) is a subgroup of \(S\) isomorphic to \(V_r\). By [6, theorem 4.3] \(\rho_N\) is an idempotent separating congruences of \(S\). We denote \(\rho_N\) by \(\rho(V_0, V_1, \ldots, V_{d-1})\).

Let \(\rho\) be an idempotent separating congruence of \(S\). Then, by [6, theorem 4.3] \(\rho = \rho_N\) where \(N\) is given in the statement of [6, theorem 4.3]. \(N_{(\nu_r, a, a)} = \{v_r, a, a : v_r \in V_r\}\), where \(V_r\) is an invariant subgroup of \(G_r\). Since \((e_{r+1}, 0, 0)(e_r, 0, 0) = (e_{r+1}, 0, 0), (e_{r+1}, 0, 0)(v_r, 0, 0) \in N_{(e_{r+1}, 0, 0)}\). Thus \(V_r \gamma_r \subseteq V_{r+1}\) for \(0 \leq r \leq d-2\). Since \((e_0, 0, 1)(e_{d-1}, 0, 0)(e_0, 1, 0) = (e_0, 0, 0), (e_0, 0, 1)(v_{d-1}, 0, 0)(e_0, 1, 0) = N_{(e_0, 0, 0)}\). Thus, \(V_d-1 \gamma_{d-1} \subseteq V_0\). Hence, \(\rho = \rho(V_0, V_1, \ldots, V_{d-1})\) and we have the desired correspondence.

**REMARK.** In the case \(d = 1\), we obtain [6, theorem 4.4].
REMARK. We may replace ‘simple I-regular semigroup’ by ‘simple regular ω-semigroup’ in theorem 3.2. The proof is analogous.

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R. J. Warne  

R. J. Warne  
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R. J. Warne  

R. J. Warne  

(Oblatum 24-XII-68) West-Virginian University  
Dept. of Mathematics  
Morgantown, W.-Virginia  
U.S.A.