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Certain theorems on unilateral and bilateral operational calculus


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Certain theorems on unilateral and bilateral operational calculus

by

B. S. Tavathia

1. Introduction

A generalization of the Laplace-transform is given [5] as

\begin{equation}
F(p) = p \int_{0}^{\infty} e^{-\frac{1}{2}pt} W_{k+\frac{1}{2}, m}(pt)(pt)^{-k-\frac{1}{2}} f(t) dt,
\end{equation}

where $W_{k, m}(t)$ is the confluent hypergeometric function. $F(p)$ is called the Meijer-transform of $f(t)$ and is symbolically denoted by

\begin{equation}
f(t) \xrightarrow{\frac{k+\frac{1}{2}}{m}} F(p) \quad \text{or} \quad F(p) \xleftarrow{\frac{k+\frac{1}{2}}{m}} f(t).
\end{equation}

For $k = m$, it reduces to the Laplace-transform.

In two variables $f(t)$ and $F(p)$ will be replaced by $f(t_1, t_2)$ and $F(p_1, p_2)$, where $F(p_1, p_2)$ is defined by the double integral

\begin{equation}
F(p_1, p_2) = p_1 p_2 \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{1}{2}p_1 t_1 - \frac{1}{2}p_2 t_2} W_{k_1+\frac{1}{2}, m_1}(p_1 t_1) W_{k_2+\frac{1}{2}, m_2}(p_2 t_2)
\times (p_1 t_1)^{-k_1-\frac{1}{2}} (p_2 t_2)^{-k_2-\frac{1}{2}} f(t_1, t_2) dt_1 dt_2,
\end{equation}

and this relation will be symbolically denoted by

\begin{equation}
f(t_1, t_2) \xrightarrow{\frac{k_i+\frac{1}{2}}{m_i}} F(p_1, p_2), \quad i = 1, 2.
\end{equation}

Further, if the range of integration in (1.3) is $-\infty$ to $\infty$ in place of $0$ to $\infty$, it will be denoted symbolically as

\begin{equation}
f(t_1, t_2) \xrightarrow{\frac{k_i+\frac{1}{2}}{m_i}} F(p_1, p_2), \quad i = 1, 2.
\end{equation}

For $k_i = m_i$, $i = 1, 2$, (1.4) and (1.5) reduce to the Laplace-transform of two variables where the range of integration is $0$ to $\infty$ and $-\infty$ to $\infty$ respectively. When the range of integration is $0$ to $\infty$, we call either transform (Laplace or Meijer) unilateral two dimensional transform and when the range of integration is

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— \infty \) to \( \infty \), it is called bilateral two dimensional transform. The right hand sides of (1.1) and (1.3) are defined by \( L^2_{\Pi}\{f\} \) and \( L^2_{\Pi}\{f\} \). The integrals are taken in the sense of Lebesgue. The domain of convergence is the domain of absolute convergence as explained in Die Dimensionale Laplace-transformation by Doetsch and Volker [6] and also in the paper of Gupta [3].

In this paper, we have proved certain theorems in unilateral and bilateral two dimensional Meijer-transform and a self-reciprocal property. Examples are given in one variable as an application.

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**THEOREM 1.** (a). Let

(i) \[ t_1^{\mu_1} t_2^{\mu_2} f(t_1, t_2) - \frac{k_i+\frac{1}{2}}{m_i} \rightarrow F(p_1, p_2), \]

where \( L^2_{\Pi}\{t_1^{\mu_1} t_2^{\mu_2} f(t_1, t_2)\} \) is absolutely convergent in a pair of associated half-planes \( H_{p_i}, H_{p_2} \) which may be defined by \( \text{Re} (p_i) > 0 \), \( i = 1, 2 \).

(ii) \[ h_i(\lambda_i, t_i) - \frac{k_i+\frac{1}{2}}{m_i} e^{-\frac{1}{2} \lambda_i \psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i \psi_i(p_i)][\lambda_i \psi_i(p_i)]^{-k_i+\frac{1}{2}}, \]

where \( \psi_i(p_i) = \phi_i^{-1}(\log p_i), \lambda_i > 0 \) and \( L^2_{\Pi}(h_i) \) is absolutely convergent in the half-planes \( D_{p_i} \) (say) defined by \( \text{Re} (p_i) > 0 \) and

(iii) \[ e^{-\frac{1}{2} \lambda_i \psi_i(p_i)} W_{k_i+\frac{1}{2}, m_i}[\lambda_i \psi_i(p_i)][\lambda_i \psi_i(p_i)]^{-k_i+\frac{1}{2}} \]

and \( h_i(\lambda_i, t_i) \) are bounded and integrable in \( (0, \infty) \) in \( p_i \) and \( t_i \) respectively and \( t_1^{\mu_1-1} t_2^{\mu_2-1} f(t_1, t_2) \) is absolutely integrable in \( t_1, t_2 \) in \( (0, \infty) \).

(iv) \( \phi_i(t_i) \) is monotonic, varying from \( -\infty \) to \( \infty \) at \( t_i \) varies from \( -\infty \) to \( \infty \).

(v) \( (F(t_1, t_2)/t_1 t_2) \) is absolutely integrable in \( t_1, t_2 \) in \( (0, \infty) \). Then

\[ G(t_1, t_2) \equiv \int_{0}^{\infty} \int_{0}^{\infty} h_1(p_1, t_1) h_2(p_2, t_2) \frac{F(t_1, t_2)}{t_1 t_2} dt_1 dt_2, \]

provided that \( L^2_{\Pi}\{G\} \) is absolutely convergent in a pair of associated convergent strips \( S_{p_1} \) and \( S_{p_2} \) which are common regions of \( H_{p_1}, D_{p_1} \) and \( H_{p_2}, D_{p_2} \) respectively.

**Proof.** Let us consider the image-integral
Suppose it to be absolutely convergent in a pair of associated convergence domains.

Let us put \( y_i = e^{\phi_i(t_i)} \). Then, by virtue of (iv), \( y_i \) varies from 0 to \( \infty \) and \( t_i = \phi_i^{-1}(\log y_i) \).

But \( \phi_i^{-1}(\log y_i) = \psi_i(y_i), \quad \therefore t_i = \psi_i(y_i), \quad i = 1, 2. \) Therefore, we have

\[
I \equiv p_1 p_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2} p_1 t_1 - \frac{1}{2} p_2 t_2} W_{k_1, m_1}(p_1 t_1) W_{k_2, m_2}(p_2 t_2) \\
\times (p_1 t_1)^{-k_1-\frac{1}{2}} (p_2 t_2)^{-k_2-\frac{1}{2}} f(e^{\phi_1(t_1)}, e^{\phi_2(t_2)}) e^{n_1 \phi_1(t_1) + m_2 \phi_2(t_2)} \\
\times \phi_1(t_1) \phi_2(t_2) dt_1 dt_2.
\]

which remains absolutely convergent for \( \Re(p_1) > 0 \) and \( \Re(p_2) > 0 \).

Now using (ii) in (2.2), we have

\[
I \equiv p_1 p_2 \int_{0}^{\infty} \int_{0}^{\infty} f(y_1, y_2) y_1^{n_1-1} y_2^{n_2-1} \left[ y_1 y_2 \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{1}{2} v_1 x_1 - \frac{1}{2} v_2 x_2} \\
\times W_{k_1, m_1}(y_1 x_1) W_{k_2, m_2}(y_2 x_2) (y_1 x_1)^{-k_1-\frac{1}{2}} (y_2 x_2)^{-k_2-\frac{1}{2}} \\
\times h_1(p_1, x_1) h_2(p_2, x_2) dx_1 dx_2 \right] dy_1 dy_2.
\]

On changing the orders of integration in (2.3), which is permissible as \( y \)- and \( x \)-integrals are absolutely and uniformly convergent due to assumptions in (i) and (ii), we get

\[
I \equiv p_1 p_2 \int_{0}^{\infty} \int_{0}^{\infty} h_1(p_1, x_1) h_2(p_2, x_2) \left[ \int_{0}^{\infty} \int_{0}^{\infty} e^{-\frac{1}{2} v_1 x_1 - \frac{1}{2} v_2 x_2} \\
\times W_{k_1, m_1}(y_1 x_1) W_{k_2, m_2}(y_2 x_2) (y_1 x_1)^{-k_1-\frac{1}{2}} (y_2 x_2)^{-k_2-\frac{1}{2}} \\
\times y_1^{n_1} y_2^{n_2} f(y_1, y_2) dy_1 dy_2 \right] dx_1 dx_2,
\]

from which the result follows by using (i).

\textbf{Theorem 1. (b).} Let

(i) \( f(t_1, t_2) \rightarrow F(p_1, p_2), \)

where \( L_H^n(f) \) is absolutely convergent in a pair of associated half-planes \( H_{p_1}, H_{p_2} \) which may be defined by \( \Re(p_i) > 0, \quad i = 1, 2. \)

(ii) \( h_i(\lambda_i, t_i) \rightarrow e^{\frac{1}{2} \lambda_i \psi_i(t_i)} W_{k_i, m_i}[\lambda_i \psi_i(p_i)] [\lambda_i \psi_i(p_i)]^{-k_i-\frac{1}{2}}, \)

where
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and $L_{\Pi}\{h_i\}$ is absolutely convergent in the half-planes $D_{p_i}$ (say) defined by $\text{Re} (p_i) > 0$ and

(iii) $e^{-\frac{1}{2} \lambda_i \psi_t(p_i)} W_{k_i t^\frac{1}{2}, m_i} [\lambda_i \psi_t(p_i)][\lambda_i \psi_t(p_i)]^{-k_i - \frac{1}{2}}$

and $h_i(\lambda_i, t_i)$ are bounded and integrable in $(0, \infty)$ in $p_i$ and $t_i$ respectively and $1/(t_1 t_2)f(t_1, t_2)$ is absolutely integrable in $t_1, t_2$ in $(0, \infty)$.

(iv) $\phi_i(t_i)$ is monotonic and $a_i^{\phi_i(t_i)}$ tends to zero as $t_i$ tends to $-\infty$ and to $\infty$ as $t_i$ tends to $\infty$.

(v) $(F(t_1, t_2))/t_1 t_2$ is absolutely integrable in $t_1, t_2$ in $(0, \infty)$. Then

\begin{align*}
G(t_1, t_2) &= f[a_1^{\phi_1(t_1)}, a_2^{\phi_2(t_2)}] \phi_1'(t_1) \phi_2'(t_2) \frac{k_i + \frac{1}{2}}{m_i} \\
T(p_1, p_2) &= \frac{p_1 p_2}{\log (a_1) \log (a_2)} \int_0^\infty \int_0^\infty h_1(p_1, t_1) h_2(p_2, t_2) \frac{F(t_1, t_2)}{t_1 t_2} dt_1 dt_2,
\end{align*}

provided that $L_{H_2}\{G\}$ is absolutely convergent in a pair of associated convergence strips $S_{p_1}, S_{p_2}$ which are common region of $H_{p_1}, D_{p_1}$ and $H_{p_2}, D_{p_2}$ respectively.

The proof is on the same lines as in Theorem 1(a).

If we substitute $k_i = m_i, i = 1, 2$ and $a_1 = a_2 = a$ in the above theorem, we get Gupta's theorem [3, p. 197].

We now give a general theorem which can be used both in unilateral and bilateral transforms.

**Theorem 2.** Let

(i) $t_1^{1/\mu_1} t_2^{1/\mu_2} f(t_1, t_2) \frac{k_i + \frac{1}{2}}{m_i} \rightarrow F(p_1, p_2),$

where $L_{H_1}^{\infty}\{t_1^{1/\mu_1} t_2^{1/\mu_2} f(t_1, t_2)\}$ is absolutely convergent in a pair of associated half-planes $H_{p_1}, H_{p_2}$ which may be defined by $\text{Re}(p_i) > 0, i = 1, 2$.

(ii) $h_i(\lambda_i, t_i) \frac{k_i + \frac{1}{2}}{m_i} \rightarrow e^{-\frac{1}{2} \lambda_i \psi_t(p_i)} W_{k_i t^\frac{1}{2}, m_i} [\lambda_i \psi_t(p_i)][\lambda_i \psi_t(p_i)]^{-k_i - \frac{1}{2}},$

where $\psi_t(p_i) = \phi_i^{-1}(p_i^{1/\mu_i}), \lambda_i > 0$ and $L_{H_1}\{h_i\}$ is absolutely convergent in the half-planes $D_{p_i}, i = 1, 2$ (say) defined by $\text{Re}(p_i) > 0$ and

(iii) $e^{-\frac{1}{2} \lambda_i \psi_t(p_i)} W_{k_i t^\frac{1}{2}, m_i} [\lambda_i \psi_t(p_i)][\lambda_i \psi_t(p_i)]^{-k_i - \frac{1}{2}}$
is bounded and integrable in \( p_i \) in \((0, \infty)\) and \( f_{1(1/p_1)}^{-1} f_{2(1/p_2)}^{-1} f(t_1, t_2) \)

is absolutely integrable in \( t_1, t_2 \) in \((0, \infty)\).

(iv) \( \phi_i(t_i) \) is monotonic in \( t_i \) and varies from 0 to \( \infty \) as \( t_i \) varies from \( -\infty \) to \( \infty \) or from 0 to \( \infty \) as the case may be. Then

\[
G(t_1, t_2) = f[\phi_1^m(t_1), \phi_2^m(t_2)] \frac{\phi_1'(t_1) \phi_2'(t_2)}{m_1}
\]

or

\[
G(t_1, t_2) = f[\phi_1^m(t_1), \phi_2^m(t_2)] \frac{\phi_1'(t_1) \phi_2'(t_2)}{m_2}
\]

(2.5)

\[
T(t_1, t_2) = \frac{p_1 p_2}{\mu_1 \mu_2} \int_0^\infty \int_0^\infty h_1(p_1, t_1) h_2(p_2, t_2) \frac{F(t_1, t_2)}{t_1 t_2} \, dt_1 \, dt_2,
\]

\[\mu_1 > 0, \mu_2 > 0,\]

provided that \( L^2_\mu(G) \) is absolutely convergent in a pair of associated strips \( S_{p_1}, S_{p_2} \) which are common regions of \( H_{p_1}, D_{p_1} \) and \( H_{p_2}, D_{p_2} \) respectively and the integral on the right hand side is absolutely convergent in \( t_1, t_2 \) in \((0, \infty)\).

A self-reciprocal property:

Let us consider the above theorem in one variable. We also take the image integral in which \( t \) varies from 0 to \( \infty \).

Let \( y = \phi^m(t) = 1/t, \) so that \( t = \phi^{-1}(y^{1/\mu}) = \psi(y). \)

\[\therefore t = \frac{1}{y} = \psi(y),\]

here \( t \to 0, y \to \infty \) and when \( t \to \infty, y \to 0.\)

Now

\[f[\phi^m(t)] \phi'(t) = f\left(\frac{1}{t}\right) \left( - \frac{1}{\mu} t^{-1-(1/\mu)} \right) \frac{k+1}{m} \frac{p}{\mu} \int_0^\infty h(p, t) \frac{F(t)}{t} \, dt\]

or

\[t^{-(1/\mu)} f\left(\frac{1}{t}\right) \frac{k+1}{m} = -p \int_0^\infty h(p, t) \frac{F(t)}{t} \, dt.\]

But

\[t^{1/\mu} f(t) = t^{-(1/\mu)} f\left(\frac{1}{t}\right) \quad \text{i.e.} \quad f\left(\frac{1}{t}\right) = t^{(2/\mu)-1} f(t),\]

So if we take

\[t^{1/\mu} f(t) = t^{-(1/\mu)} f\left(\frac{1}{t}\right) \quad \text{i.e.} \quad f\left(\frac{1}{t}\right) = t^{(2/\mu)-1} f(t),\]

we get
\[ \frac{F(p)}{p} = \int_0^\infty h(p, t) \frac{F(t)}{t} \, dt, \]

i.e. \( F(p)/p \) is self-reciprocal under the kernel \( h(p, t) \), provided \( F(p) \) and \( \int_0^\infty h(p, t)(F(t)/t) \, dt \) are continuous functions of \( p \) in \((0, \infty)\).

Now

\[ h(\lambda, t) \rightarrow \frac{k+\frac{1}{m}}{e^{-\frac{1}{2}((\lambda/\rho))} W_{k+\frac{1}{m}} \left( \frac{\lambda}{\rho}, \left( \frac{\lambda}{\rho} \right)^{-k-\frac{1}{2}}, \phi(p) = \frac{1}{p} \right). \]

\[ \therefore h(\lambda, t) = \left\{ \frac{I(-2m)I(1-3k+m)}{I(-m-k)I(1-2k)I(1-2k+2m)} \right\} \]

\[ + \frac{I(2m)I(1-3k-m)}{I(m-k)I(1-2k)I(1-2k-2m)} \]

\[ \left( \frac{k+\frac{1}{m}}{e^{-\frac{1}{2}((\lambda/\rho))} W_{k+\frac{1}{m}} \left( \frac{\lambda}{\rho}, \left( \frac{\lambda}{\rho} \right)^{-k-\frac{1}{2}}, \phi(p) = \frac{1}{p} \right)} \right). \]

provided \( 2m \) is not an integer and

\[ \text{Re}(1-3k+m) > 0, \quad \text{Re}(1-3k-m) > 0. \]

**Application of the above:**

Let \( t^{1/\mu}(t) = t^{-2k}(1+t)^{4k-1} \), which has the property that

\[ t^{1/\mu}(t) = t^{-1/\mu-1}f \left( \frac{1}{t} \right). \]

But

\[ t^{1/\mu}(t) \rightarrow \frac{k+\frac{1}{m}}{e^{-\frac{1}{2}((\lambda/\rho))} W_{k+\frac{1}{m}} \left( \frac{\lambda}{\rho}, \left( \frac{\lambda}{\rho} \right)^{-k-\frac{1}{2}}, \phi(p) = \frac{1}{p} \right)}. \]

Therefore, we have [2, p. 237]

\[ \frac{F(p)}{p} = \frac{I(1-3k+m)I(1-3k-m)}{I(1-4k)} p^{-k-\frac{1}{2}} e^{p/2} W_{3k-1, m}(p), \]

i.e. \( p^{-k-\frac{1}{2}} e^{p/2} W_{3k-1, m}(p) \) is self-reciprocal under the kernel \( h(\lambda, t) \) given by (2.7).

If we substitute \( k = m \), we see that \( p^{-m-\frac{1}{2}} e^{p/2} W_{3m-1, m}(p) \) is self-reciprocal under the kernel \( J_0(2\sqrt{\lambda}t) \) which is a known result [2, p. 84].

\[ \text{The negative sign is omitted in view of the fact that when } t \to 0, y \to \infty \text{ and when } t \to \infty, y \to 0. \]
Example on Theorem 2

We take the range of integration from 0 to $\infty$ and consider the case in one variable only.

Let $y = \phi^\mu(t) = 1/t$ so that $\psi(y) = 1/y$.

Further let $t^{1/\mu} f(t) = t^{4m-\frac{1}{2}} e^{-(a/t)}$, then taking $k = m - \frac{1}{2}$, we have [1, p. 217]

$$F(p) = \frac{2}{\sqrt{\pi}} a^{2m} p^{3-2m} [K_{2m}(\sqrt{ap})]^2.$$

From (2.7), we have

$$h(\lambda t) = \left\{ (\lambda t)^{\frac{1}{2}} \frac{\Gamma(-2m) \Gamma(\frac{5}{2}-2m)}{\Gamma(\frac{1}{2}-2m) \Gamma(2-2m)} {}_2F_3 \left[ \frac{5}{2}-2m, \frac{1}{2}+2m; 2, 1+2m, 2-2m; -\lambda t \right] 
+ (\lambda t)^{1-2m} \frac{\Gamma(2m) \Gamma(\frac{5}{2}-4m)}{\sqrt{\pi} \Gamma(2-2m) \Gamma(2-4m)} {}_2F_3 \left[ \frac{1}{2}, \frac{5}{2}-4m; 1-2m, 2-2m, 2-4m; -\lambda t \right] \right\}.$$

Then, according to Theorem 2, we have

$$t^{4m} e^{-at} \xrightarrow{m \to \infty} \frac{2a^{2m}}{\sqrt{\pi}} p \int_0^\infty \left\{ (pt)^{\frac{1}{2}} \frac{\Gamma(-2m) \Gamma(\frac{5}{2}-2m)}{\Gamma(\frac{1}{2}-2m) \Gamma(2-2m)} {}_2F_3 \left[ \frac{5}{2}-2m, \frac{1}{2}+2m; 2, 1+2m, 2-2m; -pt \right] 
+ (pt)^{1-2m} \frac{\Gamma(2m) \Gamma(\frac{5}{2}-4m)}{\sqrt{\pi} \Gamma(2-2m) \Gamma(2-4m)} {}_2F_3 \left[ \frac{1}{2}, \frac{5}{2}-4m; 1-2m, 2-2m, 2-4m; -pt \right] \right\} [K_{2m}(\sqrt{at})]^2 t^{4-2m} dt,$$

Re $p > 0$, Re $(a) > 0$, Re $(m) < \frac{1}{3}$.

Evaluating the left hand side [4, p. 387], we get after arranging properly

$$\int_0^\infty \left\{ (pt)^{\frac{1}{2}} \frac{\Gamma(-2m) \Gamma(\frac{5}{2}-2m)}{\Gamma(\frac{1}{2}-2m) \Gamma(2-2m)} {}_2F_3 \left[ \frac{5}{2}-2m, \frac{1}{2}+2m; 2, 1+2m, 2-2m; -pt \right] 
+ (pt)^{1-2m} \frac{\Gamma(2m) \Gamma(\frac{5}{2}-4m)}{\sqrt{\pi} \Gamma(2-2m) \Gamma(2-4m)} {}_2F_3 \left[ \frac{1}{2}, \frac{5}{2}-4m; 1-2m, 2-2m, 2-4m; -pt \right] \right\} [K_{2m}(\sqrt{at})]^2 t^{4-2m} dt$$

(3.1)
If we substitute \( m = \frac{1}{4} \) in (3.1), we get a known result [1, p. 182].

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\textbf{Theorem 3.} Let

(i) 
\[ f(t_1, t_2) \xrightarrow{k_i + \frac{1}{2}} F(p_1, p_2), \quad i = 1, 2 \]

where \( L^2_{\mathcal{H}}\{f\} \) is absolutely convergent in a pair of associated domains \( S_{p_1} \) and \( S_{p_2} \).

(ii) 
\[ h_i(\lambda_i, t_i) \xrightarrow{k_i + \frac{1}{2}} \phi_i(p_i) e^{-\frac{1}{2} \lambda_i \psi_i(p_i)} W_{k_i + \frac{1}{2}, m_i}[\lambda_i \psi_i(p_i)] \times [\lambda_i \psi_i(p_i)]^{-k_i - \frac{1}{2}}, \quad i = 1, 2, \]

where \( \lambda_i \) denotes a real parameter and \( L^2_{\mathcal{H}}\{h_i\} \) is absolutely convergent in \( t_i \) in the domain \( D_{p_i} \) (say) and \( \psi_i(p_i) \in S_{p_i} \) and \( \phi_i(p_i) \in S_{p_i} \). (iii) \( f(t_1, t_2) \) is absolutely convergent in \( (0, \infty) \) and \( h_1(\lambda_1, t_1) \) and \( h_2(\lambda_2, t_2) \) are bounded and integrable in \( \lambda_1, \lambda_2 \) and \( t_1, t_2 \) in \( (0, \infty) \).

Then

\[ G(t_1, t_2) = \int_0^\infty \int_0^\infty h_1(\lambda_1, t_1) h_2(\lambda_2, t_2) / (\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \]

(4.1)

\[ \frac{k_i + \frac{1}{2}}{m_i} \phi_1(p_1) \phi_2(p_2) \psi_1(p_1) \psi_2(p_2) \]

provided that \( L^2_{\mathcal{H}}\{G\} \) is absolutely convergent in a pair of associated domains \( \Omega_{p_1} \) and \( \Omega_{p_2} \) where \( \Omega_{p_1} \) is the common part (suppose it exists) of \( S_{p_1} \) and \( D_{p_1} \) in the complex \( p_1 \) plane and \( \Omega_{p_2} \) is a similar common part of \( S_{p_2} \) and \( D_{p_2} \) in the complex \( p_2 \) plane.

\textbf{Proof:} We replace \( p_1 \) and \( p_2 \) in (i) by \( \psi_1(p_1) \) and \( \psi_2(p_2) \) and rest of the proof is simple.

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