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Meromorphic functions of regular growth

by

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Let $f(z)$ be a meromorphic function of order ρ $(0 \leqq \rho < \infty)$ and lower order λ $(0 \leqq \lambda < \infty)$. Let $T(r, f)$, $n(r, a)$, $N(r, a)$, $(0 \leqq a \leqq \infty)$ have their usual meanings.

1

Let $\lambda(r)$ be proximate order H of $f(z)$ satisfying the following conditions:

1.1 $\lambda(r)$ is non-negative, continuous function of r for $r \geqq r_0$.

1.2 $\lambda(r)$ is differentiable except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist.

1.3 $\lambda(r) \to \lambda$ as $r \to \infty$

1.4 $r\lambda'(r) \log r = o(\lambda(r))$ as $r \to \infty$

1.5 $T(r, f) \geqq r^{\lambda(r)}$ for all $r \geqq r_0$

1.6 $T(r, f) = r^{\lambda(r)}$ for a sequence of values of $r \to \infty$.

From 1.1—1.4 we can easily deduce the following properties

1.7 $r^{\lambda(r)}$ is an increasing function of $r \geqq r_0$ [3]

1.8 $(ur)^{\lambda(ur)} \sim u^{\lambda} r^{\lambda(r)}$ for $r \geqq r_0$.

From 1.1—1.6 we can easily prove that

1.9 $\liminf\limits_{r \to \infty} \dfrac{T(r, f)}{r^{\lambda(r)}} = 1.$

[1] This paper was presented in the 75th annual meeting of The American Mathematical Society, at New Orleans, Louisiana, 1969.

2

Let $\rho(r)$ be proximate order D of $f(z)$ satisfying the following conditions:

2.1 $\rho(r)$ is non-negative, continuous function of r for $r \geqq r_0$

2.2 $\rho(r)$ is differentiable except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist.

2.3 $\rho(r) \to \rho$ as $r \to \infty$

2.4 $r\rho'(r) \log r = o(\rho(r))$ as $r \to \infty$

2.5 $T(r, f) \leqq r^{\rho(r)}$ for all $r \geqq r_0$

2.6 $T(r, f) = r^{\rho(r)}$ for a sequence of values of $r \to \infty$

From 2.1—2.4 we can easily deduce the following properties.

2.7 $r^{\rho(r)}$ is an increasing function of $r \geqq r_0$ [3]

2.8 $(ur)^{\rho(ur)} \sim u^{\rho} r^{\rho(r)}$ for $r \geqq r_0$

From 2.1—2.6 we can easily prove

2.9 $$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}} = 1.$$

Since in this paper we study the asymtotic properties, the behaviour of proximate orders plays no role on a finite interval and we shall assume that $r^{\lambda(r)}$ and $r^{\rho(r)}$ are monotonic functions for $r \geqq 0$ and we define them to be zero at origin, without any loss of generality.

For the existence of these proximate orders H and D for $f(z)$ see [1]. There these are defined for $\lambda > 0$, $\rho > 0$. Here our definition is true for $\lambda = \rho = 0$ as well, since we have introduced the condition 1.4 and 2.4 in place of

1.4′ $r\lambda'(r) \log r \to 0$ as $r \to \infty$

2.4′ $r\rho'(r) \log r \to 0$ as $r \to \infty$.

For $\lambda > 0$, $\rho > 0$, the conditions 1.4 and 1.4′, 2.4 and 2.4′ are equivalent.

3

For convenience we set

3.1 $n(r) = n(r, a)+n(r, b)$

3.2 $N(r) = N(r, a)+N(r, b)$
where $a \neq b$, $(0 \leqq a \leqq \infty)$, $(0 \leqq b \leqq \infty)$

3.3 $$H(r) = \exp\left(\int_{r_0}^{r} \frac{\lambda(r)}{r}\, dr\right)$$

3.4 $$D(r) = \exp\left(\int_{r_0}^{r} \frac{\rho(r)}{r}\, dr\right).$$

4

In [2] we proved

Theorem A. *If*

4.1 $$\liminf_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}} = \beta > 0$$

and

4.2 $$\frac{N(r)}{r^{\rho(r)}} \to 0 \quad as \quad r \to \infty.$$

Then for $x \neq a, b$

4.3 $$0 < \beta \leqq \liminf_{r\to\infty} \frac{N(r, n)}{r^{\rho(r)}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \leqq 1$$

and

Theorem B. *If*

4.4 $$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} = \alpha < \infty$$

and

4.5 $$\frac{N(r)}{r^{\lambda(r)}} \to 0 \quad as \quad r \to 0$$

Then for $x \neq a, b$

4.6 $$1 = \liminf_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \leqq \alpha < \infty.$$

If we analyse the conditions 4.1 and 4.4, we notice that they imply some restrictions of regularity on the growth of the function $f(z)$. In fact in proving theorem 1, we have shown that functions satisfying either 4.1 or 4.4 are functions of regular growth.

Apparently it looks that there is no direct relation between

$$\frac{T(r, f)}{r^{\lambda(r)}} \quad \text{and} \quad \frac{T(r, f)}{r^{\rho(r)}}.$$

From 1.9 and 2.9 we can conclude that

$$\liminf_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} = 1 = \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}}.$$

It is interesting to know a relation between

$$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} \quad \text{and} \quad \liminf_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}}$$

Using the properties of proximate orders given in 1 and 2 we prove

THEOREM 1.

$$\limsup_{r\to\infty} \frac{T(r, f)}{H(r)} < \infty \quad \textit{iff} \quad \liminf_{r\to\infty} \frac{T(r, f)}{D(r)} > 0.$$

THEOREM 2.

$$\limsup_{r\to\infty} \frac{T(r, f)}{H(r)} = \infty \quad \textit{iff} \quad \liminf_{r\to\infty} \frac{T(r, f)}{D(r)} = 0.$$

As an illustration let us consider the following example when

$$f(z) = \prod_{1}^{\infty} \left(1 + \frac{z}{\Delta_n}\right)^{bp_n}$$

where

$$b = [\rho]+1$$
$$\Delta_n = n^{n^n}$$
$$p_n = (\Delta_n)^{\rho+\varepsilon}.$$

Then

$$\limsup_{r\to\infty} \frac{N(r, 0)}{r^{\lambda(r)}} = \infty \qquad [2. \text{ pp. } 194].$$

Hence

$$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} = \infty$$

so that by lemma 1, 5.1

$$\limsup_{r\to\infty} \frac{T(r, f)}{H(r)} = \infty.$$

Therefore by Theorem 2,

$$\liminf_{r\to\infty} \frac{T(r, f)}{D(r)} = 0$$

which we can verify directly.

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First we prove the following easy lemmas.

LEMMA 1.

5.1 $$H(r) \sim \frac{r^{\lambda(r)}}{r_0^{\lambda(r_0)}}$$

5.2 $$D(r) \sim \frac{r^{\rho(r)}}{r_0^{\rho(r_0)}} \qquad \textit{as } r \to \infty.$$

PROOF OF LEMMA 1.

$$\begin{aligned} H(r) &= \exp\left(\int_{r_0}^{r} \frac{\lambda(r)}{r}\, dr\right) \\ &= \exp\left[\lambda(r)\log r|_{r_0}^{r} - \int_{r_0}^{r} \lambda'(r)\log r dr\right] \\ &= \exp\left[\lambda(r)\log r|_{r_0}^{r} - o\left(\int_{r_0}^{r} \frac{\lambda(r)}{r}\, dr\right)\right] \quad \text{by } 1.4 \\ &\sim \frac{r^{\lambda(r)}}{r_0^{\lambda(r)}} \quad \text{as } r \to \infty. \end{aligned}$$

The proof of 5.2 is similar, where we use property 2.4 in place of 1.4.

If $r_0 \to 1$, $r_0^{\lambda(r_0)} \to 1$ and we deduce that as $r_0 \to 1$

5.3 $$H(r) \to r^{\lambda(r)}$$

5.4 $$D(r) \to r^{\rho(r)}.$$

LEMMA 2.

5.5 $$\int_{r_0}^{r} \frac{H(r)}{r^{\alpha+1}}\, dr \sim \frac{H(r)}{(\lambda-\alpha)r^{\alpha}} \qquad 0 \leqq \alpha < \lambda$$

5.6 $$\int_{r_0}^{r} \frac{D(r)}{r^{\alpha+1}}\, dr \sim \frac{D(r)}{(\rho-\alpha)r^{\alpha}} \qquad 0 \leqq \alpha < \rho$$

5.7 $$\int_{r}^{\infty} \frac{H(r)}{r^{\alpha+1}}\, dr \sim \frac{H(r)}{(\alpha-\lambda)r^{\alpha}} \qquad \alpha > \lambda$$

5.8 $$\int_{r}^{\infty} \frac{D(r)}{r^{\alpha+1}} \sim \frac{D(r)}{(\alpha-\rho)r^{\alpha}} \qquad \alpha > \rho$$

as $r \to \infty$.

PROOF OF LEMMA 2. We have

5.9
$$\frac{d}{dr}(r^{\lambda(r)-\alpha}) = [\lambda(r)-\alpha+\lambda'(r)r\log r]r^{(\lambda(r)-\alpha-1)}$$
$$\sim (\lambda-\alpha)r^{\lambda(r)-\alpha-1} \qquad \text{by 1.3 and 1.4}$$

and so if $\alpha < \lambda$ we have by Lemma 1

$$\int_{r_0}^{r} \frac{H(r)}{r^{\alpha+1}}\,dr \sim \int_{r_0}^{r} \frac{r^{\lambda(r)-\alpha-1}}{r_0^{\lambda(r_0)}}\,dr$$
$$= \frac{r^{\lambda(r)-\alpha}}{(\lambda-\alpha)r_0^{\lambda(r_0)}} - \frac{r_0^{-\alpha}}{\lambda-\alpha} \qquad \text{by 5.9}$$
$$\sim \frac{H(r)}{(\lambda-\alpha)r^{\alpha}}$$

we omit the proofs of 5.6, 5.7 and 5.8, since they are similar to the proof of 5.5.

LEMMA 3.

5.9
$$H(kr) \sim k^{\lambda}H(r)$$

5.10
$$D(kr) \sim k^{\rho}D(r).$$

PROOF.

$$\log\frac{H(kr)}{H(r)} = \int_{r}^{kr} \frac{\lambda(r)}{r}\,dr \qquad \text{by 3.3}$$
$$= \lambda(r)\log r\,|_{r}^{kr} - \int_{r}^{kr} \lambda'(r)\log r\,dr$$
$$= \log(kr)^{\lambda(kr)} - \log(r)^{\lambda(r)} - o\left(\int_{r}^{kr} \frac{\lambda(r)}{r}\,dr\right) \qquad \text{by 1.4.}$$

Hence

$$[1+0(1)]\log\frac{H(kr)}{H(r)} = [1+o(1)]\int_{r}^{kr} \frac{\lambda(r)}{r}\,dr = \log\left[\frac{(kr)^{\lambda(kr)}}{r^{\lambda(r)}}\right]$$
$$\therefore \frac{H(kr)}{H(r)} \sim \frac{(kr)^{\lambda(kr)}}{r^{(r)}} \sim k^{\lambda} \qquad \text{by 1.8}$$

and the lemma is proved.

The proof of 5.10 is similar.

6

Proof of Theorem 1. Let

6.1 $$\limsup_{r\to\infty} \frac{T(r, f)}{H(r)} = \alpha < \infty$$

$$\therefore\ T(r, f) < (\alpha+\varepsilon)H(r) \qquad \text{for all } r \geqq r_0$$

$$\sim (\alpha+\varepsilon)\frac{r^{\lambda(r)}}{r_0^{\lambda(r_0)}} \qquad \text{by Lemma 1}$$

so that

$$\log T(r, f) < \log(\alpha+\varepsilon)+\lambda(r)\log r-\lambda(r_0)\log r_0$$

or

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \leqq \limsup_{r\to\infty} \lambda(r) = \lambda$$

i.e. $\rho \leqq \lambda$.

But $\rho \geqq \lambda$ always

6.2 Hence $\rho = \lambda$ if 6.1 is true, and function is of regular growth.

Since $r^{\lambda(r)} \leqq T(r, f) \leqq r^{\rho(r)}$ for all $r \geqq r_0$

$$\frac{T(r, f)}{D(r)} \geqq \frac{r^{\lambda(r)}}{D(r)}$$

$$\sim r^{\lambda(r)-\rho(r)} r_0^{\rho(r_0)}$$

by 5.2 of lemma 1

$$\therefore\ \liminf_{r\to\infty} \frac{T(r, f)}{D(r)} \geqq r_0^{\rho(r_0)} \liminf_{r\to\infty} [r^{\lambda(r)-\rho(r)}]$$

$$= r_0^{\rho(r_0)} \qquad (\text{since } [\lambda(r)-\rho(r)] \to \lambda-\rho = 0 \text{ by } 6.2)$$

$$> 0.$$

Conversely let

$$\liminf_{r\to\infty} \frac{T(r, f)}{D(r)} = \beta > 0$$

6.3 $$\therefore\ T(r, f) > (\beta-\varepsilon)D(r) \qquad \text{for all } r \geqq r_0$$

$$\sim (\beta-\varepsilon)\frac{r^{\rho(r)}}{r_0^{\rho(r_0)}} \qquad \text{by 5.2 of lemma 1}$$

so that

$$\log T(r, f) > \log(\beta-\varepsilon)+\rho(r)\log r-\rho(r_0)\log r_0$$

or

$$\liminf_{r\to\infty} \frac{\log T(r, f)}{\log r} \geqq \rho$$

i.e. $\lambda \geqq \rho$

But $\lambda \leqq \rho$ always

6.4 Hence $\lambda = \rho$ if 6.3 is true, and the function is of regular growth. Now

$$\frac{T(r, f)}{H(r)} \leqq \frac{r^{\rho(r)}}{H(r)} \qquad \text{by 2.5, for all } r \geqq r_0$$

$$\sim r^{\rho(r)-\lambda(r)} r_0^{\lambda(r_0)} \qquad \text{by 5.1 of lemma 1}$$

$$\therefore \limsup_{r\to\infty} \frac{T(r, f)}{H(r)} \leqq r_0^{\lambda(r_0)} \qquad \text{by 6.4}$$

$$< \infty$$

and the proof is complete.

Proof of theorem 2 follows from theorem 1 as corollary.

Now Theorem A and Theorem B, take the following form.

Theorem 3. *If $f(z)$ is a meromorphic function of regular growth and*

$$N(r) = o(r^{\rho(r)}) \text{ as } r \to \infty$$

Then for $x \neq a, b$

$$0 < \liminf_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}} \leqq \liminf_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \leqq 1.$$

Theorem 4. *If $f(z)$ is a meromorphic function of regular growth and*

$$N(r) = o(r^{\lambda(r)}) \qquad \text{as } r \to \infty.$$

Then for $x \neq a, b$

$$1 = \liminf_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \leqq \limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} < \infty.$$

Proof of Theorem 3 and 4.

These can be proved by application of same arguments as used in proving theorem 1 and then Theorem A and Theorem B.

7

Now we prove some more theorems on the distribution of a-points of $f(z)$ of order $\rho > 0$ and lower order $\lambda > 0$.

Theorem 5.

$$0 < \limsup_{r\to\infty} \frac{n(r)}{H(r)} < \infty \quad \textit{iff} \quad 0 < \limsup_{r\to\infty} \frac{N(r)}{H(r)} < \infty.$$

THEOREM 6.

$$\frac{n(r)}{H(r)} \to 0 \quad \textit{iff} \quad \frac{N(r)}{H(r)} \to 0.$$

THEOREM 7.

$$0 < \limsup_{r\to\infty} \frac{n(r)}{D(r)} < \infty \quad \textit{iff} \quad 0 < \limsup_{r\to\infty} \frac{N(r)}{D(r)} < \infty.$$

THEOREM 8.

$$\frac{n(r)}{D(r)} \to 0 \quad \textit{iff} \quad \frac{N(r)}{D(r)} \to 0.$$

PROOF OF THEOREM 5. Let

$$\limsup_{r\to\infty} \frac{n(r)}{H(r)} = \alpha \qquad 0 < \alpha < \infty.$$

Therefore

$$n(r) < (\alpha+\varepsilon)H(r) \qquad \text{for all } r \geqq r_0.$$

Now

$$\begin{aligned} N(r) &= A + \int_{r_0}^{r} \frac{n(r)}{r}\,dr \\ &< A + (\alpha+\varepsilon)\int_{r_0}^{r} \frac{H(r)}{r}\,dr \\ &\sim A + (\alpha+\varepsilon)\frac{H(r)}{\lambda} \qquad \text{by 5.5 of lemma 2.} \end{aligned}$$

Hence

$$\limsup_{r\to\infty} \frac{N(r)}{H(r)} \leqq \frac{\alpha}{\lambda}.$$

Conversely let

$$\limsup_{r\to\infty} \frac{N(r)}{H(r)} = \beta \qquad 0 < \beta < \infty$$

$$\therefore N(r) < (\beta+\varepsilon)H(r) \qquad \text{for all } r \geqq r_0.$$

Now

$$\begin{aligned} n(r)\log 2 &< \int_{r}^{2r} \frac{n(r)}{r}\,dr \\ &< N(2r) \\ &< (\beta+\varepsilon)H(2r) \quad \text{by} \\ &\sim (\beta+\varepsilon)2^{\lambda}H(r) \quad \text{by 5.9 of lemma 3.} \end{aligned}$$

Therefore

$$\limsup_{r\to\infty} \frac{n(r)}{H(r)} \leqq \frac{\beta 2^{\lambda}}{\log 2}$$

and the theorem is proved.

We omit the proof of theorems 6, 7, 8 since the proofs are similar to the proof of theorem 5.

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