

COMPOSITIO MATHEMATICA

A. ANDREOTTI

G. TOMASSINI

A remark on the vanishing of certain cohomology groups

Compositio Mathematica, tome 21, n° 4 (1969), p. 417-430

http://www.numdam.org/item?id=CM_1969__21_4_417_0

© Foundation Compositio Mathematica, 1969, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

A remark on the vanishing of certain cohomology groups

by

A. Andreotti and G. Tomassini

For a q -pseudoconvex space, cohomology with values in a coherent sheaf is finite dimensional above dimension q . Analogous results are valid for q -pseudoconcave spaces.

In this note we show that in some favorable instances these finite dimensional groups are actually zero. We prove that if \mathcal{F} is a given coherent sheaf on the space X and F is a holomorphic line bundle having a certain strong type of convexity, then $H^r(X, \mathcal{F} \otimes \Omega(F^k)) = 0$ for large values of k , and for r in the expected range given by the convexity or concavity of X .

The idea of the proof is that used for the same purpose in [1] and this note can be considered as a straightforward application of the method of [1] and there explicitly carried out for the case when the base space X is a compact manifold.

These theorems are of the same nature as the vanishing theorem of Serre ([3] n. 74 théorème 1).

1. Preliminaries

a). Let X be a (reduced) complex space and let $\pi : F \rightarrow X$ be a holomorphic line bundle over X .

Let us consider a hermitian metric on the fibres of F . If F is trivializable on the open sets of the covering $U = \{U_i\}$ of X , $F|_{U_i} \cong U_i \times \mathbf{C}$ and if $g_{ij} : U_i \cap U_j \rightarrow \mathbf{C}$ are the corresponding transition functions for F , then the hermitian metric is locally given by C^∞ functions $h_i : U_i \rightarrow \mathbf{R}$, $h_i > 0$, such that

$$h_i(z) = |g_{ji}(z)|^2 h_j(z).$$

If $v \in \pi^{-1}(U_i)$ has base coordinate $\pi(v) = z$ and fibre coordinate $\xi_i \in \mathbf{C}$ then

$$\|v\|^2 = h_i(z) |\xi_i|^2 = \chi(z, \xi)$$

is the length of v in the metric we have considered.

Let F^* be the dual bundle of F . On the same covering U , F^* is represented by the transition functions $\{g_{ij}^{-1}\}$. Given the hermitian metric $\{h_i\}$ on the fibres of F , a hermitian metric on the fibres of F^* is given by the collection of local functions $\{h_i^{-1}\}$. If $v^* \in \pi^{-1}(U_i)$ has base coordinate z and fibre coordinate $\eta_i \in \mathbb{C}$ then

$$\|v^*\|^2 = h_i^{-1}(z)|\eta_i|^2 = \chi_*(z, \eta).$$

We will call the bundle F *metrically pseudoconvex* if a hermitian metric on the fibres of F can be chosen so that the function

$$\chi(z, \xi) = \|v\|^2$$

is strongly pseudoconvex outside of the 0-section of F .¹

If X is a manifold this is certainly the case if the exterior form

$$\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log h_i(z) \in c_{\mathbb{R}}^1(F)$$

corresponds to a hermitian positive definite quadratic form $\partial \bar{\partial} \log h_i(z)$.

We will call the line bundle F *metrically pseudoconcave* if the dual bundle F^* is metrically pseudoconvex.

EXAMPLE. Consider the line bundle F associated with the \mathbb{C}^* -principal bundle (the Hopf bundle)

$$\mathbb{C}^{n+1} - \{0\} \rightarrow P_n(\mathbb{C})$$

which serves as definition for the projective space $P_n(\mathbb{C})$. On the covering $U_i = \{z = z_0, \dots, z_n \in P_n(\mathbb{C}) : z_i \neq 0\}$ of $P_n(\mathbb{C})$ the transition functions are

$$g_{ij} = \frac{z_i}{z_j} \text{ in } U_i \cap U_j$$

so that F is isomorphic to the monoidal transform of \mathbb{C}^{n+1} with center 0 :

$$F = \{(z, t) \in \mathbb{C}^{n+1} \times P_n(\mathbb{C}) : z_i t_j = z_j t_i\}.$$

Clearly F is a metrically pseudoconvex bundle since we can take the euclidean metric on \mathbb{C}^{n+1} and lift it to F to get a metric on its fibres.

¹ I.e. for each point $z_0 \in X$ we can choose a neighborhood U of z_0 such that (i) $F|_U \cong U \times \mathbb{C}$, (ii) U is realized as an analytic subset of some open set $G \subset \mathbb{C}^N$ for some N , (iii) there exists a C^∞ function $\hat{h}(z) > 0$ on G such that $\hat{\chi} = \hat{h}(z)|\hat{\xi}|^2$, $\hat{\xi} \in \mathbb{C}$, is a C^∞ strongly Levi-convex function for $\hat{\xi} \neq 0$ and $\chi = \hat{\chi}|_{U \times \mathbb{C}}$.

If on the other hand we identify \mathbf{C}^{n+1} with the open set U_{n+1} of $\mathbf{P}_{n+1}(\mathbf{C})$ so that $\mathbf{0} = (0, \dots, 0, 1)$ and

$$\mathbf{P}_n(\mathbf{C}) = \{z \in \mathbf{P}_{n+1}(\mathbf{C}) : z_{n+1} = 0\}$$

then the dual bundle of F is isomorphic to

$$F^* = \mathbf{P}_{n+1}(\mathbf{C}) - \{\mathbf{0}\}$$

the projection being the projection from $\mathbf{0}$ to the hyperplane $\mathbf{P}_n(\mathbf{C})$.

2. Filtration of cohomology

a). Let A be an analytic subset of a domain of holomorphy U of \mathbf{C}^N .

LEMMA. *Every holomorphic function f on $A \times \mathbf{C}$ admits a power series expansion*

$$f = \sum_0^\infty c_s(z) \xi^s$$

uniformly convergent on compact sets, where the coefficients $c_s(z)$ are holomorphic on A . Such an expansion is unique.

PROOF. Consider $A \times \mathbf{C}$ as an analytic subset of $U \times \mathbf{C}$. Since $U \times \mathbf{C}$ is a domain of holomorphy, there exists a holomorphic function \hat{f} on $U \times \mathbf{C}$ such that $\hat{f}|_{A \times \mathbf{C}} = f$. For \hat{f} we have a power series expansion (uniformly convergent on compact sets)

$$\hat{f} = \sum_0^\infty \hat{c}_s(z) \xi^s \quad z \in U, \quad \xi \in \mathbf{C}$$

with holomorphic $\hat{c}_s(z)$. By restriction to $A \times \mathbf{C}$ we get an expansion for f .

To prove unicity assume that $f = \sum_0^\infty c'_s(z) \xi^s$ so that for every point $a \in A$

$$\sum_0^\infty \{c_s(a) - c'_s(a)\} \xi^s \equiv 0.$$

This implies that for every $s \in \mathbf{N}$, $c_s(a) = c'_s(a)$. Thus the conclusion follows.

Let $\hat{A} = A \times \mathbf{C}$ and define

$$\Gamma(\hat{A}, O_{\hat{A}})_k = \{f \in \Gamma(\hat{A}, O_{\hat{A}}) : f = \sum_0^\infty c_s(z) \xi^s, \\ c_0(z) = \dots = c_{k-1}(z) = 0\}$$

so that we get a filtration of $\Gamma(\hat{A}, O_{\hat{A}})$

$$\Gamma(\hat{A}, O_{\hat{A}}) = \Gamma(\hat{A}, O_{\hat{A}})_0 \supset \Gamma(\hat{A}, O_{\hat{A}})_1 \supset \cdots.$$

Note that

$$\mathcal{I} = \Gamma(\hat{A}, O_{\hat{A}})_1$$

is the ideal of holomorphic functions on \hat{A} vanishing on $A \times \{0\}$, and that the filtration considered is the \mathcal{I} -adic filtration

$$\Gamma(\hat{A}, O_{\hat{A}})_k = \mathcal{I}^k.$$

We have a natural isomorphism

$$j_k : \Gamma(A, O_A) \simeq \Gamma(\hat{A}, O_{\hat{A}})_k / \Gamma(\hat{A}, O_{\hat{A}})_{k+1}$$

given by

$$c(z) \rightarrow c(z)\xi^k.$$

Let \mathcal{F} be a coherent sheaf on A and set

$$\hat{\mathcal{F}} = \pi^* \mathcal{F} \otimes_{O_A} O_{\hat{A}}$$

where $\pi : \hat{A} \rightarrow A$ is the natural projection.

We can consider $\Gamma(\hat{A}, \hat{\mathcal{F}})$ as a $\Gamma(\hat{A}, O_{\hat{A}})$ -module and thus we can consider on $\Gamma(\hat{A}, \hat{\mathcal{F}})$ the induced filtration

$$\Gamma(\hat{A}, \hat{\mathcal{F}})_k = \mathcal{I}^k \Gamma(\hat{A}, \hat{\mathcal{F}}).$$

By the nature of the sheaf $\hat{\mathcal{F}}$ obtained by inverse image by π we get natural isomorphisms

$$j_k : \Gamma(A, \mathcal{F}) \simeq \Gamma(\hat{A}, \hat{\mathcal{F}})_k / \Gamma(\hat{A}, \hat{\mathcal{F}})_{k+1}.$$

Let us assume that we have on A a finite presentation of \mathcal{F}

$$O_A^l \xrightarrow{\tau} \mathcal{F} \rightarrow 0$$

so that $\Gamma(A, O_A)^l \rightarrow \Gamma(A, \mathcal{F})$ is surjective (since A is Stein).

Let s_1, \dots, s_l be the generators of $\Gamma(A, \mathcal{F})$, the images by τ of the sections $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ of $\Gamma(A, O_A)^l$. By tensoring over O_A by $O_{\hat{A}}$ we get then the exact sequence

$$O_{\hat{A}}^l \xrightarrow{\hat{\tau}} \hat{\mathcal{F}} \rightarrow 0$$

and correspondingly a surjective map $\Gamma(\hat{A}, O_{\hat{A}})^l \rightarrow \Gamma(\hat{A}, \hat{\mathcal{F}})$. If $\hat{s} \in \Gamma(\hat{A}, \hat{\mathcal{F}})$ then

$$\hat{s}(z, \xi) = \sum_{\mathbf{1}}^l \alpha_i(z, \xi) s_i(z).$$

One has thus

$$\Gamma(\hat{A}, \hat{\mathcal{F}})_k = \{s \in \Gamma(\hat{A}, \hat{\mathcal{F}}) : s = \sum_1^l \alpha_i s_i, \alpha_i \in \Gamma(\hat{A}, O_{\hat{A}})_k, 1 \leq i \leq l\}.$$

b). Let $U = \{U_i\}_{i \in I}$ be a covering of the space X by open sets U_i which are holomorphically complete and such that $F|_{U_i}$ is trivial for each $i \in I$.

Let $\hat{U}_i = \pi^{-1}(U_i) \cong U_i \times \mathbb{C}$. Then $\hat{U} = \{\hat{U}_i\}_{i \in I}$ is a Stein covering of the bundle space F .

Let \mathcal{F} be a coherent sheaf on X and let $\hat{\mathcal{F}} = \pi^* \mathcal{F} \otimes_{O_X} O_F$ be the inverse image sheaf on F .

We consider the cochain groups

$$C^r(\hat{U}, \hat{\mathcal{F}}) = \prod_{(i_0 \dots i_r)} \Gamma(\hat{U}_{i_0} \cap \dots \cap \hat{U}_{i_r}, \hat{\mathcal{F}}).$$

On each of the factors we have defined a filtration. This induces a filtration

$$C^r(\hat{U}, \hat{\mathcal{F}}) = C^r(\hat{U}, \hat{\mathcal{F}})_0 \supset C^r(\hat{U}, \hat{\mathcal{F}})_1 \supset \dots$$

of the cochain group which is compatible with coboundary operator δ .

If $\mathcal{F} = O$ we get a split exact sequence for every $k \geq 1$

$$0 \rightarrow C^r(\hat{U}, O)_{k-1} \rightarrow C^r(\hat{U}, O)_k \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\sigma} \end{matrix} C^r(U, \Omega(F^{*k})) \rightarrow 0$$

the map α being defined by

$$f_{i_0 \dots i_r} = \sum_{s=k}^{\infty} c_{i_0 \dots i_r, s}(z) \xi_{i_r}^s \rightarrow c_{i_0 \dots i_r, k}(z).$$

The splitting σ is obviously given as a left inverse of α by

$$c_{i_0 \dots i_r}(z) \rightarrow c_{i_0 \dots i_r}(z) \xi_{i_r}^k.$$

For a general sheaf \mathcal{F} on X we thus have analogously split exact sequences

$$0 \rightarrow C^r(\hat{U}, \hat{\mathcal{F}})_{k-1} \rightarrow C^r(\hat{U}, \hat{\mathcal{F}})_k \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\sigma} \end{matrix} C^r(U, \mathcal{F} \otimes \Omega(F^{*k})) \rightarrow 0.$$

The homomorphisms here considered are homomorphisms of the cochain complexes.

The filtration on the cochain complexes induce a filtration on the cohomology groups

$$H^r(F, \hat{\mathcal{F}}) = H^r(F, \hat{\mathcal{F}})_0 \supset H^r(F, \hat{\mathcal{F}})_1 \supset \dots$$

where $H^r(F, \hat{\mathcal{F}})_k$ is the image of the cocycles of $C^r(\hat{U}, \hat{\mathcal{F}})_k$. From the above remark it follows that $H^r(F, \hat{\mathcal{F}})_k$ is also the r -th cohomology group of the cochain complex $\{C^*(\hat{U}, \hat{\mathcal{F}})_k, \delta\}$.

We thus have split exact sequences

$$0 \rightarrow H^r(F, \hat{\mathcal{F}})_{k-1} \rightarrow H^r(F, \hat{\mathcal{F}})_k \begin{matrix} \xrightarrow{\alpha_*} \\ \xleftarrow{\sigma_*} \end{matrix} H^r(X, \mathcal{F} \otimes \Omega(F^{**k})) \rightarrow 0.$$

In particular we get the following conclusions:

(i) for any coherent sheaf \mathcal{F} on X the graded group associated to the filtration of $H^r(F, \hat{\mathcal{F}})$ is

$$GH^r(F, \hat{\mathcal{F}}) \simeq \coprod_{k=0}^{\infty} H^r(X, \mathcal{F} \otimes \Omega(F^{**k}))$$

(ii) we have a natural injection

$$\coprod_{k=0}^{\infty} H^r(X, \mathcal{F} \otimes \Omega(F^{**k})) \rightarrow H^r(F, \hat{\mathcal{F}})$$

so that

$$\dim_{\mathbb{C}} GH^r(F, \hat{\mathcal{F}}) \leq \dim_{\mathbb{C}} H^r(F, \hat{\mathcal{F}}).$$

3. Compact base space (cf. [1])

If X is compact then

(i) if F is metrically pseudoconcave, given any coherent sheaf \mathcal{F} on X there exists an integer $k_0 = k_0(\mathcal{F}, F)$ such that

$$H^r(X, \mathcal{F} \otimes \Omega(F^k)) = 0$$

for $k \geq k_0$ and any $r > 0$

(ii) if F is metrically pseudoconvex, given any coherent sheaf \mathcal{F} on X there exists an integer $k_0 = k_0(\mathcal{F}, F)$ such that

$$H^r(X, \mathcal{F} \otimes \Omega(F^k)) = 0$$

for $k \geq k_0$ and $r < \text{prof}(\mathcal{F})$.

In case (i) the bundle space of F^* is a strongly 0-pseudoconvex space. Thus by [1]

$$\dim_{\mathbb{C}} H^r(F^*, \hat{\mathcal{F}}) < +\infty$$

for $r > 0$. From the end of the previous section we derive the conclusion.

In case (ii) the bundle space F^* is a strongly 0-pseudoconcave space. Moreover we have

$$\text{prof}(\hat{\mathcal{F}}) = \text{prof}(\mathcal{F}) + 1.$$

The conclusion then follows from the finiteness theorem [1]:

$$\dim_{\mathbf{C}} H^r(F^*, \hat{\mathcal{F}}) < +\infty$$

for $r < \text{prof}(\hat{\mathcal{F}}) - 1$.

4. Pseudoconvex base space

a). For a C^∞ function on a complex manifold we adopt the notion of "strongly q -pseudoconvex" as given in [1] or [2].

As usual this notion is carried to complex spaces as follows.

A function $\phi : X \rightarrow \mathbf{R}$ will be called *strongly q -pseudoconvex* at a point $x_0 \in X$ if we can find

(i) an embedding τ of a neighborhood V of $x_0 \in X$ into an analytic set of some open neighborhood U of the origin of the Zariski tangent space T_{x_0} to X at x_0

(ii) a strongly q -pseudoconvex function $\phi : U \rightarrow \mathbf{R}$ such that

$$(a_1) \quad \phi|_V = \phi \circ \tau$$

$$(a_2) \quad \overline{\{x \in V : \phi(x) < c\}} = \{x \in V : \phi(x) \leq c\} \text{ for every } c \in \mathbf{R}.$$

A complex space X is called *strongly q -pseudoconvex* if we can find a compact set $K \subset X$ and a C^∞ function $\phi : X \rightarrow \mathbf{R}$ such that

(i) for every $c < \sup_X \phi$ the sets

$$B_c = \{x \in X : \phi(x) < c\}$$

are relatively compact

(ii) on $X - K$, ϕ is strongly q -pseudoconvex.

For a strongly q -pseudoconvex space one has the following theorem of finiteness for cohomology (cf. [1]): for any coherent sheaf \mathcal{F} on X

$$\dim_{\mathbf{C}} H^r(X, \mathcal{F}) < +\infty$$

if $r > q$.

b). **PROPOSITION 1.** *The bundle space of a metrically pseudoconvex line bundle F over a strongly q -pseudoconvex space X is a strongly q -pseudoconvex space.*

PROOF. With the notations introduced we consider on F the function

$$\Theta = \pi^* \phi + \mu(\chi)$$

where μ is a C^∞ function defined for $0 \leq t < +\infty$ such that

$$\mu(t) \geq 0, \quad \mu'(t) > 0, \quad \mu''(t) \geq 0$$

(i.e. positive, increasing, convex).

Without loss of generality we may assume $\phi \geq 0$ and $\sup_X \phi = +\infty$.

For every $c \in \mathbf{R}$ we have

$$B_c = \{v \in F : \Theta(v) < c\} \\ \subset \pi^{-1}\{x \in X : \phi(x) < c\} \cap \{v \in F : \chi(v) < \mu^{-1}(c)\}.$$

Since $\lim_{t \rightarrow +\infty} \mu(t) = +\infty$ and since π restricted on the set $\{v \in F : \chi(v) \leq \text{const}\}$ is a proper map, it follows that the sets B_c are relatively compact.

Let U be an open set in X . If U is sufficiently small we may assume that

$$(i) \quad F|_U \cong U \times \mathbf{C}$$

(ii) U is an analytic subset of some open set $G \subset \mathbf{C}^N$ and there exists a C^∞ strongly q -pseudoconvex function $\hat{\phi} : G \rightarrow \mathbf{R}$ such that $\phi = \hat{\phi}|_U$

(iii) $\chi|_{U \times \mathbf{C}} = h(z)|\xi|^2$ and exists a C^∞ function \hat{h} on G such that $\hat{\chi} = \hat{h}(z)|\xi|^2$ is strongly 0-pseudoconvex on $G \times \mathbf{C}$ ($z \in G$, $\xi \in \mathbf{C}$, $\xi \neq 0$).

Let $G' \Subset G$; there exists a constant $c = c(G')$ such that if $\mu' > c$ the function

$$\hat{\Theta} = \hat{\phi} + \mu(\hat{\chi})$$

is strongly 0-pseudoconvex on the set

$$\{(z, \xi) \in G' \times \mathbf{C} : \hat{\chi} \geq 1\}.$$

This is straightforward verification.

Making use of this remark we may select μ in such a way that on

$$\pi^{-1}(K) \cap \{v \in F : \chi(v) \geq 1\}$$

the function Θ be strongly 0-pseudoconvex.

On each point of F outside of the compact set

$$C = \{v \in F : \chi(v) \leq 1\} \cap \pi^{-1}(K)$$

the function Θ is strongly q -pseudoconvex. This follows from the fact that everywhere on F (including the 0-section) the Levi form $\mathcal{L}(\hat{\chi})$ has one positive eigenvalue in the direction of the fibre.

Finally the verification that $\hat{\Theta}$ has the property (a₂) stated in a) follows from the fact that $\hat{\chi}$ is an open map on each fibre.

Arguing now as we did for the case where X is compact and using the theorem of finiteness quoted before for the bundle space of F , we get the following

THEOREM 1. *Let X be a strongly q -pseudoconvex space. Let F be a metrically pseudoconcave line bundle on X and let \mathcal{F} be a coherent sheaf on X . There exists an integer $k_0 = k(\mathcal{F}, F)$ such that*

$$H^r(X, \mathcal{F} \otimes \Omega(F^k)) = 0$$

for $k \geq k_0$ and $r > q$.

5. Pseudoconcave base space

a). A complex space X is called *strongly q -pseudoconcave* if we can find a compact set $K \subset X$ and a C^∞ function $\phi : X \rightarrow \mathbf{R}$ such that

(i) for any $c > \inf_X \phi$ the sets

$$B_c = \{x \in X : \phi(x) > c\}$$

are relatively compact in X ,

(ii) for each point $x_0 \in X - K$ we can find a positive constant $k(x_0)$ such that $e^{k\phi}$ is strongly q -pseudoconvex for $k \geq k(x_0)$ in a neighborhood of x_0 .

As in [1] one establishes for strongly q -pseudoconcave spaces the following theorem of finiteness for cohomology: for any coherent sheaf \mathcal{F} on X one has

$$\dim_c H^r(X, \mathcal{F}) < +\infty$$

for $r < \text{prof}(\mathcal{F}) - q - 1$.

b). **PROPOSITION 2.** *The bundle space of a metrically pseudoconcave line bundle F over a strongly q -pseudoconcave space X is strongly $(q+1)$ -pseudoconcave space.*

PROOF. By the hypothesis we can choose on F^* a metric on the fibres such that the function

$$\|v^*\|^2 = \chi_*(z, \eta) = h(z)|\eta|^2$$

is 0-pseudoconvex outside of the 0-section.

On F we can consider the function

$$\|v\|^{-2} = h(z)|\xi|^{-2}$$

where ξ is the fibre coordinate on F .

Let $\mu(t)$ be a C^∞ function defined for $0 < t < +\infty$ such that

$\mu(t) > 0$, $\mu(t) = t$ if $0 < t < \frac{1}{2}$, $\mu(t) = \frac{1}{2}$ if $t \geq 1$, $\mu'(t) \geq 0$, and set $g = \mu(|v|^{-2})$.

On the bundle space F we then define the function

$$\Theta = \frac{\pi^*(\phi) \cdot g}{e^{\pi^*(\phi)+g}}$$

It is no loss of generality to assume that $\inf_X \phi = 0$ and $\phi \leq \frac{1}{2}$ on X .

First we remark that $e^{\pi^*(\phi)+g} > \sup \{\pi^*(\phi), g\}$ so that

$$\Theta < \frac{\pi^*(\phi) \cdot g}{\sup \{\pi^*(\phi), g\}} = \inf \{\pi^*(\phi), g\}.$$

Therefore the sets

$$\{v \in F : \Theta(v) > c\}$$

are relatively compact for $c > 0$.

For U sufficiently small on X we may make the assumptions (i), (ii), (iii) given in the proof of proposition 1. We thus have to compute the pseudoconvexity of the function

$$\hat{\Theta} = \frac{\hat{\phi}\hat{g}}{e^{\hat{\phi}+\hat{g}}}$$

where $\hat{g} = \mu(\hat{h}(z)|\xi|^{-2})$.

Writing $\hat{\Theta} = \exp \log \hat{\Theta}$ we get for the Levi form the expression

$$\begin{aligned} \mathcal{L}(\hat{\Theta}) = \hat{\Theta} \left\{ |\partial \log \hat{\Theta}|^2 + \left(\frac{1}{\hat{\phi}} - 1\right) \partial \bar{\partial} \hat{\phi} + \left(\frac{1}{\hat{g}} - 1\right) \partial \bar{\partial} \hat{g} \right. \\ \left. - \frac{1}{\hat{\phi}^2} |\partial \hat{\phi}|^2 - \frac{1}{\hat{g}^2} |\partial \hat{g}|^2 \right\}. \end{aligned}$$

Let $v = (z, \xi) \in \pi^{-1}(U)$ and restrict the Levi form to the directions at v for which

$$\partial \hat{\phi} = \partial \hat{g} = 0.$$

If ξ is sufficiently large, then the eigenvalues of $((1/\hat{g})-1) \partial \bar{\partial} \hat{g}$ will be prevalent and the Levi form will be positive definite. If on the other hand $U \cap K = \emptyset$, the Levi form of \hat{g} on $\partial \hat{g} = 0$ is positive so that we do keep the $n-q-1$ positive eigenvalues coming from $\hat{\phi}$.

In conclusion, we can find a compact set $C \in \pi^{-1}(K)$ on F such that for any point $v_0 \notin C$ for large values of $k \geq k(v_0)$ the function $e^{k\Theta}$ is strongly $(q+1)$ -pseudoconvex.

Arguing now as in the compact case we obtain the following

THEOREM 2. *Let X be a strongly q -pseudoconcave space and let F be a metrically pseudoconvex line bundle on X . Let \mathcal{F} be a coherent sheaf on X . There exists an integer $k_0 = k_0(\mathcal{F}, F)$ such that*

$$H^r(X, \mathcal{F} \otimes \Omega(F^k)) = 0$$

for $k \geq k_0$ and $r < \text{prof}(\mathcal{F}) - q - 1$.

6. An application

a). By a q -corona we mean a complex space X endowed with a C^∞ and positive function $\phi : X \rightarrow \mathbf{R}$ having the following properties

(i) for every $\varepsilon > 0$, $c > 0$ the sets

$$X_{\varepsilon, c} = \{x \in X : \varepsilon < \phi(x) < c\}$$

are relatively compact

(ii) ϕ is a strongly q -pseudoconvex function outside of some compact subset $K \subset X$.

For a q -corona one has the following theorem of finiteness for cohomology

THEOREM 3. *Let X be a q -corona and let \mathcal{F} be a coherent sheaf on X . Then*

$$\dim_{\mathbf{C}} H^r(X, \mathcal{F}) < +\infty$$

if $q < r < \text{prof}(\mathcal{F}) - q - 1$.

The proof of this theorem is obtained by the procedure of [1] combining the arguments for the pseudoconvex and pseudoconcave case.

Precisely one establishes (cf. [1] propositions 16 and 17) that given ε, c , with $\varepsilon < \inf_K \phi$ and $c > \sup_K \phi$ we can find $0 < \varepsilon' < \varepsilon$, $c' > c$ such that the restriction map

$$H^r(X_{\varepsilon', c'}, \mathcal{F}) \rightarrow H^r(X_{\varepsilon, c}, \mathcal{F})$$

is surjective if $q < r < \text{prof}(\mathcal{F}) - q - 1$.

This implies the finiteness of $\dim_{\mathbf{C}} H^r(X_{\varepsilon, c}, \mathcal{F})$ for

$$q < r < \text{prof}(\mathcal{F}) - q - 1.$$

Then one establishes the "Runge theorem" and obtains the complete statement of the theorem (cf. theorems 12 and 13 of [1]).

Can we establish a vanishing theorem on a q -corona for the

same ranges of r for which we have the theorem of finiteness? To this question we give a partial answer in what follows.

b). We consider the following situation: Z is a compact complex space, Y_1 and Y_2 are two open subsets of Z such that

- (i) $Z = Y_1 \cup Y_2$
- (ii) Y_1 is q -complete and Y_2 is q -pseudoconvex
- (iii) $X = Y_1 \cap Y_2$ is a q -corona.

THEOREM 4. *Let X, Y_1, Y_2, Z be as above, let \mathcal{F} be a coherent sheaf on Z , let F be a metrically pseudoconvex line bundle on Z . Then there exists an integer $k_0 = k_0(\mathcal{F}, F)$ such that*

$$H^r(X, \mathcal{F} \otimes \Omega(F^k)) = 0$$

for $k \geq k_0$ and $q < r < \text{prof}_Z(\mathcal{F}) - q - 1$.

PROOF. From the Mayer-Vietoris sequence we get the exact cohomology sequence

$$\begin{aligned} \cdots &\rightarrow H^r(Z, \mathcal{F} \otimes \Omega(F^k)) \\ &\rightarrow H^r(Y_1, \mathcal{F} \otimes \Omega(F^k)) \oplus H^r(Y_2, \mathcal{F} \otimes \Omega(F^k)) \\ &\rightarrow H^r(X, \mathcal{F} \otimes \Omega(F^k)) \rightarrow H^{r+1}(Z, \mathcal{F} \otimes \Omega(F^k)) \rightarrow \cdots \end{aligned}$$

If k is large by the theorem of *n. 3*

$$H^r(Z, \mathcal{F} \otimes \Omega(F^k)) = 0 = H^{r+1}(Z, \mathcal{F} \otimes \Omega(F^k))$$

if $r < \text{prof}_Z(\mathcal{F}) - 1$.

Since Y_1 is q -complete

$$H^r(Y_1, \mathcal{F} \otimes \Omega(F^k)) = 0$$

if $r > q$.

Therefore

$$H^r(X, \mathcal{F} \otimes \Omega(F^k)) \cong H^r(Y_2, \mathcal{F} \otimes \Omega(F^k)).$$

Now the second of these groups vanishes for large k if $r < \text{prof}_{Y_2}(\mathcal{F}) - q - 1$ according to theorem 2.

Collecting together all the above information we get the statement of theorem 4.

EXAMPLE. Let

$$X = \{z \in \mathbb{C}^n : a < \sum_1^n z_i \bar{z}_i < b, a < b\}.$$

It can be obtained as intersection of two open sets in $\mathbb{P}_n(\mathbb{C})$, a ball and the complement of a concentric ball. Let F be the Hopf

bundle. Since $F|_X$ is trivial we obtain for any coherent sheaf \mathcal{F} on $\mathbf{P}_n(\mathbf{C})$

$$H^r(X, \mathcal{F}) = 0$$

for $0 < r < \text{prof}_{\mathbf{P}_n}(\mathcal{F}) - 1$.

c) Another vanishing theorem for a q -corona is as follows.

Let X be a q -pseudoconvex space for which we adopt the notations of $n \cdot 4$. Let $c < \inf_K \phi$ and let

$$X_c = \{x \in X : \phi(x) > c\}.$$

If $\inf_K \phi > \inf_X \phi$ this is a q -corona. For this type of coronas we have:

THEOREM 5. *Let F be a metrically pseudoconcave line bundle over X and \mathcal{F} any coherent sheaf on X . There exists an integer $k_0 = k_0(\mathcal{F}, F)$ such that*

$$H^r(X_c, \mathcal{F} \otimes \Omega(F^k)) = 0$$

if $k \geq k_0$ and $q < r < \text{prof } \mathcal{F} - q - 1$.

PROOF. First one establishes that for $\varepsilon > 0$ and sufficiently small

$$(1) \quad H^r(X_{c-\varepsilon}, \mathcal{F} \otimes \Omega(F^k)) \cong H^r(X_c, \mathcal{F} \otimes \Omega(F^k))$$

for $r < \text{prof } \mathcal{F} - q - 1$ (cf. a)) (actually we need only the surjectivity of the restriction map).

Secondly by theorem 1 we can find $k_0 = k_0(\mathcal{F}, F)$ such that

$$(2) \quad H^r(X, \mathcal{F} \otimes \Omega(F^k)) = 0$$

for $k \geq k_0$ and $r > q$.

Let

$$B_{c-\varepsilon/2} = \left\{x \in X : \phi(x) < c - \frac{\varepsilon}{2}\right\}.$$

Let $\xi \in H^r(X_c, \mathcal{F} \otimes \Omega(F^k))$. By (1) we can find

$$\eta \in H^r(X - B_{c-\varepsilon/2}, \mathcal{F} \otimes \Omega(F^k))$$

such that by the natural restriction map

$$r_{X_c}^{X - B_{c-\varepsilon/2}}(\eta) = \xi.$$

Now by theorem 15 of [1] if $r < \text{prof } \mathcal{F} - q - 1$ we can find

$$\hat{\eta} \in H^r(X, \mathcal{F} \otimes \Omega(F^k))$$

such that

$$r_{X_c}^X(\hat{\eta}) = \xi.$$

By (2) $\hat{\eta} = 0$ thus also $\xi = 0$.

For instance in the exemple given in b) if we apply this theorem we get the same conclusion but under less restrictive conditions i.e. \mathcal{F} needs to be defined only in the ball

$$B = \{z \in \mathbf{C}^n : \sum_1^n z_i \bar{z}_i < b\}$$

and the vanishing of $H^r(X, \mathcal{F})$ is for $0 < r < \text{prof}_B \mathcal{F} - 1$.

REFERENCES

A. ANDREOTTI AND H. GRAUERT

- [1] *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, 90 (1962) p. 193—259.

A. ANDREOTTI AND E. VESENTINI

- [2] *On deformations of discontinuous groups*, Acta Math., 112 (1964) p. 249—298.

J. P. SERRE

- [3] *Faisceaux algébriques cohérents*, Ann. of. Math., 61 (1955) p. 197—278.

(Oblatum 12-5-69)