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Asymptotic expansions in renewal theory

by

B. B. van der Genugten

1. Introduction and statement of the results

Let μ be a probability measure defined on the Borel sets of $(-\infty, \infty)$ with $\int |x|\mu\{dx\} < \infty$ and $\mu_1 = \int x\mu\{dx\} > 0$. Then the renewal measure ν belonging to μ , defined by $\nu = \sum_0^\infty \mu^{n*}$, assigns finite measure to bounded Borel sets.

In this paper our aim is to get approximations of $\nu\{x+E\}$, E some Borel set, for $x \rightarrow -\infty$ if $\mu\{(-\infty, x)\}$ decreases exponentially, and for $x \rightarrow \infty$ if $\mu\{(x, \infty)\}$ has this property. Work on this has been done in Stone [1] and [2]. Results are obtained for μ lattice and for the case that some μ^{m*} is non-singular (we call μ lattice with span d if μ is concentrated on $\{nd : -\infty < n < \infty\}$ but not on $\{nd' : -\infty < n < \infty\}$ for any $d' > d$, and we call μ^{m*} non-singular if it contains an absolutely continuous component).

Let $g(s)$ be the moment generating function of μ , defined by $g(s) = \int e^{sx}\mu\{dx\}$, the domain being all complex numbers for which the integral exists absolutely. As far as defined let $A(s_0)$ denote the residue of $1/(1-g(s))$ at $s = s_0$.

THEOREM 1. *Let μ be lattice with span 1.*

a) *If $g(s)$ exists for some s with $\operatorname{Re} s = -R < 0$, then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = -r$, the set*

$$S = \{s_0 : g(s_0) = 1, -r < \operatorname{Re} s_0 < 0, -\pi < \operatorname{Im} s_0 \leq \pi\}$$

is finite, $A(s_0)$ exists for $s_0 \in S$ and for integer $k \rightarrow -\infty$

$$(1.1) \quad \nu\{k\} = \sum_{s_0 \in S} A(s_0)e^{-s_0 k} + o(e^{rk})$$

$$(1.2) \quad \nu\{(-\infty, k]\} = \sum_{s_0 \in S} (1-e^{s_0})^{-1} A(s_0)e^{-s_0 k} + o(e^{rk}).$$

b) *If $g(s)$ exists for some s with $\operatorname{Re} s = R > 0$, then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = r$, the set*

$$S' = \{s_0 : g(s_0) = 1, 0 < \operatorname{Re} s_0 < r, -\pi < \operatorname{Im} s_0 \leq \pi\}$$

is finite, $A(s_0)$ exists for $s_0 \in S'$ and for integer $k \rightarrow \infty$

$$(1.3) \quad v\{k\} = \mu_1^{-1} - \sum_{s_0 \in S} A(s_0)e^{-s_0 k} + o(e^{-rk}).$$

Moreover, if $\mu_2 = \int x^2 \mu\{dx\} < \infty$ then

$$(1.4) \quad \begin{aligned} v\{(-\infty, k)\} \\ = k/\mu_1 + \frac{1}{2}(\mu_2/\mu_1)^2 + \sum_{s_0 \in S'} (1 - e^{-s_0})^{-1} A(s_0)e^{-s_0 k} + o(e^{-rk}). \end{aligned}$$

Under mild conditions S is not empty and contains even one real point which provides the leading term. This does not hold for the set S' .

THEOREM 2. Let μ be lattice with span 1, $\mu\{(-\infty, 0)\} > 0$ and let I be the interior of the interval I of real points $s < 0$ for which $g(s)$ exists. Suppose I is not empty.

a) If $I = I$ or if there exists some s with $g(s) = 1$ and $\text{Re } s \in I$, or even $\text{Re } s \in I$ and $\text{Im } s \neq 2\pi k$, $k = 0, \pm 1, \dots$, then there exists exactly one real $s_0 \in I$ with $g(s_0) = 1$. Moreover, $g'(s_0) < 0$ and for some $r > -s_0$

$$(1.5) \quad v\{k\} = -e^{-s_0 k}/g'(s_0) + o(e^{rk}), \quad k \rightarrow -\infty$$

$$(1.6) \quad v\{(-\infty, k]\} = -e^{-s_0 k}/\{g'(s_0)(1 - e^{s_0})\} + o(e^{rk}), \quad k \rightarrow -\infty.$$

b) If $I \neq I$ and there does not exist such an $s_0 \in I$ then for any $-r \in I$

$$\begin{aligned} v\{k\} &= o(e^{rk}), \quad k \rightarrow -\infty \\ v\{(-\infty, k]\} &= o(e^{rk}), \quad k \rightarrow -\infty. \end{aligned}$$

Moreover, if even there does not exist such an $s_0 \in I$ then these order relations hold for $r = R$, where $-R$ is the (finite) left boundary of I .

The corresponding theorems for μ non-lattice are:

THEOREM 3. Let μ^{m*} be non-singular.

a) If $g(s)$ exists for some s with $\text{Re } s = -R < 0$, then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\text{Re } s = -r$, for which the singular part ζ of μ^{m*} satisfies

$$(1.7) \quad \int_{-\infty}^0 e^{-rx} \zeta\{dx\} + \int_0^\infty (1+x) \zeta\{dx\} < 1,$$

the set

$$S = \{s_0 : g(s_0) = 1, -r < \text{Re } s_0 < 0\}$$

is finite, $A(s_0)$ exists for $s_0 \in S$ and for $x \rightarrow -\infty$

$$(1.8) \quad \nu\{x+E\} = \sum_{s_0 \in S} A(s_0)e^{-s_0x} \int_E e^{-s_0t} dt + o(e^{rx})$$

for every Borel set E bounded from above. In particular, for $x \rightarrow -\infty$

$$(1.9) \quad \nu\{(-\infty, x)\} = - \sum_{s_0 \in S} s_0^{-1} A(s_0)e^{-s_0x} + o(e^{rx}).$$

b) If $g(s)$ exists for some s with $\operatorname{Re} s = R > 0$, then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = r$, for which the singular part ζ of μ^{m*} satisfies

$$(1.10) \quad \int_{-\infty}^{\infty} (1-x)\zeta\{dx\} + \int_0^{\infty} e^{rx}\zeta\{dx\} < 1,$$

the set

$$S' = \{s_0 : g(s_0) = 1, 0 < \operatorname{Re} s_0 < r\}$$

is finite, $A(s_0)$ exists for $s_0 \in S'$ and for $x \rightarrow \infty$

$$(1.11) \quad \nu\{x+E\} = |E|/\mu_1 - \sum_{s_0 \in S'} A(s_0)e^{-s_0x} \int_E e^{-s_0t} dt + o(e^{-rx}),$$

for every Borel set E bounded from below of finite length $|E|$. Moreover, if $\mu_2 = \int x^2 \mu\{dx\} < \infty$ then

$$(1.12) \quad \nu\{(-\infty, x)\} = x/\mu_1 + \frac{1}{2}(\mu_2/\mu_1)^2 + \sum_{s_0 \in S'} s_0^{-1} A(s_0)e^{-s_0x} + o(e^{-rx}).$$

THEOREM 4. Let μ^{m*} be non-singular, $\mu\{(-\infty, 0)\} > 0$, let the singular part of μ^{m*} be restricted to $(-\infty, 0]$, let I be the interior of the interval I of real points $s < 0$ for which $g(s)$ exists and let E be a Borel set bounded from above. Suppose I is not empty.

a) If $I = I$ or if there exists some s with $g(s) = 1$ and $\operatorname{Re} s \in I$, or even $\operatorname{Re} s \in I$ and $\operatorname{Im} s \neq 0$, then there exists exactly one real $s_0 \in I$ with $g(s_0) = 1$. Moreover, $g'(s_0) < 0$ and for some $r > -s_0$

$$(1.13) \quad \nu\{x+E\} = -e^{-s_0x} \int_E e^{-s_0t} dt/g'(s_0) + o(e^{rx}), \quad x \rightarrow -\infty.$$

In particular

$$(1.14) \quad \nu\{(-\infty, x)\} = e^{-s_0x}/e_0 g'(s_0) + o(e^{rx}), \quad x \rightarrow -\infty$$

b) If $I \neq I$ and there does not exist such an $s_0 \in I$ then for any $-r \in I$

$$\begin{aligned} \nu\{x+E\} &= o(e^{rx}), \quad x \rightarrow -\infty \\ \nu\{(-\infty, x)\} &= o(e^{rx}), \quad x \rightarrow -\infty. \end{aligned}$$

Moreover, if even there does not exist such an $s_0 \in I$ then these order relations hold for $r = R$, where $-R$ is the (finite) left boundary of I .

2. Proof of the theorems

PROOF OF THEOREM 1a). $g(s)$ is analytic for $\operatorname{Re} s \in (-R, 0)$, continuous for $\operatorname{Re} s \in [-R, 0]$, $g(i\theta) \neq 1$ for $|\theta| \in (0, 2\pi)$ and

$$(2.1) \quad g(s) = 1 + \mu_1 s + o(|s|), \text{ for } |s| \rightarrow 0 \text{ and } \operatorname{Re} s \leq 0.$$

Therefore, for any $r \in (0, R]$ with $g(s) \neq 1$ on $\operatorname{Re} s = -r$ and $\varepsilon > 0$ sufficiently small the function $1/(1-g(s))$ is continuous on Γ , and analytic within Γ with the exception of a finite number of poles. Here Γ is the contour in the complex s -plane shown in fig. 1. If for one or more s_0 with $\operatorname{Re} s_0 \in (-r, 0)$ it occurs that

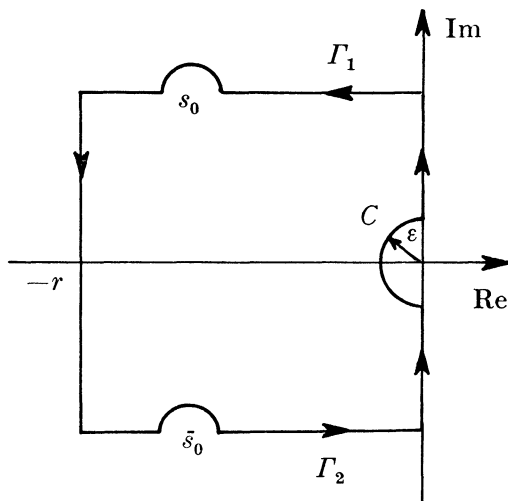


Figure 1

$g(s_0) = 1$ with $\operatorname{Im} s_0 = \pi$ and also $g(\bar{s}_0) = 1$ with $\operatorname{Im} \bar{s}_0 = -\pi$ then the parts Γ_1 and Γ_2 of Γ are slightly deformed as indicated.

Setting

$$\psi(s) = \{1-g(s)\}^{-1} + \{\mu_1(1-e^s)\}^{-1}$$

we get with the Cauchy residue theorem

$$(2.2) \quad \frac{1}{2\pi i} \int_{\Gamma} e^{-sk} \Psi(s) ds = \sum_{s_0 \in S} A(s_0) e^{-s_0 k}.$$

According to Stone [3], (20), for $k < 0$ we have

$$(2.3) \quad \nu\{k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \{e^{-ik\theta} \Psi(i\theta)\} d\theta = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\substack{\varepsilon \leq |\operatorname{Im} s| \leq \pi \\ \operatorname{Re} s = 0}} e^{-sk} \Psi(s) ds.$$

With the Riemann-Lebesgue lemma

$$(2.4) \quad \frac{1}{2\pi i} \int_{-r-i\pi}^{-r+i\pi} e^{-sk} \Psi(s) ds = \frac{1}{2\pi} e^{rk} \int_{-\pi}^{\pi} e^{-i\theta k} \Psi(i\theta - r) d\theta = o(e^{rk}),$$

$k \rightarrow -\infty.$

Since $g(s+2\pi i) = g(s)$, the contributions of Γ_1 and Γ_2 to the integral in (2.2) cancel out. With (2.1) we see that the contribution of C to the integral in (2.2) tends to zero for $\varepsilon \rightarrow 0$. So (1.1) follows from (2.2)–(2.4) and (1.2) follows from (1.1).

b) The proof of (1.3) is similar to that of (1.1). Use Stone [1], (20), for $k \geq 0$. With (1.3)

$$v\{k, N\} = \frac{N-k}{\mu_1} + \sum_{s_0 \in S'} (1 - e^{-s_0})^{-1} A(s_0) e^{-s_0 k} + o(e^{-rk}) + o(e^{-rN}),$$

$k \rightarrow \infty, N \rightarrow \infty$

and, as is well-known,

$$\lim_{N \rightarrow \infty} \left[v\{(-\infty, N)\} - \frac{N}{\mu_1} \right] = \frac{1}{2} (\mu_2 / \mu_1)^2$$

we get (1.4).

LEMMA A. *Let I and I be defined as in theorem 2. Suppose I is not empty and $\mu\{(-\infty, 0)\} > 0$. If $I = I$ or if $g(s_1) = 1$ for some s_1 with $\text{Re } s_1 \in I$ then there exists exactly one real $s_0 \in I$ with $g(s_0) = 1$. We have $g'(s_0) < 0$.*

PROOF. Let

$$(2.5) \quad g_1(s) = \int_{[0, \infty)} (e^{sx} - 1) \mu\{-dx\}, \quad -s \in I$$

$$g_2(s) = \int_{(0, \infty)} (1 - e^{-sx}) \mu\{dx\}, \quad -s \in I.$$

Since $g_1(0) = g_2(0) = 0$, $0 < g'_1(0^+) < g'_2(0^+)$ and g_1 is convex and g_2 concave, there is at most one s_0 with

$$0 = g_1(-s_0) - g_2(-s_0) = g(s_0) - 1,$$

and then $g'(s_0) < 0$. If $g(s_1) = 1$ with $\text{Re } s_1 \in I$ then $g(\text{Re } s_1) \geq 1$. But $g'(0^-) > 0$ and so there exists $s_0 \in I$ with $g(s_0) = 1$. Finally, if $I = I$ i.e. I is open to the left, then $g_1(-s) \rightarrow \infty$ if s tends to the left boundary of I . This also assures that there is $s_0 \in I$ with $g(s_0) = 1$.

PROOF OF THEOREM 2.

a) According to lemma A the set S in theorem 1 contains exactly one real $s_0 \in I$ with $g'(s_0) < 0$ and $s_0 \geq \text{Re } s_1$ for any $s_1 \in S$. But μ has span 1 and so $s_0 > \text{Re } s_1$ and $s_0 \in I$. With $A(s_0) = -1/g'(s_0)$ and theorem 1 we see that (1.5) holds for some $r > -s_0$. (1.6) follows from (1.5).

b) This part follows immediately from theorem 1.

In the following for any signed measure ψ let $|\psi|$ denote its variation. We call ψ finite if the measure $|\psi|$ is finite.

LEMMA B. Let μ^{m*} be non-singular, ζ the singular part of μ^{m*} , and let $K(x)$ and $L_s(x)$, $s \in T$ with T an arbitrary index-set, be non-negative Borel functions in x so that

(2.6) for every fixed finite interval I

$$\int_{\{x+I\}} K(y-x)\mu\{dy\} \text{ is bounded in } x, \quad -\infty < x < \infty$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\{x+I\}} |K(y+\varepsilon) - K(y)|(\mu^{m*} - \zeta)\{dy\} = 0, \quad -\infty > x > \infty$$

(2.7) $\int K(x)\mu\{dx\} < \infty$

(2.8) $L_s(x) \leq K(x), \quad -\infty < x < \infty, \quad s \in T$

(2.9) $L_s(x+y) \leq L_s(x)L_s(y), \quad -\infty < x, y < \infty, s \in T$

(2.10) $\sup_{s \in T} \int L_s(x)\zeta\{dx\} < 1.$

Then for any $\varepsilon > 0$ there exist an integer $n_0 \geq 1$, a measure φ with infinitely often differentiable density with compact support, and a signed measure φ' such that

(2.11) $\mu^{n_0*} = \varphi + \varphi',$

(2.12) $|\varphi'| \{(-\infty, \infty)\} < \varepsilon,$

(2.13) $1 - \varepsilon \leq \varphi\{(-\infty, \infty)\} \leq 1,$

(2.14) $\sup_{s \in T} \int L_s(x)\varphi\{dx\} < \infty,$

(2.15) $\sup_{s \in T} \int L_s(x)|\varphi'| \{dx\} < \varepsilon.$

Moreover, for $\varepsilon < 1$ the renewal measure

$$v = \sum_0^\infty \mu^{k*}$$

can be written as

$$(2.16) \quad \nu = \nu' + \nu''$$

with

$$\begin{aligned} \nu'' &= (\mu^{0*} + \dots + \mu^{(n_0-1)}) * \sum_0^\infty \varphi'^{k*} \\ \nu' &= \varphi * \nu'' * \sum_0^\infty \mu^{kn_0*}. \end{aligned}$$

Here ν'' is a finite signed measure with

$$(2.17) \quad \sup_{s \in T} \int L_s(x) |\nu''| \{dx\} < \infty.$$

PROOF. With $\zeta\{(-\infty, \infty)\} < 1$, (2.9) and (2.10) it follows that for n sufficiently large

$$(2.18) \quad \zeta^{n*}\{(-\infty, \infty)\} < \frac{\varepsilon}{4},$$

$$(2.19) \quad \sup_{s \in T} \int L_s(x) \zeta^{n*}\{dx\} < \frac{\varepsilon}{4}.$$

Setting $\xi = \mu^{m*} - \zeta$ and $n_0 = nm$ we get

$$(2.20) \quad \mu^{n_0*} = \zeta^{n*} + \sum_{k=1}^n \binom{n}{k} \cdot \xi^{k*} * \zeta^{(n-k)*}.$$

The second term on the right hand side of (2.20) is absolutely continuous. Let $h(x)$ be its density. With (2.7), (2.8) and (2.9) for $A > 0$

$$\sup_{s \in T} \int_{|x| \geq A} L_s(x) \mu^{n_0*}\{dx\} \leq n_0 \left[\int K(x) \mu\{dx\} \right]^{n_0-1} \cdot \int_{|x| \geq A/n_0} K(x) \mu\{dx\}$$

and so with (2.7) and (2.20) for A sufficiently large

$$(2.21) \quad \int_{|x| \geq A} h(x) dx < \frac{\varepsilon}{4}$$

$$(2.22) \quad \sup_{s \in T} \int_{|x| \geq A} L_s(x) h(x) dx < \frac{\varepsilon}{4}.$$

Set

$$q_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{1}{2}\sigma^{-2}x^2 \right\}, \quad \sigma > 0$$

$$h_\sigma(x) = \int q_\sigma(x-t)h(t)dt$$

and let, for $\delta > 0$, $\theta(x)$ be some infinitely often differentiable function with

$$\begin{aligned} \theta(x) &= 1, & |x| &\leq A - \delta \\ 0 &\leq \theta(x) \leq 1, & A - \delta &\leq |x| \leq A \\ \theta(x) &= 0, & |x| &\geq A. \end{aligned}$$

With (2.6),

$$(2.33) \quad \int_{|x| \leq A} K(x) \mu^{n_0 *} \{dx\} < \infty.$$

So with (2.20)

$$\int_{|x| \leq A} K(x) h(x) dx < \infty$$

and therefore for δ sufficiently small, again with (2.6)

$$(2.24) \quad \int_{-A}^A |h(x) - h_\sigma(x)| dx < \frac{\varepsilon}{4}$$

$$(2.25) \quad \int_{-A}^A K(x) |h(x) - h_\sigma(x)| dx < \frac{\varepsilon}{4}.$$

Finally, for δ sufficiently small

$$(2.26) \quad \int_{A-\delta \leq |x| \leq A} (1 - \theta(x)) h_\sigma(x) dx < \frac{\varepsilon}{4}$$

$$(2.27) \quad \int_{A-\delta \leq |x| \leq A} K(x) (1 - \theta(x)) h_\sigma(x) dx < \frac{\varepsilon}{4}.$$

Let φ be the measure with density

$$\begin{aligned} p_\varphi(x) &= \theta(x) h_\sigma(x), & |x| &\leq A \\ &= 0 & |x| &> A \end{aligned}$$

and φ' the sum of the measure ζ^{n*} and the signed measure with density $h(x) - p_\varphi(x)$. Then (2.11) holds, φ and φ' are finite with $\varphi\{(-\infty, \infty)\} \leq 1$, and p_φ is infinitely often differentiable with compact support $[-A, A]$.

With (2.18), (2.21), (2.24), (2.26)

$$\begin{aligned} |\varphi'\{(-\infty, \infty)\}| &\leq \zeta^{n*}\{(-\infty, \infty)\} + \int_{|x| \geq A} h(x) dx \\ &+ \int_{-A}^A |h(x) - h_\sigma(x)| dx + \int_{A-\delta \leq |x| \leq A} (1 - \theta(x)) h_\sigma(x) dx < \varepsilon, \end{aligned}$$

which proves (2.12). With (2.11) this gives (2.13). From (2.8),

(2.20) and (2.23) we get (2.14). With (2.19), (2.8), (2.22), (2.25), (2.27)

$$\begin{aligned} \sup_{s \in T} \int L_s(|\varphi'| \{dx\}) &\leq \sup_{s \in T} \int L_s(x) \zeta^{n*} \{dx\} + \sup_{s \in T} \int_{|x| \geq A} L_s(x) h(x) dx \\ &+ \int_{-A}^A K(x) |h(x) - h_\sigma(x)| dx + \int_{A-\delta \leq |x| \leq A} K(x) (1 - \theta(x)) h_\sigma(x) dx < \varepsilon \end{aligned}$$

which proves (2.15).

Moreover, if $\varepsilon < 1$ then from (2.12) it follows that ν'' is a finite signed measure. So $\nu - \nu''$ is defined, and with (2.11),

$$\begin{aligned} \nu - \nu'' &= (\mu^{0*} + \dots + \mu^{(n_0-1)*}) * \sum_{k=1}^\infty (\mu^{kn_0*} - \varphi'^{k*}) \\ &= (\mu^{0*} + \dots + \mu^{(n_0-1)*}) * \sum_{k=1}^\infty \sum_{j=0}^{k-1} \varphi^{*j} \mu^{jn_0*} * \varphi'^{(k-1-j)*} \\ &= (\mu^{0*} + \dots + \mu^{(n_0-1)*}) * \sum_{j=0}^\infty \sum_{k=j+1}^\infty \varphi * \mu^{jn_0*} * \varphi'^{(k-1-j)*} \\ &= \varphi * \nu'' * \sum_0^\infty \mu^{kn_0*}, \end{aligned}$$

which proves (2.16). Note that the summations with respect to j and k may be interchanged since ν'' is finite.

Finally, (2.17) follows with (2.7), (2.9) and (2.15).

PROOF OF THEOREM 3.

a) Let $r \in (0, R]$ with $g(s) \neq 1$ on $\text{Re } s = -r$. We apply Lemma B for $T = [-r, 0]$,

$$\begin{aligned} L_s(x) &= e^{sx}, & x < 0 \\ &= 1 + x, & x \geq 0 \\ K(x) &= L_{-r}(x), & -\infty < x < \infty \end{aligned}$$

and keep the same notations. Denoting the moment generating function of any finite measure or finite signed measure ψ different from μ by ψ_1 , we get that $\int |x| |\nu''| \{dx\}$, $\varphi_1(-r)$ and $|\nu''|_1(-r)$ are finite.

In (1.8) and (1.9) we may replace ν by ν' since

$$e^{-rx} |\nu''| \{(-\infty, x)\} \leq \int_{(-\infty, x)} e^{-ry} |\nu''| \{dy\} \rightarrow 0 \text{ if } x \rightarrow -\infty.$$

Note that $g(s)$ and $g^{n_0}(s)$ are analytic for $\text{Re } s \in (-R, 0)$, continuous for $\text{Re } s \in [-R, 0]$, that $g^{n_0}(i\theta) \neq 1$ for $\theta \neq 0$ and that

$$(2.28) \quad g^{n_0}(s) = 1 + n_0 \mu_1 s + o(|s|), \text{ for } |s| \rightarrow 0 \text{ and } \text{Re } s \leq 0.$$

Since $\varphi_1(s)$ tends to zero if $|\text{Im } s| \rightarrow \infty$, uniformly in $\text{Re } s \in [-R, 0]$ and $|\varphi'_1(-r)$ can be made arbitrary small

(2.29) $|g^{n_0}(s)| \leq C < 1, \text{Re } s \in [-R, 0],$ for $|\text{Im } s|$ sufficiently large.

Therefore, for N sufficiently large and ε sufficiently small the function $1/(1-g^{n_0}(s))$ is continuous on Γ and analytic within Γ with the exception of a finite number of poles. Here Γ is the contour in the proof of theorem with π replaced by N .

If $\chi = n_0^{-1} \cdot \varphi * \nu'$, then χ is a finite signed measure with $\chi\{(-\infty, \infty)\} = 1$, and $\chi_1(s)$ is continuous on Γ and analytic within Γ . Setting

$$\Psi(s) = \chi_1(s)[\{1-g^{n_0}(s)\}^{-1} + (n_0\mu_1s)^{-1}]$$

we get with the Cauchy residue theorem

$$(2.30) \quad \frac{1}{2\pi i} \int_{\Gamma} e^{-sx} \Psi(s) ds = \sum_{s_0 \in Z} B(s_0) \chi_1(s_0) e^{-s_0 x}.$$

Here $B(s_0)$ is the residue of $1/(1-g^{n_0}(s))$ at $s = s_0$ and Z is defined by

$$Z = \{s_0 : g^{n_0}(s) = 1, -r < \text{Re } s_0 < 0\}.$$

But $\chi_1(s_0) = 1$ if $g(s_0) = 1$ and $\chi_1(s_0) = 0$ if $g^{n_0}(s_0) = 1$ and $g(s_0) \neq 1$. If $s_0 \in S \subset Z$ then $B(s_0) = n_0^{-1} A(s_0)$. So we get

$$(2.31) \quad \sum_{s_0 \in Z} B(s_0) \chi_1(s_0) e^{-s_0 x} = n_0^{-1} \sum_{s_0 \in S} A(s_0) e^{-s_0 x}$$

Let $p(x)$ be the density of ν' . In the same way as in the proof of Stone [2], Theorem, it follows that

$$\int |\chi(i\theta + s)| d\theta < \infty, s \in [-R, 0].$$

and

$$(2.32) \quad p(x) - \mu_1^{-1} \cdot \chi\{(-\infty, x)\} \\ \frac{n_0}{2\pi} \int \text{Re} \{e^{-ix\theta} \Psi(i\theta)\} d\theta = \lim_{\varepsilon \rightarrow 0} \frac{n_0}{2\pi i} \int_{\substack{|\text{Im } s| \geq \varepsilon \\ \text{Re } s = 0}} e^{-sx} \Psi(s) ds.$$

It follows easily that

$$(2.33) \quad \chi\{(-\infty, x)\} = o(e^{rx}), \quad x \rightarrow -\infty.$$

With (2.29) and the Riemann-Lebesgue lemma

$$\begin{aligned}
 (2.34) \quad & \lim_{N \rightarrow \infty} \frac{n_0}{2\pi i} \int_{-r-iN}^{-r+iN} e^{-sx} \Psi(s) ds \\
 & = \frac{n_0}{2\pi} e^{rx} \int e^{-i\theta x} \Psi(i\theta - r) d\theta = o(e^{rx}), \quad x \rightarrow -\infty.
 \end{aligned}$$

The contributions of Γ_1 and Γ_2 to the integral of (2.30) tend to zero for $N \rightarrow \infty$. This follows with (2.29) and the fact that $\chi_1(s)$ tends to zero for $|\text{Im } s| \rightarrow \infty$, uniformly in $\text{Re } s \in [-R, 0]$. With (2.28) we see that the contribution of C to the integral in (2.30) tends to zero for $\varepsilon \rightarrow 0$. Therefore, from (2.30)–(2.34)

$$(2.35) \quad p(x) = \sum_{s_0 \in S} A(s_0) e^{-s_0 x} + o(e^{rx}), \quad x \rightarrow -\infty$$

and (1.8), (1.9) follow from (2.35).

b) Compare the corresponding part of the proof of theorem 1.

PROOF OF THEOREM 4. Compare the proof of theorem 2. Use Lemma A and theorem 3. Since $g(s) < 1$ for real $s \in (s_0, 0)$ and $g(s_0) = 1$ the condition (1.7) is fulfilled for some $r > -s_0$.

3. Final remarks

REMARK 1. Let μ be lattice or some μ^{m*} be non-singular. Suppose $\mu\{(-\infty, 0)\} > 0$ and let $g(s)$ exist for some $s < 0$. Then there exists always a finite real number $r < 0$ such that $\int e^{sx} \nu\{dx\}$ converges for $s \in (r, 0)$ and diverges for $s \in (-\infty, r)$.

This follows from theorem 2 and $\nu\{k\}$ bounded, and from theorem 4, (2.35) and $p(x)$ bounded.

REMARK 2.

a) Suppose $g(s)$ exists for $\text{Re } s \leq 0$. If

$$(3.1) \quad \liminf_{r \rightarrow \infty} e^{rk_0} \int_0^\pi |g(i\theta - r) - 1|^{-1} d\theta = 0$$

then the sum in (1.1) converges for $r \rightarrow \infty$ and equals $\nu\{k\}$, $k \leq k_0 < 0$. This follows from the fact that the left side of (2.4) tends to zero for $r \rightarrow \infty$, uniformly in $k \leq k_0$. Note that the sum remains a finite one and (3.1) holds if the number of lattice-points of μ in $(-\infty, 0)$ is finite.

b) Suppose $g(s)$ exists for $\text{Re } s \geq 0$. Similarly, if

$$(3.2) \quad \liminf_{r \rightarrow \infty} e^{-rk_0} \int_0^\pi |g(i\theta + r) - 1|^{-1} d\theta = 0$$

then the sum in (1.3) converges for $r \rightarrow \infty$ and equals $\nu\{k\} - \mu_1^{-1}$, $k \geq k_0 \geq 0$. Note that the sum remains a finite one and (3.2) holds if the number of lattice-points of μ in $(0, \infty)$ is finite.

Postscript. Further investigations have led to the stronger result that theorem 3 continues to hold if (1.7) and (1.10) are replaced by $\zeta_1(-r) < 1$ and $\zeta_1(r) < 1$. The condition in theorem 4 that $\zeta\{(0, \infty)\} = 0$ can be dropped. We refer to van der Genugten [4].

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