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# Certain theorems on unilateral and bilateral operational calculus

by

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## 1. Introduction

A generalization of the Laplace-transform is given [5] as

$$F(p) = p \int_0^\infty e^{-\frac{1}{2}pt} W_{k+\frac{1}{2}, m}(pt)(pt)^{-k-\frac{1}{2}} f(t) dt, \tag{1.1}$$

where $W_{k,m}(t)$ is the confluent hypergeometric function. $F(p)$ is called the Meijer-transform of $f(t)$ and is symbolically denoted by

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} F(p) \quad \text{or} \quad F(p) \xleftarrow[m]{k+\frac{1}{2}} f(t). \tag{1.2}$$

For $k = m$, it reduces to the Laplace-transform.

In two variables $f(t)$ and $F(p)$ will be replaced by $f(t_1, t_2)$ and $F(p_1, p_2)$, where $F(p_1, p_2)$ is defined by the double integral

(1.3)
$$F(p_1, p_2) = p_1 p_2 \int_0^\infty \int_0^\infty e^{-\frac{1}{2}p_1 t_1 - \frac{1}{2}p_2 t_2} W_{k_1+\frac{1}{2}, m_1}(p_1 t_1) W_{k_2+\frac{1}{2}, m_2}(p_2 t_2) \times (p_1 t_1)^{-k_1-\frac{1}{2}} (p_2 t_2)^{-k_2-\frac{1}{2}} f(t_1, t_2) dt_1 dt_2,$$

and this relation will be symbolically denoted by

$$f(t_1, t_2) \xrightarrow[m_i]{k_i+\frac{1}{2}} F(p_1, p_2), \qquad i = 1, 2. \tag{1.4}$$

Further, if the range of integration in (1.3) is $-\infty$ to $\infty$ in place of 0 to $\infty$, it will be denoted symbolically as

$$f(t_1, t_2) \overset{k_i+\frac{1}{2}}{\underset{m_i}{\twoheadrightarrow}} F(p_1, p_2), \qquad i = 1, 2. \tag{1.5}$$

For $k_i = m_i$, $i = 1, 2$, (1.4) and (1.5) reduce to the Laplace-transform of two variables where the range of integration is 0 to $\infty$ and $-\infty$ to $\infty$ respectively. When the range of integration is 0 to $\infty$, we call either transform (Laplace or Meijer) unilateral two dimensional transform and when the range of integration is

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$-\infty$ to $\infty$, it is called bilateral two dimensional transform. The right hand sides of (1.1) and (1.3) are defined by $L_{II}\{f\}$ and $L_{II}^2\{f\}$. The integrals are taken in the sense of Lebesgue. The domain of convergence is the domain of absolute convergence as explained in Die Dimensionale Laplace-transformation by Doetsch and Voelker [6] and also in the paper of Gupta [3].

In this paper, we have proved certain theorems in unilateral and bilateral two dimensional Meijer-transform and a self-reciprocal property. Examples are given in one variable as an application.

## 2

THEOREM 1. (a). Let

(i) $$t_1^{n_1} t_2^{n_2} f(t_1, t_2) \xrightarrow[m_i]{k_i+\frac{1}{2}} F(p_1, p_2),$$

where $L_{II}^2\{t_1^{n_1} t_2^{n_2} f(t_1, t_2)\}$ is absolutely convergent in a pair of associated half-planes $H_{p_1}$, $H_{p_2}$ which may be defined by $\mathrm{Re}\,(p_i) > 0$, $(i = 1, 2)$.

(ii) $$h_i(\lambda_i, t_i) \xrightarrow[m_i]{k_i+\frac{1}{2}} e^{-\frac{1}{2}\lambda_i \psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i \psi_i(p_i)][\lambda_i \psi_i(p_i)]^{-k_i-\frac{1}{2}},$$

where $\psi_i(p_i) = \phi_i^{-1}(\log p_i)$, $\lambda_i > 0$ and $L_{II}(h_i)$ is absolutely convergent in the half-planes $D_{p_i}$ (say) defined by $\mathrm{Re}\,(p_i) > 0$ and

(iii) $$e^{-\frac{1}{2}\lambda_i \psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i \psi_i(p_i)][\lambda_i \psi_i(p_i)]^{-k_i-\frac{1}{2}}$$

and $h_i(\lambda_i, t_i)$ are bounded and integrable in $(0, \infty)$ in $p_i$ and $t_i$ respectively and $t_1^{n_1-1} t_2^{n_2-1} f(t_1, t_2)$ is absolutely integrable in $t_1$, $t_2$ in $(0, \infty)$.

(iv) $\phi_i(t_i)$ is monotonic, varying from $-\infty$ to $\infty$ at $t_i$ varies from $-\infty$ to $\infty$.

(v) $(F(t_1, t_2))/t_1 t_2$ is absolutely integrable in $t_1$, $t_2$ in $(0, \infty)$. Then

(2.1)
$$G(t_1, t_2) \equiv f\{e^{\phi_1(t_1)}, e^{\phi_2(t_2)}\} e^{n_1\phi_1(t_1)+n_2\phi_2(t_2)} \phi_1'(t_1)\phi_2'(t_2) \xrightarrow[m_i]{k_i+\frac{1}{2}} T(p_1, p_2)$$
$$\equiv p_1 p_2 \int_0^\infty \int_0^\infty h_1(p_1, t_1) h_2(p_2, t_2) \frac{F(t_1, t_2)}{t_1 t_2}\, dt_1\, dt_2,$$

provided that $L_{II}^2\{G\}$ is absolutely convergent in a pair of associated convergent strips $S_{p_1}$ and $S_{p_2}$ which are common regions of $H_{p_1}$, $D_{p_1}$ and $H_{p_2}$, $D_{p_2}$ respectively.

PROOF. Let us consider the image-integral

$$I \equiv p_1 p_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2}p_1 t_1 - \frac{1}{2}p_2 t_2} W_{k_1+\frac{1}{2}, m_1}(p_1 t_1) W_{k_2+\frac{1}{2}, m_2}(p_2 t_2)$$
$$\times (p_1 t_1)^{-k_1-\frac{1}{2}} (p_2 t_2)^{-k_2-\frac{1}{2}} f\{e^{\phi_1(t_1)}, e^{\phi_2(t_2)}\} e^{n_1 \phi_1(t_1) + n_2 \phi_2(t_2)}$$
$$\times \phi_1'(t_1) \phi_2'(t_2) dt_1 dt_2.$$

Suppose it to be absolutely convergent in a pair of associated convergence domains.

Let us put $y_i = e^{\phi_i(t_i)}$. Then, by virtue of (iv), $y_i$ varies from 0 to $\infty$ and $t_i = \phi_i^{-1}(\log y_i)$.

But $\phi_i^{-1}(\log y_i) = \psi_i(y_i)$, $\therefore\ t_i = \psi_i(y_i)$, $i = 1, 2$. Therefore, we have

$$I \equiv p_1 p_2 \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}p_1 \psi_1(y_1) - \frac{1}{2}p_2 \psi_2(y_2)} W_{k_1+\frac{1}{2}, m_1}[p_1 \psi_1(y_1)]$$
$$\times W_{k_2+\frac{1}{2}, m_2}[p_2 \psi_2(y_2)] [p_1 \psi_1(y_1)]^{-k_1-\frac{1}{2}} [p_2 \psi_2(y_2)]^{-k_2-\frac{1}{2}} \tag{2.2}$$
$$\times f(y_1, y_2) y_1^{n_1-1} y_2^{n_2-1} dy_1 dy_2,$$

which remains absolutely convergent for $\text{Re}\,(p_1) > 0$ and $\text{Re}\,(p_2) > 0$.

Now using (ii) in (2.2), we have

$$I \equiv p_1 p_2 \int_0^{\infty} \int_0^{\infty} f(y_1, y_2) y_1^{n_1-1} y_2^{n_2-1} \Big[ y_1 y_2 \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}y_1 x_1 - \frac{1}{2}y_2 x_2}$$
$$\times W_{k_1+\frac{1}{2}, m_1}(y_1 x_1) W_{k_2+\frac{1}{2}, m_2}(y_2 x_2) (y_1 x_1)^{-k_1-\frac{1}{2}} (y_2 x_2)^{-k_2-\frac{1}{2}} \tag{2.3}$$
$$\times h_1(p_1, x_1) h_2(p_2, x_2) dx_1 dx_2 \Big] dy_1 dy_2.$$

On changing the orders of integration in (2.3), which is permissible as $y$- and $x$-integrals are absolutely and uniformly convergent due to assumptions in (i) and (ii), we get

$$I \equiv p_1 p_2 \int_0^{\infty} \int_0^{\infty} h_1(p_1, x_1) h_2(p_2, x_2) \Big[ \int_0^{\infty} \int_0^{\infty} e^{-\frac{1}{2}y_1 x_1 - \frac{1}{2}y_2 x_2}$$
$$\times W_{k_1+\frac{1}{2}, m_1}(y_1 x_1) W_{k_2+\frac{1}{2}, m_2}(y_2 x_2) (y_1 x_1)^{-k_1-\frac{1}{2}} (y_2 x_2)^{-k_2-\frac{1}{2}}$$
$$\times y_1^{n_1} y_2^{n_2} f(y_1, y_2) dy_1 dy_2 \Big] dx_1 dx_2,$$

from which the result follows by using (i).

THEOREM 1. (b). Let

(i) $$f(t_1, t_2) \xrightarrow[m_i]{k_i+\frac{1}{2}} F(p_1, p_2),$$

where $L_{\Pi}^2\{f\}$ is absolutely convergent in a pair of associated half-planes $H_{p_1}$, $H_{p_2}$ which may be defined by $\text{Re}\,(p_i) > 0$, $i = 1, 2$.

(ii) $h_i(\lambda_i, t_i) \xrightarrow[m_i]{k_i+\frac{1}{2}} e^{-\frac{1}{2}\lambda_i \psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i \psi_i(p_i)] [\lambda_i \psi_i(p_i)]^{-k_i-\frac{1}{2}}$,

where

$$\psi_i(p_i) = \phi_i^{-1}\left\{\frac{\log p_i}{\log a_i}\right\}, \quad \lambda_i > 0$$

and $L_{II}\{h_i\}$ is absolutely convergent in the half-planes $D_{p_i}$ (say) defined by $\mathrm{Re}\,(p_i) > 0$ and

(iii) $$e^{-\frac{1}{2}\lambda_i\psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i\psi_i(p_i)][\lambda_i\psi_i(p_i)]^{-k_i-\frac{1}{2}}$$

and $h_i(\lambda_i, t_i)$ are bounded and integrable in $(0, \infty)$ in $p_i$ and $t_i$ respectively and $1/(t_1t_2)f(t_1, t_2)$ is absolutely integrable in $t_1$, $t_2$ in $(0, \infty)$.

(iv) $\phi_i(t_i)$ is monotonic and $a_i^{\phi_i(t_i)}$ tends to zero as $t_i$ tends to $-\infty$ and to $\infty$ as $t_i$ tends to $\infty$.

(v) $(F(t_1, t_2))/t_1t_2$ is absolutely integrable in $t_1, t_2$ in $(0, \infty)$. Then

(2.4)

$$G(t_1, t_2) \equiv f[a_1^{\phi_1(t_1)}, a_2^{\phi_2(t_2)}]\phi_1'(t_1)\phi_2'(t_2) \xrightarrow[m_i]{k_i+\frac{1}{2}}$$

$$T(p_1, p_2) \equiv \frac{p_1p_2}{\log(a_1)\log(a_2)} \int_0^\infty \int_0^\infty h_1(p_1, t_1)h_2(p_2, t_2)\frac{F(t_1, t_2)}{t_1t_2}\, dt_1 dt_2, \quad a_i > 0,$$

provided that $L_{II}^2\{G\}$ is absolutely convergent in a pair of associated convergence strips $S_{p_1}$, $S_{p_2}$ which are common region of $H_{p_1}$, $D_{p_1}$ and $H_{p_2}$, $D_{p_2}$ respectively.

The proof is on the same lines as in Theorem 1(a).

If we substitute $k_i = m_i$, $i = 1, 2$ and $a_1 = a_2 = a$ in the above theorem, we get Gupta's theorem [3, p. 197].

We now give a general theorem which can be used both in unilateral and bilateral transforms.

THEOREM 2. Let

(i) $$t_1^{1/\mu_1}t_2^{1/\mu_2}f(t_1, t_2) \xrightarrow[m_i]{k_i+\frac{1}{2}} F(p_1, p_2),$$

where $L_{II}^2\{t_1^{1/\mu_1}t_2^{1/\mu_2}f(t_1, t_2)\}$ is absolutely convergent in a pair of associated half-planes $H_{p_1}$, $H_{p_2}$ which may be defined by $Re(p_i) > 0$, $i = 1, 2$.

(ii) $$h_i(\lambda_i, t_i) \xrightarrow[m_i]{k_i+\frac{1}{2}} e^{-\frac{1}{2}\lambda_i\psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i\psi_i(p_i)][\lambda_i\psi_i(p_i)]^{-k_i-\frac{1}{2}},$$

where $\psi_i(p_i) = \phi_i^{-1}(p_i^{1/\mu_i})$, $\lambda_i > 0$ and $L_{II}\{h_i\}$ is absolutely convergent in the half-planes $D_{p_i}$, $i = 1, 2$ (say) defined by $\mathrm{Re}\,(p_i) > 0$ and

(iii) $$e^{-\frac{1}{2}\lambda_i\psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i\psi_i(p_i)][\lambda_i\psi_i(p_i)]^{-k_i-\frac{1}{2}}$$

is bounded and integrable in $p_i$ in $(0, \infty)$ and $t_1^{(1/\mu_1)-1} t_2^{(1/\mu_2)-1} f(t_1, t_2)$ is absolutely integrable in $t_1$, $t_2$ in $(0, \infty)$.

(iv) $\phi_i(t_i)$ is monotonic in $t_i$ and varies from 0 to $\infty$ as $t_i$ varies from $-\infty$ to $\infty$ or from 0 to $\infty$ as the case may be. Then

$$G(t_1, t_2) \equiv f[\phi_1^{\mu_1}(t_1), \phi_2^{\mu_2}(t_2)]\, \phi_1'(t_1)\, \phi_2'(t_2) \quad \xrightarrow[m_i]{k_i+\frac{1}{2}}\!\!\!\!\!\rightarrow \ \text{or} \ \xrightarrow[m_i]{k_i+\frac{1}{2}}$$

$$(2.5) \qquad T(t_1, t_2) \equiv \frac{p_1 p_2}{\mu_1 \mu_2} \int_0^\infty \int_0^\infty h_1(p_1, t_1) h_2(p_2, t_2) \frac{F(t_1, t_2)}{t_1 t_2}\, dt_1 dt_2,$$

$$\mu_1 > 0, \ \mu_2 > 0,$$

provided that $L_H^2\{G\}$ is absolutely convergent in a pair of associated strips $S_{p_1}$, $S_{p_2}$ which are common regions of $H_{p_1}$, $D_{p_1}$ and $H_{p_2}$, $D_{p_2}$ respectively and the integral on the right hand side is absolutely convergent in $t_1$, $t_2$ in $(0, \infty)$.

*A self-reciprocal property:*

Let us consider the above theorem in one variable. We also take the image integral in which $t$ varies from 0 to $\infty$.

Let $y = \phi^\mu(t) = 1/t$, so that $t = \phi^{-1}(y^{1/\mu}) = \psi(y)$.

$$\therefore t = \frac{1}{y} = \psi(y),$$

here $t \to 0$, $y \to \infty$ and when $t \to \infty$, $y \to 0$.

Now

$$f[\phi^\mu(t)]\phi'(t) = f\left(\frac{1}{t}\right)\left(-\frac{1}{\mu}\, t^{-1-(1/\mu)}\right) \xrightarrow[m]{k+\frac{1}{2}} \frac{p}{\mu}\int_0^\infty h(p, t)\, \frac{F(t)}{t}\, dt$$

or

$$t^{-(1/\mu)-1} f\left(\frac{1}{t}\right) \xrightarrow[m]{k+\frac{1}{2}} -p \int_0^\infty h(p, t)\, \frac{F(t)}{t}\, dt.$$

But

$$t^{1/\mu} f(t) \xrightarrow[m]{k+\frac{1}{2}} F(p).$$

So if we take

$$t^{1/\mu} f(t) = t^{-(1/\mu)-1} f\left(\frac{1}{t}\right) \quad \text{i.e.} \quad f\left(\frac{1}{t}\right) = t^{(2/\mu)+1} f(t),$$

we get

$$(2.6) \qquad \frac{F(p)}{p} = \int_0^\infty h(p, t)\, \frac{F(t)}{t}\, dt,\ ^{2}$$

i.e. $F(p)/p$ is self-reciprocal under the kernel $h(p, t)$, provided $F(p)$ and $\int_0^\infty h(p, t)(F(t)/t)\, dt$ are continuous functions of $p$ in $(0, \infty)$.

Now

$$h(\lambda, t) \xrightarrow[m]{k+\frac{1}{2}} e^{-\frac{1}{2}(\lambda/p)} W_{k+\frac{1}{2}, m}\left(\frac{\lambda}{p}\right)\left(\frac{\lambda}{p}\right)^{-k-\frac{1}{2}}, \quad \text{where } \psi(p) = \frac{1}{p}\cdot$$

$$(2.7) \qquad \begin{aligned} \therefore h(\lambda, t) = \Bigg\{ & (\lambda t)^{m-k} \frac{\Gamma(-2m)\Gamma(1-3k+m)}{\Gamma(-m-k)\Gamma(1-2k)\Gamma(1-2k+2m)} \\ & \quad {}_2F_3\left[\begin{matrix} 1+m-3k,\ 1+m+k; \\ 1+2m,\ 1-2k,\ 1+2m-2k; \end{matrix}\ -\lambda t\right] \\ & + (\lambda t)^{-m-k} \frac{\Gamma(2m)\Gamma(1-3k-m)}{\Gamma(m-k)\Gamma(1-2k)\Gamma(1-2k-2m)} \\ & \quad {}_2F_3\left[\begin{matrix} 1-m-3k,\ 1-m+k; \\ 1-2m,\ 1-2k,\ 1-2m-2k; \end{matrix}\ -\lambda t\right]\Bigg\} \end{aligned}$$

provided $2m$ is not an integer and

$$\operatorname{Re}(1-3k+m) > 0, \quad \operatorname{Re}(1-3k-m) > 0.$$

*Application of the above:*

Let $t^{1/\mu}f(t) = t^{-2k}(1+t)^{4k-1}$, which has the property that

$$t^{1/\mu}f(t) = t^{-(1/\mu)-1} f\left(\frac{1}{t}\right)\cdot$$

But

$$t^{1/\mu}f(t) \xrightarrow[m]{k+\frac{1}{2}} F(p).$$

Therefore, we have [2, p. 237]

$$\frac{F(p)}{p} = \frac{\Gamma(1-3k+m)\Gamma(1-3k-m)}{\Gamma(1-4k)}\, p^{-k-\frac{1}{2}} e^{p/2} W_{3k-\frac{1}{2}, m}(p),$$

i.e. $p^{-k-\frac{1}{2}} e^{p/2} W_{3k-\frac{1}{2}, m}(p)$ is self-reciprocal under the kernel $h(\lambda, t)$ given by (2.7).

If we substitute $k = m$, we see that $p^{-m-\frac{1}{2}} e^{p/2} W_{3m-\frac{1}{2}, m}(p)$ is self-reciprocal under the kernel $J_0(2\sqrt{\lambda t})$ which is a known result [2, p. 84].

[2] The negative sign is omitted in view of the fact that when $t \to 0$, $y \to \infty$ and when $t \to \infty$, $y \to 0$.

## 3

*Example on Theorem 2*

We take the range of integration from 0 to $\infty$ and consider the case in one variable only.

Let $y = \phi^{\mu}(t) = 1/t$ so that $\psi(y) = 1/y$.

Further let $t^{1/\mu}f(t) = t^{4m-\frac{3}{2}}e^{-(a/t)}$, then taking $k = m-\frac{1}{2}$, we have [1, p. 217]

$$F(p) = \frac{2}{\sqrt{\pi}}\, a^{2m} p^{\frac{3}{2}-2m} [K_{2m}(\sqrt{ap})]^2.$$

From (2.7), we have

$$\begin{aligned} h(\lambda t) = \Bigg\{ &(\lambda t)^{\frac{1}{2}} \frac{\Gamma(-2m)\Gamma(\frac{5}{2}-2m)}{\Gamma(\frac{1}{2}-2m)\Gamma(2-2m)}\, {}_2F_3\left[\begin{matrix}\frac{5}{2}-2m,\ \frac{1}{2}+2m; \\ 2,\ 1+2m,\ 2-2m;\end{matrix}\ -\lambda t\right] \\ &+(\lambda t)^{\frac{1}{2}-2m} \frac{\Gamma(2m)\Gamma(\frac{5}{2}-4m)}{\sqrt{\pi}\Gamma(2-2m)\Gamma(2-4m)} \\ &{}_2F_3\left[\begin{matrix}\frac{1}{2},\ \frac{5}{2}-4m; \\ 1-2m,\ 2-2m,\ 2-4m;\end{matrix}\ -\lambda t\right]\Bigg\}. \end{aligned}$$

Then, according to Theorem 2, we have

$$\begin{aligned} t^{\frac{1}{2}-4m}e^{-at} \xrightarrow[m]{\ m\ } \frac{2a^{2m}}{\sqrt{\pi}}\, p\int_0^{\infty} \Bigg\{ &(pt)^{\frac{1}{2}} \frac{\Gamma(-2m)\Gamma(\frac{5}{2}-2m)}{\Gamma(\frac{1}{2}-2m)\Gamma(2-2m)} \\ &{}_2F_3\left[\begin{matrix}\frac{5}{2}-2m,\ \frac{1}{2}+2m; \\ 2,\ 1+2m,\ 2-2m;\end{matrix}\ -pt\right] \\ &+(pt)^{\frac{1}{2}-2m} \frac{\Gamma(2m)\Gamma(\frac{5}{2}-4m)}{\sqrt{\pi}\Gamma(2-2m)\Gamma(2-4m)} \\ &{}_2F_3\left[\begin{matrix}\frac{1}{2},\ \frac{5}{2}-4m; \\ 1-2m,\ 2-2m,\ 2-4m;\end{matrix}\ -pt\right]\Bigg\} [K_{2m}(\sqrt{at})]^2 t^{\frac{1}{2}-2m}\,dt, \end{aligned}$$

$$\mathrm{Re}\,(p) > 0,\ \mathrm{Re}\,(a) > 0,\ \mathrm{Re}\,(m) < \tfrac{1}{3}.$$

Evaluating the left hand side [4, p. 387], we get after arranging properly

$$\begin{aligned} \int_0^{\infty} \Bigg\{ &(pt)^{\frac{1}{2}} \frac{\Gamma(-2m)\Gamma(\frac{5}{2}-2m)}{\Gamma(\frac{1}{2}-2m)\Gamma(2-2m)}\, {}_2F_3\left[\begin{matrix}\frac{5}{2}-2m,\ \frac{1}{2}+2m; \\ 2,\ 1+2m,\ 2-2m;\end{matrix}\ -pt\right] \\ &+(pt)^{\frac{1}{2}-2m} \frac{\Gamma(2m)\Gamma(\frac{5}{2}-4m)}{\sqrt{\pi}\Gamma(2-2m)\Gamma(2-4m)} \\ &{}_2F_3\left[\begin{matrix}\frac{1}{2},\ \frac{5}{2}-4m; \\ 1-2m,\ 2-2m,\ 2-4m;\end{matrix}\ -pt\right]\Bigg\} [K_{2m}(\sqrt{at})]^2 t^{\frac{1}{2}-2m}\,dt \end{aligned} \tag{3.1}$$

$$(3.1)\qquad = \frac{\sqrt{\pi}\Gamma(2-4m)\Gamma(2-6m)}{2a^{2m}\Gamma(\frac{5}{2}-6m)} p^{4m-\frac{3}{2}}\, {}_2F_1\left[\begin{matrix}2-6m, 2-4m; \\ \frac{5}{2}-6m;\end{matrix} \; -\frac{a}{p}\right],$$

$\mathrm{Re}\,(p) > 0$, $\mathrm{Re}\,(a) > 0$, $\mathrm{Re}\,(m) < \frac{1}{3}$.

If we substitute $m = \frac{1}{4}$ in (3.1), we get a known result [1, p. 182].

## 4

THEOREM 3. Let

$$\text{(i)}\qquad f(t_1, t_2) \xrightarrow[m_i]{k_i+\frac{1}{2}} F(p_1, p_2), \qquad i = 1, 2$$

where $L_{II}^2\{f\}$ is absolutely convergent in a pair of associated domains $S_{p_1}$ and $S_{p_2}$.

$$\text{(ii)}\quad h_i(\lambda_i, t_i) \xrightarrow[m_i]{k_i+\frac{1}{2}} \phi_i(p_i)e^{-\frac{1}{2}\lambda_i\psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i\psi_i(p_i)] \times [\lambda_i\psi_i(p_i)]^{-k_i-\frac{1}{2}}, \qquad i = 1, 2,$$

where $\lambda_i$ denotes a real parameter and $L_{II}\{h_i\}$ is absolutely convergent in $t_i$ in the domain $D_{p_i}$ (say) and $\psi_i(p_i) \in S_{p_i}$ and $\phi_i(p_i) \in S_{p_i}$. (iii) $f(t_1, t_2)$ is absolutely convergent in $(0, \infty)$ and $h_1(\lambda_1, t_1)$ and $h_2(\lambda_2, t_2)$ are bounded and integrable in $\lambda_1, \lambda_2$ and $t_1, t_2$ in $(0, \infty)$.

Then

$$(4.1)\qquad \begin{aligned} G(t_1, t_2) &\equiv \int_0^\infty\int_0^\infty h_1(\lambda_1, t_1)h_2(\lambda_2, t_\lambda)f(\lambda_1, \lambda_2)\,d\lambda_1 d\lambda_2 \\ &\xrightarrow[m_i]{k_i+\frac{1}{2}} \frac{\phi_1(p_1)\phi_2(p_2)}{\psi_1(p_1)\psi_2(p_2)} F[\psi_1(p_1), \psi_2(p_2)], \end{aligned}$$

provided that $L_{II}^2\{G\}$ is absolutely convergent in a pair of associated domains $\Omega_{p_1}$ and $\Omega_{p_2}$ where $\Omega_{p_1}$ is the common part (suppose it exists) of $S_{p_1}$ and $D_{p_1}$ in the complex $p_1$ plane and $\Omega_{p_2}$ is a similar common part of $S_{p_2}$ and $D_{p_2}$ in the complex $p_2$ plane.

PROOF: We replace $p_1$ and $p_2$ in (i) by $\psi_1(p_1)$ and $\psi_2(p_2)$ and rest of the proof is simple.

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