COMPOSITIO MATHEMATICA

D. B. A. EPSTEIN

The simplicity of certain groups of homeomorphisms

Compositio Mathematica, tome 22, nº 2 (1970), p. 165-173
<http://www.numdam.org/item?id=CM_1970__22_2_165_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 2, 1970, pag. 165-173
Wolters-Noordhoff Publishing
Printed in the Netherlands

THE SIMPLICITY OF CERTAIN GROUPS OF HOMEOMORPHISMS

by

D. B. A. Epstein

In 1961 Anderson [1] published a proof that the group of stable homeomorphisms of a manifold is simple. Fisher [2] had previously shown this for manifolds of low dimension. The question was also considered by Ulam and von Neumann [5], who proved the same result for S^2. The recent work of Chernavski, Kirby and Edwards shows that the group of stable homeomorphisms is the same as the group of homeomorphisms isotopic to the identity.

Recently Smale has asked whether the group of diffeomorphisms which are differentiably isotopic to the identity is simple. In this paper we show that the commutator subgroup of such C^r diffeomorphisms is simple for $1 \leqq r \leqq \infty$. We also show that the group of such C^1 diffeomorphisms has no closed normal subgroups. In the last part of the paper we examine the piecewise linear situation, where simplicity is proved for PL homeomorphisms of $\mathbb{R}^1$ and S^1.

If the group of diffeomorphisms, which are isotopic to the identity, is simple, then any class of such diffeomorphisms which is closed under conjugation will generate the whole group. For example we would know that any such diffeomorphism is the product of Morse-Smale diffeomorphisms. Or we could deduce that any such diffeomorphism was the product of time one diffeomorphisms of flows.

The partial result we have proved does not enable us to go so far. However we can say that any class of diffeomorphisms closed under conjugation generates a subgroup which is C^1 dense in all diffeomorphisms isotopic to the identity.

I would like to thank M. W. Hirsch and J. Palis for helpful conversations.

1

Let X be a paracompact Hausdorff space, G a group of homeomorphisms of X and $\mathfrak{U}$ a basis of open neighbourhoods of X satisfying the following axioms.

AXIOM 1. If $U \in \mathfrak{U}$ and $g \in G$, then $gU \in \mathfrak{U}$.

AXIOM 2. G acts transitively on $\mathfrak{U}$.

If g is a homeomorphism of X we define *supp g*, the *support* of g, to be the closure of $\{x | x \neq gx\}$.

AXIOM 3. Let $g \in G$, $U \in \mathfrak{U}$ and let $\mathfrak{V} \subseteq \mathfrak{U}$ be a covering of X. Then there exists an integer n, elements $g_1, \cdots, g_n \in G$ and $V_1, \cdots, V_n \in \mathfrak{V}$ such that $g = g_n g_{n-1} \cdots g_1$, $\operatorname{supp} g_i \subset V_i$ and

$$\operatorname{supp} g_i \cup (g_{i-1} \cdots g_i \bar{U}) \neq X \text{ for } 1 \leqq i \leqq n.$$

1.1. THEOREM. *If these axioms are satisfied then* $[G, G]$, *the commutator subgroup of* G, *is simple.*

The proof of this theorem will occupy the first half of this paper.

Let X be a paracompact connected C^r manifold of dimension n $(1 \leqq r \leqq \infty)$. Let $\mathfrak{U}$ be the set of subsets of X of the form φB where $\varphi : \mathbb{R}^n \to X$ is a C^r embedding and B is the open unit ball in $\mathbb{R}^n$. Let $G = \operatorname{Diff}(X, r)$ be the group of those C^r diffeomorphisms f of X for which there exists a compact subset K of X and an isotopy H of f to the identity, with H fixed outside K.

1.2. PROPOSITION. *In the situation just described, the axioms are satisfied.*

COROLLARY. *If* $G = \operatorname{Diff}(X, r)$, *then* $[G, G]$ *is simple.*

PROOF OF PROPOSITION 1.2. We have only to check Axiom 3. It is well-known (see for example Palis and Smale [3], Lemma 3.1) that we can write $g = g_n \cdots g_1$ with $\operatorname{supp} g_i \subset V_i \in \mathfrak{V}$. It follows that we need only prove the following lemma in order to complete the proof of the proposition.

1.3. LEMMA. *Let* $g \in \operatorname{Diff}(X, r)$ *where* X *is a connected paracompact manifold and let* $\operatorname{supp} g \subset V$ *where* V *is an open ball as described just before Proposition* 1.2. *Let* U *be an open subset with* $\bar{U} \neq X$. *Then we can write* $g = g_2 g_1$ *with* $g_i \in \operatorname{Diff}(X, r)$, $\operatorname{supp} g_i \subset V$ $(i = 1, 2)$,

$$\operatorname{supp} g_1 \cup \bar{U} \neq X \text{ and } \operatorname{supp} g_2 \cup g_1 \bar{U} \neq X.$$

PROOF. If $\operatorname{supp} g \subseteq \bar{U}$ then we can take $g_1 = id$ and $g_2 = g$. Otherwise let $x_1 \notin \bar{U}$ with $gx_1 \neq x_1$ and choose $x_2 \notin \bar{U}$ with x_2 distinct from x_1 and gx_1. Let $h \in \operatorname{Diff}(X, r)$ be the identity in a small neighbourhood N_2 of x_2 and let $hx_1 = gx_1$ and $Dh(x_1) = Dg(x_1)$ and $\operatorname{supp} h \subseteq V$. (If X is one dimensional, we must choose x_2 so as not to separate x_1 and gx_1 in V.)

Let $f : \mathbb{R}^n, 0 \to \mathbb{R}^n, 0$ with $Df(0) = id$. Let $\varphi : \mathbb{R}^n, 0 \to \mathbb{R}$ be a bump function equal to 1 in a neighbourhood of 0 and equal to 0 outside a compact set. We define

$$f_\lambda(x) = \varphi(\lambda x)f(x)+(1-\varphi(\lambda x))x$$

where $\lambda > 0$. It is easy to see that for λ very large f_λ is a diffeomorphism of $\mathbb{R}^n$ which is the identity outside a very small compact set. Moreover if f is C^r then $f_\lambda \in \mathrm{Diff}(\mathbb{R}^n, r)$.

Hence we can define $\rho : X \to X$ to be a diffeomorphism which is the identity outside a small neighbourhood of x_1 and is equal to $h^{-1}g$ on an even smaller neighbourhood N_1 of x_1. We put $g_1 = h\rho$ and $g_2 = g(g_1)^{-1}$.

Now $g_1|N_2 = id$ and $\mathrm{supp}\, g_1 \subseteq V$. Hence $\mathrm{supp}\, g_2 \subseteq V$ and $x_2 \notin (\mathrm{supp}\, g_1 \cup \overline{U})$. Finally $g_1 x_1 \notin (\mathrm{supp}\, g_2 \cup g_1\overline{U})$, for $x_1 \notin \overline{U}$ and

$$g_2(g_1x_1) = gx_1 = hx_1 = h\rho x_1 = g_1x_1.$$

This completes the proof of the lemma and hence of proposition 1.2.

1.4. Theorem. *Let $(X, G, \mathfrak{U})$ satisfy the three axioms above and let N be a non-trivial subgroup of G which is normalized by $[G, G]$ (i.e. if $n \in N$ and $g \in [G, G]$ then $[n, g] = n^{-1}g^{-1}ng \in N$). Then $[G, G] \subseteq N$.*

Theorem 1.1 is obviously a corollary. Another corollary is that every normal subgroup of G contains the commutator subgroup. We start with some remarks.

1.4.1. If G has more than one element (which we may obviously assume), then X is connected. For if $X = U \cup V$ and U and V are disjoint open subsets, then by Axiom 3, G is generated by elements supported either in U or in V and this contradicts Axiom 2. In particular no open subset is finite, for otherwise X would be discrete by Axiom 2.

1.4.2. Let $V \in \mathfrak{U}$ and let $h \in G$ with $\mathrm{supp}\, h \subseteq V$. Let $W \in \mathfrak{U}$ with $W \cap \mathrm{supp}\, h = \emptyset$. By Axiom 2 we can find $g \in G$ with $gW = V$. Then

$$[g, h] = g^{-1}h^{-1}gh \in [G, G]$$

is equal to h on $\mathrm{supp}\, h$, to $g^{-1}h^{-1}g$ on $g^{-1}(\mathrm{supp}\, h) \subseteq W$, and to the identity elsewhere.

1.4.3. We now show that $(G, G]$ acts transitively on $\mathfrak{U}$. Let U and U_1 be elements of $\mathfrak{U}$ and let $g \in G$ with $gU = U_1$. By Axiom 3, we can find an integer n, and $V_1, \cdots, V_n \in \mathfrak{U}$ and $h_1, \cdots, h_n \in G$ such that $g = h_n \cdots h_1$, $\mathrm{supp}\, h_i \subset V_i$ and $K_i\, \mathrm{supp}\, h_i \cup (h_{i-1} \cdots h_1\overline{U}) \neq X$. Let $W_i \subset X - X_i$ be an element of $\mathfrak{U}$. Let $g_i \in G$ with $gW_i = V_i$.

Then applying 1.4.2., it is easy to see that

$$[g_n, h_n] \cdots [g_1, h_1]U = h_n \cdots h_1 U = gU = U_1.$$

1.4.4. Let $id \neq \alpha \in N$. Let $x \in X$ with $\alpha x \neq x$. Let $U \in \mathfrak{U}$ with $U \cap \alpha^{-1} U = \emptyset$. Let $V, W \in \mathfrak{U}$ be as in 1.4.2., with $V \cap W = \emptyset$, $\overline{V} \cup \overline{W} \subset U$ and V a neighbourhood of x. Then with g and h as in 1.4.2 $\rho = [\alpha,[g, h]] \in N$. Then ρ is equal to h on V, $g^{-1}h^{-1}g$ on W, $\alpha^{-1}h\alpha$ on $\alpha^{-1}V$, $\alpha^{-1}g^{-1}h^{-1}g\alpha$ on $\alpha^{-1}W$ and the identity elsewhere.

Using 1.4.3 and conjugating ρ by an element of $[G, G]$, we may assume that V is an arbitrary element of $\mathfrak{U}$ and h is an arbitrary element of G supported on V and $\rho \in N$.

1.4.5. We now show that the orbits of N are the same as the orbits of G (which are dense in X by Axiom 2). Let $gx = y$ and let $y = g_n \cdots g_1 x$ be an expression with n minimal, $g_1 \in G$ and supp g_i contained in some element V_i of $\mathfrak{U}$ for each i. Since n is minimal, $g_{i-1} \cdots g_1 x \in V_i$ for each i $(1 \leqq i \leqq n)$. Constructing elements of N as in 1.4.4, we can find $\rho_i \in N$ such that $\rho_i | V_i = g_i | V_i$ $(1 \leqq i \leqq n)$. We obviously have

$$y = g_n \cdots g_1 x = \rho_n \cdots \rho_1 x \in Nx.$$

1.4.6. Let $x \in X$ and choose by 4.5 $\alpha_1, \alpha_2 \in N$ with x, $\alpha_1^{-1}x$ and $\alpha_2^{-1}x$ distinct. Let $U \in \mathfrak{U}$ be a small neighbourhood of x with U, $\alpha_1^{-1}U$ and $\alpha_2^{-1}U$ disjoint. Let $g_1, g_2 \in G$ with x, $g_1^{-1}x$, $g_2^{-1}x$ distinct elements of U. Let $V \in \mathfrak{U}$ be a neighbourhood of x such that V, $g_1^{-1}V$ and $g_2^{-1}V$ are disjoint subsets of U. We can construct elements ρ_i of N as in 1.4.4 using $\alpha_i, g_i, W_i = g_i^{-1}V$ and with h_i an arbitrary element of G supported on V $(i = 1, 2)$. ρ_1 is supported on the four sets V, $g_1^{-1}V$, $\alpha_1^{-1}V$, $\alpha_1^{-1}g_1^{-1}V$, and ρ_2 is supported on the four sets V, $g_2^{-1}V$, $\alpha_2^{-1}V$, $\alpha_2^{-1}g_2^{-1}V$. All of the seven sets occurring in these two lists are disjoint. Hence

$$[h_1, h_2] = [\rho_1, \rho_2] \in N.$$

What we have shown is that if h_1 and h_2 are arbitrary elements of G supported on V, then $[h_1, h_2] \in N$. Since N is normalised by $[G, G]$ and by 1.4.3. $[G, G]$ acts transitively on $\mathfrak{U}$, we may assume moreover that V is an arbitrary element of $\mathfrak{U}$.

1.4.7. Since X is paracompact, we can find a covering $\mathfrak{B} \subseteq \mathfrak{U}$ such that if $V_1, V_2 \in \mathfrak{B}$ and $V_1 \cap V_2 \neq \emptyset$, then there is an element $U \in \mathfrak{U}$ with $V_1 \cup V_2 \subseteq U$. By Axiom 3, G is generated by elements h for which there exists $W \in \mathfrak{B}$ with h supported on W. Let h_1 and h_2 be two such generators, supported on W_1 and W_2 respectively. Then either $W_1 \cap W_2 = \emptyset$ in which case $[h_1, h_2] = id \in N$ or $W_1 \cap W_2 \neq \emptyset$ in which case both h_1 and h_2 are supported on some element U of $\mathfrak{U}$, in which case $[h_1, h_2] \in N$ by 1.4.6.

To complete the proof of Theorem 1.4 we need only show that all conjugates in G of $[h_1, h_2]$ are in N (for then such conjugates generate a normal subgroup S of G and G/S is abelian since the generators commute). Let $g \in G$. We wish to show that $g[h_1, h_2]g^{-1} \in N$. By Axiom 3 there is an integer n, elements $g_1, \cdots, g_n \in G$ and $V_1, \cdots, V_n \in \mathfrak{B}$, with $g = g_n \cdots g_1$, supp $g_i \subset V_i$ and

$$K_i = \operatorname{supp} g_i \cup (g_{i-1} \cdots g_1 \bar{U}) \neq X.$$

(Here h_1 is supported on W_1, h_2 supported on W_2, $W_1 \cup W_2 \subseteq U$ as in the preceding paragraph.)

Choose $U_i \in \mathfrak{U}$ with U_i in the complement of K_i. Let $\beta_i \in G$ with $\beta_i U_i = V_i$. Put $\gamma_i = [\beta_i, g_i]$. As in 4.2 γ_i is equal to g_i on supp g_i, to $\beta_i^{-1} g_i^{-1} \beta_i$ on U_i and to the identity elsewhere. It follows that

$$\begin{aligned} g[h_1, h_2]g^{-1} &= g_n \cdots g_1[h_1, h_2]g_1^{-1} \cdots g_n^{-1} \\ &= \gamma_n \cdots \gamma_1[h_1, h_2]\gamma_1^{-1} \cdots \gamma_n^{-1} \end{aligned}$$

and this is an element of N.

This completes the proof of Theorem 1.4.

2

In this part of the paper, we restrict ourselves to the case where X is a paracompact connected C^r manifold and $G_r = \operatorname{Diff}(X, r)$. We topologize G_r with the fine C^r topology.

Conjecture 1. G_r is a simple group for $1 \leqq r \leqq \infty$.

This implies the weaker conjecture:

Conjecture 2(r). $[G_r, G_r]$ is dense in G_r.

Conjecture 2(r) obviously implies Conjecture 2(s) for $s < r$. We have been able to prove conjecture 2(r) in the case $r = 1$ only.

2.1. Theorem. *$[G_1, G_1]$ is dense in G_1 (with respect to the fine C^1 topology).*

Corollary. *G_1 has no closed normal subgroups except for e and G_1.*

Proof. We have to show that $[G_1, G_1] = G_1$. Let $\theta \in G_1$. Without loss of generality we may assume that θ is near the identity and is supported on a small ball, so that we can work in a single coordinate chart. (This is because G_1 is generated by such elements.) Hence there is no loss of generality in assuming that our manifold is $\mathbb{R}^n$ and that θ is supported on the ball B of radius 1 with centre at $(4, 0, \cdots, 0)$.

We take the usual norm on $\mathbb{R}^n$

$$||x||^2 = x_1^2 + \cdots + x_n^2$$

and the associated operator norm on $(n \times n)$ matrices. Let

$$||\theta|| = \max \{||x-\theta x||, \ ||id - D\theta(x)|| : X \in \mathbb{R}^n\}$$

We may assume without loss of generality that $||\theta|| < \frac{1}{8}$, and we wish to prove that $\theta \in \overline{[G_1, G_1]}$. Since $||\theta|| < \frac{1}{8}$.

$$\varphi_t(x) = tx + (1-t)\theta(x)$$

defines an isotopy of θ to the identity.

We first find a bound for $||\theta^{-1}||$.

$$||x-\theta^{-1}x|| = ||\theta(\theta^{-1}x) - \theta^{-1}x|| \leqq ||\theta||.$$

Let $D\theta(x) = id - \alpha(x)$ so that $||\alpha(x)|| \leqq \frac{1}{8}$. Then

$$\begin{aligned} D\theta(x)^{-1} &= D\theta^{-1}(\theta x) \\ &= id + \alpha(x) + \alpha(x)^2 + \cdots \end{aligned}$$

Therefore $||D\theta^{-1}(y) - id|| \leqq ||\theta|| + ||\theta||^2 + \cdots = ||\theta||/(1-||\theta||)$. It follows that $||\theta^{-1}|| \leqq ||\theta||/(1-||\theta||)$.

Let $\theta_1 = \varphi_{\frac{1}{2}}$, so that $\theta_1(x) = (x+\theta x)/2$. Then we see immediately that $||\theta_1|| \leqq ||\theta||/2$.

Let $\theta_2 = \theta\theta_1^{-1}$. We compute a bound for $||\theta_2^{-1}||$.

$$\theta_2^{-1}(x) = \theta_1\theta^{-1}(x) = (x+\theta^{-1}x)/2.$$

So θ_2^{-1} bears the same relationship to θ^{-1} as θ_1 does to θ. Hence

$$||\theta_2^{-1}|| \leqq ||\theta^{-1}||/2 \leqq ||\theta||/2(1-||\theta||).$$

Since

$$||\theta_2|| \leqq ||\theta_2^{-1}||/(1-||\theta_2^{-1}||),$$

we see that

$$||\theta_2|| \leqq ||\theta||/(2-3||\theta||).$$

If $||\theta|| < \frac{1}{8}$, we have $\theta = \theta_2\theta_1$ with

$$||\theta_1|| < \tfrac{3}{4}||\theta|| \text{ and } ||\theta_2|| < \tfrac{3}{4}||\theta||.$$

We can now factorize $\theta_1 = \theta_{12}\theta_{11}$ and $\theta_2 = \theta_{22}\theta_{21}$ in the same way. Formally suppose we have defined θ_I for all r-tuples $(1 \leqq r < n)$ of 1's and 2's. If $I = (i_1, \cdots, i_{n-1})$ we define $J = (i_1, \cdots, i_{n-1}, 1)$ and $K = (i_1, \cdots, i_{n-1}, 2)$ and we put $\theta_J = (\theta_I)_1$ and $\theta_K = (\theta_I)_2$, so that $\theta_I = \theta_K\theta_J$. By induction we have

$$||\theta_K|| \leqq (\tfrac{3}{4})^n||\theta|| \text{ and } ||\theta_J|| \leqq (\tfrac{3}{4})^n||\theta||.$$

2.2. Remarks. Let $\varphi \in \text{Diff}\,(X, r)$ where X is a connected paracompact C^r manifold. Let $\varphi = \sigma\rho$ and let $\sigma, \rho \in \text{Diff}\,(X, r)$ supported on a ball U. Let $g, h \in \text{Diff}\,(X, r)$ be diffeomorphisms such that U, $g^{-1}U$ and $h^{-1}U$

are disjoint. Then in the commutator subgroup of Diff (X, r), we have the element $[g, \sigma][h, \rho]$ and this element is equal to φ on U, $g^{-1}\sigma^{-1}g$ on $g^{-1}U$, $h^{-1}\rho^{-1}h$ on $h^{-1}U$ and the identity elsewhere. In particular we can take $\varphi = \theta_I$, $\sigma = \theta_K$ and $\rho = \theta_J$.

We now construct a sequence of diffeomorphisms

$$\theta, \theta_1^{-1}, \theta_2^{-1}, \theta_1, \theta_{11}^{-1}, \theta_{12}^{-1}, \theta_2, \theta_{21}^{-1}, \theta_{22}^{-1}, \theta_{11}, \theta_{111}^{-1}, \theta_{112}^{-1}, \cdots.$$

The sequence is constructed in blocks of three. The first diffeomorphism in a block has the form θ_I, the second has the form θ_J^{-1} and the third has the form θ_K^{-1}, where I, J, K are as described above. Suppose that we have already constructed i blocks of three. To construct the $(i+1)$st block of three, read from left to right along the list already constructed, and take the first diffeomorphism of the form θ_I^{-1}, such that θ_I does not already appear further to the right. This θ_I will be the first term in the $(i+1)$st block.

Call the sequence which we have constructed $\varphi_0, \theta_1, \varphi_2, \cdots$. We are supposing that $\theta = \varphi_0$ is supported on a ball of radius one, with centre at $(4, 0, \cdots, 0)$. Then each φ_i is supported on the same ball. Let $B_n = (\frac{1}{2}^n)B$. That is, B_n has radius $\frac{1}{2}^n$ and centre at $(2^{2-n}, 0, \cdots, 0)$. We define $\psi : \mathbb{R}^n \to \mathbb{R}^n$ by putting φ_n in B_n. More formally, we set

$$\begin{aligned} \psi(x) &= 2^{-n}\varphi_n(2^n x) && \text{if } x \in B_n \\ &= x && \text{if } x \notin \bigcup_{n \geqslant 0} B_n. \end{aligned}$$

Obviously $\psi(B_n) = B_n$ so ψ is a homeomorphism, which is a diffeomorphism everywhere except possibly at 0. But since φ_n tends to *id* in the C^1-topology as n tends to infinity, we have

$$\begin{aligned} D\psi(x) &= 2^{-n} \cdot 2^n(D\varphi_n)(2^n x) \quad \text{if } x \in B_n \\ &= (D\varphi_n)(2^n x) \end{aligned}$$

which tends to *id* as n tends to infinity. Hence ψ is differentiable at 0 and is therefore a diffeomorphism. (Formally, we apply the mean value theorem to $(\psi - id)$ in order to show ψ is differentiable at 0).

NOTE. ψ is not C^2, except in exceptional circumstances.

The proof of Theorem 2.1 is completed by proving that both ψ and $\theta^{-1}\psi$ are contained in the closure in $G_1 = \text{Diff}(\mathbb{R}^n, 1)$ of $[G_1, G_1]$ with respect to the fine C^1-topology.

According to Remark 2.2, if we work modulo $[G_1, G_1]$ we may delete any finite number of blocks of the form θ_I, θ_J^{-1}, θ_K^{-1}, where I, J and K are as described just before the remark. This shows that ψ is in the closure of $[G_1, G_1]$.

According to 4.2, if we work modulo $[G_1, G_1]$, we may delete from ψ a finite number of pairs of the form θ_I, θ_I^{-1}. Hence $\theta^{-1}\psi$ is in the closure of $[G_1, G_1]$. This completes the proof of Theorem 2.1.

3

Throughout this section of the paper, let X be a connected piecewise linear n-dimensional manifold and let $G = PL(X)$ be the group of those PL homeomorphisms which are PL isotopic to the identity by an isotopy fixed outside a compact set. Let $\mathfrak{U}$ be the set of all PL n-dimensional balls in X. According to Hudson and Zeeman [4] every element of G is the product of elements of G, each element supported on a ball. As in Proposition 1.2, we see that Axioms 1, 2 and 3 are true. Hence we have as a consequence of Theorem 1.4,

3.1. THEOREM. *Let $G = PL(X)$. Then $[G, G]$ is simple and every non-trival normal subgroup of G contains $[G, G]$.*

We can also prove

3.2. THEOREM. *Let $X = \mathbb{R}^1$ or S^1. Then $PL(X)$ is simple.*

PROOF. By Theorem 3.1, we need only show that $PL(X)$ is equal to its commutator subgroup. Since every element of $PL(S^1)$ is the product of elements supported on intervals, we may restrict ourselves to $PL(\mathbb{R}^1)$.

Let $0 < \alpha < \beta < \frac{1}{2}$. Let $\sigma(\alpha) : \mathbb{R}^1 \to \mathbb{R}^1$ be the PL homeomorphism whose graph lies on the diagonal outside $[0, 1] \times [0, 1]$ and which has corners at $(0, 0)$, $(1, 1)$ and $(\frac{1}{2}, \alpha)$. Let $\mu = \sigma(\alpha)^{-1}\sigma(\beta)^{-1}\sigma(\alpha)\sigma(\beta)$. Then the graph of μ has corners at

$$(\tfrac{1}{2}, \tfrac{1}{2}),\ ((3-4\beta)/4(1-\beta),\ (\alpha+2\beta-4\alpha\beta)/4\beta(1-\alpha))$$

and

$$((3-4\alpha)/4(1-\alpha),\ (3-4\alpha)/4(1-\alpha)).$$

Conjugating μ by an element which is linear on $[0, 1]$, we obtain an element λ in the commutator subgroup, such that the graph of λ has exactly three corners, and these are at $(0, 0)$, $(1, 1)$ and

$$((1-2\beta)(1-\alpha)/(1-2\alpha)(1-\beta),\ \alpha(1-2\beta)/\beta(1-2\alpha)).$$

Writing $u = \alpha/(1-2\alpha)$ and $v = \beta/(1-2\beta)$ so that $0 < u < v < \infty$, the corner for λ is at

$$\left(\frac{1+u}{1+v}, \frac{u}{v}\right).$$

Writing $u = \gamma v$ with $0 < \gamma < 1$, this point becomes $((1+\gamma v)/(1+v), \gamma)$. For fixed γ, $(1+\gamma v)/(1+v)$ varies between γ and 1.

This shows that the commutator subgroup contains all elements λ whose graphs have corners at $(0,0)$, $(1,1)$ and (x,y) with $x > y$ and no other corners. Taking inverses and conjugating, we see that the commutator subgroup contains all elements λ whose graphs have exactly three corners. But such elements obviously generate $PL(\mathbb{R})$. This completes the proof of Theorem 3.2.

The above proof raises the following question. We define a *glide* on a connected piecewise linear n-manifold M to be a PL homeomorphism $h : M \to M$ such that there is a ball B in M with $h|M-B = id$. Further, with respect to some PL coordinate system, B is the linear join of S^{n-2} with I, $h(x) = y$ where $x, y \in I$ and h is extended to $S^{n-2} * I$ by mapping each ray ax with $a \in S^{n-2} * \partial I$ linearly to the ray ay. (Note that λ has this form in the preceding proof.)

The subgroup of $PL(M)$ generated by glides is obviously normal. The question is, is it the whole of $PL(M)$? In other words, is every element of $PL(M)$ a product of glides? An affirmative answer is equivalent to the simplicity of $PL(M)$.

BIBLIOGRAPHY

R. D. Anderson

[1] "On homeomorphisms...", Topology of 3-Manifolds, edited by M. K. Fort, Prentice Hall 1961.

G. M. Fisher

[2] "On the group of all homeomorphisms of a manifold", T.A.M.S. **97** (1960). pp. 193–212.

J. Palis and S. Smale

[3] "Structural Stability Theorem", (to appear).

J. F. P. Hudson and E. C. Zeeman

[4] "On Combinatorial Isotopy", Publication Mathematiques No. 19, I.H.E.S..

S. Ulam and J. von Neumann

[5] "On the group of homeomorphisms of the surface of a sphere", B.A.M.S. **53** (1947).

(Oblatum 20–XI–1969)

Mathematics Institute
University of Warwick
Coventry, England