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Laplace transform of functions satisfying the Lipschitz condition

by

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Introduction

In this paper we shall treat the Laplace transform of a function $\varphi(t)$, $L_I[\varphi, x]$ defined by

$$L_I[\varphi, x] \equiv f(x) = \int_0^\infty e^{-xt}\varphi(t)dt \tag{1.1}$$

where $\varphi(t) \in L_1(0, R)$ for all $R > 0$.

We shall show that the Jump Operator defined by

$$J[f, k; t] \equiv \frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(-\frac{k}{t}\right)^{k+1}\left[\frac{k+1}{k} f^{(k)}\left(\frac{k}{t}\right)+\frac{1}{t} f^{(k+1)}\left(\frac{k}{t}\right)\right] \tag{1.2}$$

(see also [2, p. 91] and [1, p. 369]) is $0(k^{-\gamma/2})k \to \infty$, $0 < \gamma \leqq 1$ whenever $\varphi(t)$ satisfies at t

$$\int_t^{t+h} [\varphi(t+y)-\varphi(t)]dy = 0(h^{1+\gamma}) \qquad h \to 0 \tag{1.3}$$

which is a generalization of Lipschitz condition of order α (see [4, p. 42]).

We shall prove that $f(x) = L_I[\varphi, x]$ and the asymptotic behavior $J[f, k; t] = 0(k^{-\frac{1}{2}})$ uniformly in some interval of t, implies that $\varphi(t)$ (or an equivalent in Lebesgue sense) satisfies the Lipschitz condition there.

Similar results can be achieved for the Laplace Stieltjes transform $LS_I[\alpha, x]$ which is defined by

$$LS_I[\alpha, x] \equiv f(x) = \int_0^\infty e^{-xt}\, d\,\alpha(t) \tag{1.4}$$

where $\alpha(t) \in B\cdot V\,[0, R]$ (is of bounded variation in $[0, R]$) for all $R \geqq 0$.

In the case of the Laplace-Stieltjes transform one should define $I[f, k; t]$ by

$$I[f,k; t] = \left(-\frac{e}{t}\right)^k f^{(k)}\left(\frac{k}{t}\right). \tag{1.5}$$

and results analogous to those for the Laplace transform will be achieved in section 4.

2. The behavior of the Jump operator

In this section we shall state and prove properties of $J[f, k; t]$ as a result of a Lipschitz condition on $\varphi(t)$.

Theorem 2.1. *Suppose* $f(x) = L_I[\varphi; x]$, $t > 0$ *and* $\gamma \geqq 0$. *Then:*

$$\text{(a)} \qquad \int_t^{t+h} [\varphi(t+y)-\varphi(t)]dy = O(h^{1+\gamma}) \qquad h \to 0$$

implies

$$J[f, k; t] = 0(k^{-\gamma/2}) \qquad k \to \infty. \tag{2.1}$$

$$\text{(b)} \qquad \int_t^{t+h} [\varphi(t+y)-\varphi(t)]dy = o(h^{1+\gamma}) \qquad h \to 0$$

implies

$$J[f, k; t] = 0(k^{-\gamma/2}) \qquad k \to \infty. \tag{2.2}$$

The effect of the Lipschitz condition in an interval on $J[f, k; t]$ is described by the following theorem.

Theorem 2.2. *Suppose* $f(x) \equiv L_I[\varphi, x]$ *and that for some real* a, b *satisfying* $0 \leqq a < b < \infty$ *there exist* $k > 0$, $\delta_1 > 0$ *and* γ, $0 \leqq \gamma \leqq 1$, *such that for each* t_1 *and* t_2, $a < t_1 \leqq t_2 < b$, *satisfying* $|t_1-t_2| < \delta_1$

$$|\varphi(t_1)-\varphi(t_2)| \leqq K|t_1-t_2|^{\gamma}. \tag{2.3}$$

Then there exist M *so that for* $k \geqq k_0$

$$|J[f, k, t]| \leqq K \cdot M \cdot k^{-\gamma/2} \tag{2.4}$$

uniformly for $t \in [c, d]$ $a < c < d < b$ *and in case* $a = 0$ (2.4) *will be valid for* $t \in (0\ d]$. M *depends on* c *and* d.

Remark 2.3. If we replace in assumption (2.3) of Theorem 2.2. K by ε the result will be for $k \geqq k_0$ $|J[f, k; t]| \leqq \varepsilon \cdot M_1 k^{-\gamma/2}$ for $0 \leqq \gamma < 1$. The result is not interesting here for $\gamma \geqq 1$ neither is Theorem 2.2 for $\gamma > 1$ since in these cases $\varphi(t)$ would be a constant in (a, b).

Remark 2.4. Assumption (1.3) uniformly in an interval implies (2.3) uniformly in that interval. Therefore the employment of (2.3) in Theorem 2.2 is not a restriction but a simplification of the notation. The implication in the opposite direction is trivial.

We shall prove Theorems 2.1 and 2.2 together since the skeleton of the proof is the same.

PROOF OF THEOREMS 2.1 AND 2.2.

One can first establish the following equality by the integral definition of the Γ-function

$$\frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(-\frac{k}{t}\right)^{k+1}\int_0^\infty e^{-k\,u/t}\left[\frac{k+1}{k}(-u)^k+t^{-1}(-u)^{k+1}\right]du \tag{2.5}$$
$$= \frac{(2\pi k)^{\frac{1}{2}}}{k!}\left[-\frac{k+1}{k}\,\Gamma(k+1)+\frac{1}{k}\,\Gamma(k+2)\right] = 0.$$

Using (2.5), (1.2) and the known formula

$$\frac{d^n}{dx^n}L_I[\varphi, x] = L_I[\psi, x] \quad \text{where} \quad \psi(t) = (-t)^n\varphi(t)$$

one may write

$$J[f, k; t] = J[f, k; t] - 0\cdot\varphi(t)$$
$$= \frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(-\frac{k}{t}\right)^{k+1}\left\{\int_0^{\max(t-\delta,0)} + \int_{\max(t-\delta,0)}^{t+\delta} + \int_{t+\delta}^{\infty}\right\}\cdot e^{-k\,u/t}$$
$$\left[\left(\frac{k+1}{k}\right)(-u)^k+t^{-1}(-u)^{k+1}\right](\varphi(u)-\varphi(t))\,du \equiv I_1+I_2+I_3.$$

(in case $\delta \geqq t_1$, which will occur in the choice $a = c = 0$, obviously $I_1 = 0$).

Since the Laplace transform converges at some real C and $\varphi(t)$ is bounded in $[c, d]$ or in $(0, d]$ in case $a = 0$ ($\varphi(t)$ is just a number for proving Theorem 2.1) we have $|\alpha_i(u, t, \delta)| \leqq M$ $i = 1, 2$ where

$$\alpha_1(u, t, \delta) \equiv \int_u^{t-\delta} e^{-Cv}[\varphi(v)-\varphi(t)]dv$$

and

$$\alpha_2(u, t, \delta) \equiv \int_{t+\delta}^{u} e^{-Cv}[\varphi(v)-\varphi(t)]dv.$$

Using the above we estimate I_1 in case $t-\delta > 0$.

$$I_1 = \frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(-\frac{k}{t}\right)^{k+1}$$
$$\cdot\int_0^{t-\delta}\frac{d}{du}\left\{e^{-k\,u/t}e^{Cu}\left[\frac{k+1}{k}(-u)^k+t^{-1}(-u)^{k+1}\right]\right\}\cdot\alpha_1(u, t, \delta)du.$$

For $k \geqq k_1(C)$ and for fixed C, $d/du\,\{\,\}$ is of fixed sign and therefore

$$|I_1| \leqq \frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(\frac{k}{t}\right)^{k+1} M$$

$$\cdot e^{-k(t-\delta)/t} e^{C(t-\delta)}\left[\left(1+\frac{1}{k}\right)(t-\delta)^k - t^{-1}(t-\delta)^{k+1}\right]$$

$$\leqq e^{C(t-\delta)}\cdot Mt^{-1}k\left\{e^{\delta/t}\left(1-\frac{\delta}{t}\right)\right\}^k\cdot (t-\delta)^{k+1}.$$

This implies for fixed C, $\delta < 1/2c$ and $k \geqq k_2 > k_1(C)$ that $|I_1| \leqq q^k$ for some $q < 1$ (when under the assumptions of Theorem 2.4, uniformly in $[c, d]$).

Similar considerations yield for fixed C and $k \geqq k_3$ the result $|I_3| \leqq q^k$, $q < 1$ (when under the assumptions of Theorem 2.2, uniformly in $[c, d]$ or if in addition $a = 0$, uniformly in $(0d]$).

We have to estimate I_2 and we do it first for Theorem 2.1, for which we define $\alpha(u) \equiv \int_t^u [\varphi(v)-\varphi(t)]dv$, and have therefore

$$I_2 = -\frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(-\frac{k}{t}\right)^{k+1}\int_{t-\delta}^{t+\delta}\frac{d}{du}\left\{e^{-ku/t}\left[\frac{k+1}{k}(-u)^k + t^{-1}(-u)^{k+1}\right]\right\}.$$

$\alpha(u)du + 0(q^k)$ $k \to \infty$ $q < 1$ fixed for any fixed choice of $\delta > 0$. We choose δ so that for $|u-t| < \delta$ in case (a) $|\alpha(u)| \leqq K|u-t|^{1+\gamma}$ and in case (b) $|\alpha(u)| \leqq \varepsilon|u-t|^{1+\gamma}$. A simple calculation shows that

$$\frac{d}{du}\left\{e^{-ku}\left(\frac{k+1}{k}(-u)^k + t^{-1}(-u)^{k+1}\right)\right\}$$

has changes of sign at

$$u = t\left(1+\frac{1}{k} \pm \sqrt{k^{-1}+k^{-2}}\right)$$

denoted by $u_1(k)$ and $u_2(k)$ $(u_1(k) < u_2(k))$. Therefore in case (a) (case (b) is similar) we have

$$|I_2| \leqq K\frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(\frac{k}{t}\right)^{k+1}\left\{\int_{t-\delta}^{u_1(k)} + \int_{u_1(k)}^{0} + \int_{0}^{u_2(k)} + \int_{u_2(k)}^{t+\delta}\right\}$$

$$\left|\frac{d}{du}\left\{e^{-kv}\left(\frac{k+1}{k}(-u)^k + t^{-1}(-u)^{k+1}\right)\right\}\right| |u-t|^{1+\gamma}du$$

$$\equiv I_{21}+I_{22}+I_{23}+I_{24}.$$

We shall estimate I_{24} (the rest are similar or easier)

$$
\begin{aligned}
|I_{24}| \leqq K(1+\gamma)\, &\frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(\frac{k}{t}\right)^{k+1}\frac{1}{t}\int_{u_2(k)}^{t+\delta} e^{-ku/t}u^k(u-t)^\gamma \\
&\cdot\left[(u-t)+\frac{1}{k}\,t\right]du \\
&+K\,\frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(\frac{k}{t}\right)^{k+1}\{e^{-k(t+\delta)}(t+\delta)^k\}\,\delta^{1+\gamma}\left(\frac{\delta}{t}+\frac{1}{k}\right) \\
&+K\,\frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(\frac{k}{t}\right)^{k+1}\left(e^{-u_2(k)t/}u_2(k)\right)^k\left(u_2(k)-t\right)^{1+\gamma} \\
&\cdot\left(\frac{u_2(k)}{t}-\frac{k+1}{k}\right) \equiv J_1+J_2+J_3.
\end{aligned}
$$

Since $ze^{-z+1} \leqq 1$ and therefore $e^k(e^{-u_2(k)/t}(u_2(k)/t))^k \leqq 1$; also $u_2(k)-t < 2t\,k^{-\frac{1}{2}}$ and

$$
\frac{u_2(k)}{t}-\frac{k+1}{k} < 2k^{-\frac{1}{2}} \quad \text{for} \quad k > k_4;
$$

so we have

$$
J_3 \leqq 8K\,t^\gamma\,k^{-\gamma/2}.
$$

By a method already employed here $J_2 \leqq q^k$ where $q < 1$.

Let us estimate J_1 now

$$
\begin{aligned}
J_1 &\leqq K(1+\gamma)\,\frac{(2\pi k)^{\frac{1}{2}}}{k!}\left(\frac{k}{t}\right)^{k+1}\left\{\frac{1}{t}\int_t^{t+\delta} e^{-ku/t}u^k(u-t)^{1+\gamma}du\right. \\
&\qquad\left.+\frac{1}{k}\int_t^{t+\delta} e^{-ku/t}u^k(u-t)^\gamma\,du\right\} \\
&= (1+\eta(k))K(1+\gamma)t^\gamma k\left\{\int_1^{1+(\delta/t)} e^{-k(z-1)}z^k(z-1)^{1+\gamma}dz\right. \\
&\qquad\left.+\frac{1}{k}\int_1^{1+(\delta/t)} e^{-k(z-1)}z^k(z-1)^\gamma\,dz\right\}
\end{aligned}
$$

where $\eta(k) = o(1)$ $k \to \infty$.

Since $z\,e^{-(z-1)} \leqq e^{-(z-1)^2/4}$ in $0 \leqq z \leqq 2$ which is derived from the fact that in $0 \leqq z \leqq 2$ the only relative maximum of $z\exp\left(-(z-1)+(z-1)^2/4\right)$ is at $z = 1$) we have

$$J_1 \leqq (1+\eta(k))K(1+\gamma)t^\gamma k \left\{ \int_1^{\min(2,1+(\delta/t))} \exp(-k(z-1)^2/4) \right.$$

$$\cdot \left\{ (z-1)^{1+\gamma} + \frac{1}{k}(z-1)^\gamma \right\} dz$$

$$\left. + \int_{\min(2,1+(\delta/t))}^{1+(\delta/t)} e^{-k(z-1)} z^k \left\{ (z-1)^{1+\gamma} + \frac{1}{k}(z-1)^\gamma \right\} dz \right\} \leqq (1+\eta(k))$$

$$\cdot k^{-\gamma/2} K(1+\gamma) t^{\gamma/2} \int_0^\infty e^{-v^2/4} \{v^{1+\gamma} + k^{-\frac{1}{2}} v^\gamma\} dv + q^k$$

$$\leqq 2K(1+\gamma)t^\gamma k^{-\gamma/2} \cdot L.$$

Where $L = \int_0^\infty e^{-v^2/4} v^{1+\gamma} dv$ which concludes the proof of Theorem 2.1.

Estimating I_2 for Theorem 2.2 we choose as δ

$$\delta = \min(\delta_1, c-a, d-b)$$

or in the case when $a = 0$ $\delta = \min(\delta_1, d-b)$ where δ_1 is defined in Theorem 2.2.

$$|I_2| \leqq \left| K \frac{(2\pi k)^{\frac{1}{2}}}{k!} \left(\frac{k}{t}\right)^{k+1} \left\{ \int_{\max(t-\delta,0)}^t + \int_t^{t+\delta} \right\} \right.$$

$$\left. \cdot e^{-ku/t} \left(\frac{k+1}{k}(u)^k + t^{-1}(-u)^{k+1} (\varphi(u) - \varphi(t)) \right) du \right|$$

$$\leqq K \frac{(2\pi k)^{\frac{1}{2}}}{k!} \left(\frac{k}{t}\right)^{k+1} \left(\frac{k+1}{k}\right) \left\{ \int_{\max(t-\delta,0)}^t e^{-ku/t} u^k (t-u)^{1+\delta} du \right.$$

$$\left. + \frac{1}{t} \int_t^{t+\delta} e^{-ku/t} u^{k+1} (u-t)^{1+\delta} du \right\}.$$

Following similar calculations to those used in estimating J_3 we conclude the proof. Q.E.D.

3. The behaviour of the determining function and a representation theorem

For the Laplace transform $f(x) = L_I[\varphi, x]\varphi$ is called the determining function and $f(x)$ the generating function of the transform. (See also [3]).

The following is the main result of the section.

THEOREM 3.1. *Suppose* $f(x) = L_I[\varphi, x]$ *and let*

(3.1) $\quad |J[f, k; t]| \leqq K k^{-\frac{1}{2}}$ *for* $t \in (a, b)$ *for some* $K > 0$

then there exists a function $\psi(t)$ *which is equal to* $\varphi(t)$ *in* $L_1[a, b]$ *norm such that for every* $\delta > 0$ *there exists a* K *that satisfies*

$$(3.2) \qquad |\psi(t_1)-\psi(t_2)| \leqq K\left(\frac{1}{2\pi}\right)^{\frac{1}{2}} |t_1-t_2| \cdot \min\left(\frac{2}{t_1}, \frac{2}{t_2}\right)$$

$$t_i \in (a, b) \; i = 1, 2.$$

We shall need for the proof of Theorem 3.1 the following representation theorem for the Laplace transform (which I have not been able to find in this form) which is a corollary of some well known results.

For the result one has to define the operator $L_{k,t}[f]$ as follows:

$$(3.3) \qquad L_{k,t}[f] = \frac{(-1)^k}{k!}\left(\frac{k}{t}\right)^{k+1} f^{(k)}\left(\frac{k}{t}\right).$$

THEOREM 3.2. *Necessary and sufficient conditions for* $f(x)$ *to be a Laplace transform of* $\varphi(t)$ *with* $\varphi(t) \in L_1(0, R)$ *for all* $R > 0$ *which converges for* $x > C$ *are*:

(1) $$\int_{(k+1)/R}^{\infty} \frac{u^k}{k!} |f^{(k+1)}(u)| du < N_R$$

$$k = k_0(R, C)+1, k_0+2, \cdots \text{ for every } R > 0$$

(2) $$\lim_{\substack{j\to\infty \\ k\to\infty}} \int_0^R |L_{j,t}[f]-L_{k,t}[f]| dt = 0 \text{ for every } R > 0.$$

(3) *For each* $\varepsilon > 0$ *there exists an* M_ε *such that*

$$\left|\int_x^{\infty} \frac{u^k}{k!} f^{(k+1)}(u+C) du\right| \leqq M_\varepsilon \left(\frac{x}{x-\varepsilon}\right)^{k+1} \quad \textit{for } x > \varepsilon.$$

PROOF. This Theorem is a corollary of various Theorems proved by D. V. Widder. By a slight modification of Theorem 18.b of [3, pp. 322—324] conditions (1) and (3) are necessary and sufficient for $f(x)$ to be a Laplace Stieltjes transform of $\alpha(t)$ of bounded variation in every finite interval.

To prove sufficiency we observe that by Theorem 10 of [3, pp. 299—301] formula (1) is equivalent now to

$$\int_0^R |L_{k,t}[f]| dt \leqq N_R^t \text{ for all } R > 0.$$

By formula (2) we obtain that $L_{k,t}[f]$ is a Cauchy sequence in $L_1(0, R)$ and therefore there exists a function $\varphi(t)$ which is their limit and $\varphi(t) \in L_1(0, R)$, (for all $R > 0$).

By substitution we have

$$\int_0^v L_{k,t}[f]dt = \frac{(-1)^k}{k!}\int_0^v \left(\frac{k}{t}\right)^{k+1} f^{(k)}\left(\frac{k}{t}\right) dt$$
$$= \frac{(-1)^{k-1}}{(k-1)!}\int_0^\infty u^{k-1} f^{(k)}(u)du$$

and therefore $\alpha(t) = \lim_{k\to\infty} \int_0^t L_{k,v}[f]dv = \int_0^t \varphi(v)dv$.

For all finite N, $x > \varepsilon > 0$ one has by the proof of Theorem 8.b of [3. p. 322]

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t) = \lim_{k\to\infty} x\int_0^\infty e^{-xt}\left\{\int_0^t L_{k,v}[f]dv\right\} dt$$
$$= \lim_{k\to\infty} x\int_0^N e^{-xt}\left\{\int_0^t L_{k,v}[f]dv\right\} dt + \lim_{k\to\infty} x\int_N^\infty e^{-xt}\left\{\int_0^t L_{k,v}[f]dv\right\} dt$$
$$= \lim_{k\to\infty} \int_0^N e^{-xt}L_{k,t}[f]dt + \lim_{k\to\infty} e^{-xN}\int_0^N L_{k,v}[f]dv$$
$$+ \lim_{k\to\infty} x\int_N^\infty e^{-xt}\left\{\int_0^t L_{k,v}[f]dv\right\} dt = \int_0^N e^{-xt}\varphi(t)dt + 0(1) \quad N\to\infty$$

which concludes the proof of sufficiency.

To prove necessity one has to show only that (2) is necessary. We estimate $||L_{k,t}[f]-\varphi(t)||_{L_1(0,R)}$ as follows

$$\int_0^R |L_{k,t}[f]-\varphi(t)|\,dt \leqq \frac{k^{k+1}}{k!}\int_0^R dt\left\{\int_0^2 + \int_2^\infty\right\} e^{-ku}u^k|\varphi(tu)-\varphi(t)|\,du$$
$$\leqq \frac{k^{k+1}}{k!}\int_0^2 e^{-ku}u^k\left\{\int_0^R |\varphi(tu)-\varphi(t)|\,dt\right\} du + R\max_{0<t<R} \frac{k^{k+1}}{k!}\int_2^\infty e^{-ku}u^k$$
$$\cdot\, |\varphi(tu)-\varphi(t)|du.$$

While the second term is obviously tending to zero (as q^k $q < 1$) the first term tends also to zero as can be proved by repeating almost verbatim the proof of Theorem 17 (a) of [3, p. 318].

Q.E.D.

Proof of Theorem 3.1. By Theorem 3.2 $L_{k,t}[f]$ converges to $\varphi(t)$ in $L_1(0, R)$ (for every $R > 0$) and therefore in $L_1(a, b)$ and therefore in measure on (a, b). This implies that there exists a subsequence of $L_{k,t}[f]$, say $L_{k(i),t}[f]$, that converges to $\varphi(t)$ almost everywhere in (a, b). We shall prove that the sequence $L_{k(i),t}[f]$ converges pointwise in (a, b) to $\psi(t)$ which obviously is equal to $\varphi(t)$ in $L_1(a, b)$. Suppose at $t_0 L_{k(i),t}[f]$ does not converge, then since $L_{k(i),t}[f]$ converges to $\varphi(t)$ a.e. in (a, b) there exists a sequence

$t_j \varepsilon (a, b)$, $t_j \to t_0$ where $\lim_{j\to\infty} t_j = t_0$ and the limits $\lim_{i\to\infty} L_{k(i), t_j}[f]$ exist. We shall prove $\lim_{j\to\infty} \lim_{i\to\infty} L_{k(i), t_j}[f]$ exists and is equal to $\lim_{i\to\infty} L_{k(i), t_0}[f]$ (which also exists). To show that $\lim_{i\to\infty} L_{k(i), t_j}[f]$ is Cauchy sequence in j we use the mean value theorem as follows

(3.3)

$$L_{k, t_{j(1)}}[f] - L_{k, t_{j(2)}}[f] = (t_{j(1)} - t_{j(2)}) \left\{ \frac{(-1)^{k+1}}{k!} \left(\frac{k}{\zeta}\right)^{k+1} \cdot \left[\frac{k+1}{k} f^{(k)}\left(\frac{k}{\zeta}\right) + \frac{k}{\zeta^2} f^{k+1}\left(\frac{k}{\zeta}\right) \right] \right\} = (t_{j(1)} - t_{j(2)}) \left(\frac{k}{2\pi}\right)^{\frac{1}{2}} \frac{1}{\zeta} J[f, k, \zeta]$$

where ζ is between $t_{j(1)}$ and $t_{j(2)}$.

Formula (3.3) is valid for every k and therefore using (3.1) we obtain

$$(3.4) \qquad |L_{k(i), t_{j(1)}}[f] - L_{k(i), t_{j(2)}}[f]| \leqq K |t_{j(1)} - t_{j(2)}| \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \frac{2}{t_0}$$

and therefore

$$(3.5) \qquad |\lim_{i\to\infty} L_{k(i) t_{j(1)}}[f] - \lim_{i\to\infty} L_{k(i) t_{j(2)}}[f]| \leqq K |t_{j(1)} - t_{j(2)}| \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \frac{2}{t_0}.$$

Inequality (3.5) implies that $\lim_{i\to\infty} L_{k_{(i)} t_j}[f]$ is a Cauchy sequence in j.

Define $\lim_{j\to\infty} \lim_{i\to\infty} L_{k_{(i)}, t_j}[f] = A$. Since one can replace $t_{j(1)}$ by t_j and $t_{j(2)}$ by t_0 in (3.4) it is easy to see that $\lim_{i\to\infty} L_{k(i), t_0}[f] = A$. Now every point in (a, b) can replace $t_{j(i)}$ and $t_{j(2)}$ in (3.4) and (3.5) and therefore (3.2). Q.E.D.

Remark 3.3. A combination of Theorems 2.2, 3.1 and 3.2 form a kind of representation theorem for functions satisfying Lipschitz conditions.

4. The Laplace Stieltjes transform

Analoguos to the theorems of sections 2 and 3 for the Laplace Stieltjes transform can also be obtained.

Theorem 4.1. *Suppose* $f(s) = LS_I[\alpha, x]$ *and for some real* a, b *satisfying* $0 \leqq a \leqq b \leqq \infty$ *there exists a* δ, $0 \leqq \delta \leqq 1$, $K > 0$ *and* $\delta_1 > 0$ *such that for each* t_1 *and* t_2, $a < t_1 \leqq t_2 < b$ *satisfying* $|t_1 - t_2| < \delta_1$

$$(4.1) \qquad |\alpha(t_1) - \alpha(t_2)| < K |t_1 - t_2|^\gamma$$

Then there exists an M *so that for* $k \geqq k_0$

$$|I[f, k, t]| \leqq K \cdot M \cdot k^{-\gamma/2} \tag{4.2}$$

uniformly for $t \in [c, d]$ $a < c < d < b$.

THEOREM 4.2. *Suppose* $f(x) = LS_I[\alpha, x]$ *and let for some* $K > 0$

$$|I[f, k; t]| \leqq Kk^{-\frac{1}{2}} \text{ for } t \in (a, b) \ 0 < a < b < \infty, \tag{4.3}$$

then $\beta(t)$ *satisfies for some* $M < 0$

$$|\alpha(t_1) - \alpha(t_2)| < KM|t_1 - t_2| \quad t_i \in (a, b) \ i = 1, 2 \tag{4.4}$$

The proof of Theorem 4.1 is similar to that of Theorem 2.2 but a little simpler. The proof of Theorem 4.2 is similar to part of the proof of Theorem 3.1 using here Theorem 8.b of [3, p. 322] instead of Theorem 3.2 there.

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