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Some applications of Henderson's open embedding theorem of F -manifolds

by

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The primary purpose of this paper is to apply the following two theorems to the study of certain subsets of F -manifolds. These theorems can be found, for example, in [1].

THEOREM 0.1. *Each F -manifold can be embedded as an open subset of Hilbert space.*

THEOREM 0.2. *Two F -manifolds are homeomorphic if and only if they have the same homotopy type.*

The major results in this paper are an engulfing theorem for certain subsets of an F -manifold (Theorem 2.1) and an annulus theorem for disjoint, bicollared spheres in an F -manifold (Theorem 3.2).

H will be used to denote separable Hilbert space, and M will denote an arbitrary F -manifold. By an F -manifold is meant a separable metric space which is a manifold modeled on a separable infinite-dimensional Fréchet space.

$B_r(x)$ will be the ball in H of radius r centered at x ; $S_r(x) = \text{Bd } B_r(x)$; $B_r = B_r(\theta)$; and $S_r = S_r(\theta)$, where θ is the zero element of H .

By an *open H -cell* (or open cell) in a space X is meant an open subset of X which is the homeomorphic image of H . A *closed H -cell* (or cell) is a subset C of the space X such that there is a homeomorphism from the pair (B_1, S_1) in H onto the pair $(C, \text{Bd } C)$ in X . A closed subset K of X is a *collar* of C if there exists a homeomorphism h from the pair (B_2, B_1) in H onto the pair $(K \cup C, C)$ in X such that $h(S_2) = \text{Bd } (K \cup C)$.

1. When F -manifolds are homeomorphic to H

The monotone union property for an F -manifold M is the property that the union of an increasing sequence of copies of M , which are open in the union, must be homeomorphic to M .

The following theorem gives several necessary and sufficient conditions for an F -manifold M to be Hilbert space.

THEOREM 1.1. *The following are equivalent for an F -manifold M .*

- (1) M is homeomorphic to H .
- (2) M is contractible.
- (3) M has trivial homotopy groups.
- (4) M is an AR .
- (5) M has the monotone union property.

PROOF. (1) is equivalent to (3) follows from Theorem 0.2. That (2), (3), and (4) are equivalent can be found in [4]. (1) implies (5) follows from the Monotone Union Theorem for Hilbert Space found in [2]. Finally to show that (5) implies (1), by Theorem 0.1, let h be an embedding of M as an open subset of H . Choose $\delta > 0$ and $x \in h(M)$ so that $B_\delta(x) \subset h(M)$. Let g be a homeomorphism of H onto itself such that $g[B_\delta(x)] = B_1$. For each $n = 1, 2, \dots$, let f_n be a homeomorphism from H onto $\text{Int } B_{n+1}$ such that $f_n(B_1) = B_n$, and let $M_n = f_n g h(M)$. Then $H = \bigcup_{n=1}^{\infty} M_n$ is a monotone union of open copies of M , so that M is homeomorphic to H .

2. An engulfing theorem for F -manifolds

LEMMA 2.1. *If U and V are open cells in H such that $U \cap V$ is a cell, then $U \cup V$ is a cell.*

PROOF. U , V , and $U \cap V$ are AR 's, so that $U \cup V$ is an AR . Then by Theorem 1.1, $U \cup V$ is a cell.

Lemma 2.1 can be generalized slightly using Van Kampen's Theorem.

LEMMA 2.2. *Let U be a connected open subset of H . Then every two points in U can be joined by a piecewise linear arc lying in U .*

PROOF. Let $x, y \in U$. Since U is connected and open in H , it will be arcwise connected. Because an arc between x and y is compact, there exists $B_{\delta_1}(x_1), \dots, B_{\delta_n}(x_n)$ such that each $B_{\delta_i}(x_i) \subset U$, $B_{\delta_i}(x_i) \cap B_{\delta_{i+1}}(x_{i+1}) \neq \emptyset$ for $i = 1, \dots, n-1$, and $x_1 = x$ and $x_n = y$. Then $[x_i : x_{i+1}] \subset B_{\delta_i}(x_i) \cup B_{\delta_{i+1}}(x_{i+1})$ for $i = 1, \dots, n-1$, so that $\bigcup_{i=1}^{n-1} [x_i : x_{i+1}]$ is a piecewise linear arc joining x and y , where $[x_i : x_{i+1}]$ is the line segment from x_i to x_{i+1} .

LEMMA 2.3. *Let U be a connected open subset of H containing B_1 , let $x \in U - B_1$, and let $y \in S_1$. Then there is a piecewise linear arc joining x and y lying in $(U - B_1) \cup y$.*

PROOF. Let $\delta > 0$ be such that $B_\delta(y) \subset U$. Let z be the point such that $\text{Ray} [\theta : y] \cap (B_\delta(y) - \text{Int } B_1) = [y : z]$, where $\text{Ray} [\theta : y]$ is the ray emanating from θ and passing through y . By Lemma 2.2, there exists a piecewise linear arc α from x to z lying in $U - B_1$. Let w be the point of $\alpha \cap [y : z]$ such that the portion of α lying between w and x , call it β , intersects $[y : z]$ only at w . Then $[y : w] \cup \beta$ is the desired piecewise linear arc.

LEMMA 2.4. *Let U be a connected open subset of H containing B_1 , and let $x \in U - B_1$. Then there exists an open cell contained in U and containing $\text{Int } B_1 \cup x$.*

PROOF. Let $y \in S_1$. Then by Lemma 2.3 there exists a piecewise linear arc α joining x and y and lying in $(U - B_1) \cup y$. Let $\alpha_1, \dots, \alpha_n$ be the linear pieces of α , starting at y and going to x . Let $\delta_1 > 0$ be such that $N_{\delta_1}(\alpha_1) \cap (\bigcup_{i=3}^n \alpha_i) = \emptyset$, where $N_{\delta_1}(\alpha_1)$ is the open δ_1 neighborhood of α_1 . Then for each $i = 2, \dots, n$, inductively define δ_i so that

$$N_{\delta_i}(\alpha_i) \cap \left(\left[\bigcup_{j=1}^{i-2} N_{\delta_j}(\alpha_j) \right] \cup \left[\bigcup_{j=i+2}^n \alpha_j \right] \cup \text{Int } B_1 \right) = \emptyset.$$

Each $N_{\delta_i}(\alpha_i)$ is an open cell, and $N_{\delta_1}(\alpha_1) \cap \text{Int } B_1$ and $N_{\delta_i}(\alpha_i) \cap N_{\delta_{i+1}}(\alpha_{i+1})$ are convex and are hence open cells. Then by repeated applications of Lemma 2.1, $\left[\bigcup_{i=1}^n N_{\delta_i}(\alpha_i) \right] \cup \text{Int } B_1$ is an open cell containing $\text{Int } B_1 \cup x$ and contained in U .

THEOREM 2.1. *Let X be a subset of a connected F -manifold M , which is contained in some collared cell in M , and let U be an open subset of M . Then there exists a homeomorphism h of M onto itself and a collared cell C such that $X \subset h(U)$ and $h|(M - C) = \text{identity}$.*

PROOF. By Theorem 0.1, consider M as an open subset of H . By hypothesis, X is contained in a collared cell C' which is contained in M . Let $x \in U$ and $x \in \text{Int } C'$. From an application of Lemma 2.4, $x \cup y$ is contained in a collared cell C'' in H such that $C'' \subset M$. Then let f be a homeomorphism of H onto itself so that $f(C'') = B_1$. Define g to be a homeomorphism of H onto itself so that $gf(y) = f(x)$ and $g|(H - B_1) = \text{identity}$. Then define the homeomorphism φ of M onto itself by $\varphi(x) = f^{-1}gf(x)$ if $x \in C'$, and $\varphi(x) = x$ otherwise. Now $\varphi(C')$ is a collared cell in M such that $U \cap \text{Int } \varphi(C) \neq \emptyset$. If K is a collar of $\varphi(C')$ in M , then $K \cup \varphi(C')$ is homeomorphic to H . Since the image of $\varphi(C')$ under this homeomorphism is collared in H , the theorem can be established using the Engulfing Theorem for Hilbert Space found in [3].

3. Bicollared spheres in F -manifolds

A closed subset A of a space X is an *annulus* if there exists a homeomorphism h from $B_2 - \text{Int } B_1$ onto A such that

$$\text{Bd } A = h(S_1 \cup S_2).$$

A *bicollared sphere* in X is a closed set S such that there exists a homeomorphism g from $B_3 - \text{Int } B_1$ onto an annulus in X such that $g(S_2) = S$.

THEOREM 3.1. *A closed complementary domain of a bicollared sphere in an F -manifold M is a closed H -cell if and only if it satisfies one of conditions (1) through (5) in Theorem 1.1.*

The proof of Theorem 3.1 is similar to the proof of the following theorem.

THEOREM 3.2. *The closed region between two disjoint, bicollared spheres in an F -manifold M is an annulus if and only if it satisfies one of conditions (1) through (5) in Theorem 1.1.*

PROOF. The annular region plus the collars of the spheres can be seen to be a contractible F -manifold, which is therefore homeomorphic to H by Theorem 1.1. The images of the spheres in H under this homeomorphism are tame since they are bicollared (see [5]), so that the region between them is an annulus by Corollary 1 of [3].

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