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H. L. RESNIKOFF

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Supplement to “Some remarks on Poincaré series”

by

H. L. Resnikoff

In his study of Poincaré series, Clifford Earle [3] considered those bounded homogeneous subdomains B of C^n whose Bergman kernel function k_B satisfies the condition:

For each $w \in B$ there exist an open set $U \subset B$ and a positive number α such that for all $z \in B$ and $w' \in U$,

$$(1) \quad \alpha^{-1} \text{abs } k_B(z, w) \leq \text{abs } k_B(z, w') \leq \alpha \text{abs } k_B(z, w).$$

We will show

THEOREM 1. *If B is analytically equivalent to a bounded symmetric domain, then the inequalities (1) are valid.*

Since the Bergman kernel of a product domain is the product of the kernels of the component factor domains [2], it is enough to consider equivalence classes of irreducible domains. If the map $z \rightarrow \tilde{z}$ defines an analytic equivalence of B with a domain \tilde{B} , then a basic property of the Bergman kernel together with (1) imply

$$(2) \quad \begin{aligned} & \alpha^{-1} \text{abs } k_B(\tilde{z}, \tilde{w}) \\ & \leq \text{abs } \det \left(\frac{\partial \tilde{w}'}{\partial w'} \frac{\partial w}{\partial \tilde{w}} \right) \text{abs } k_B(\tilde{z}, \tilde{w}') \leq \alpha \text{abs } k_B(\tilde{z}, \tilde{w}). \end{aligned}$$

The absolute value of the determinant appearing in (2) is bounded both from above and below for an appropriate choice of U . Therefore the validity of (1) for B entails its validity for \tilde{B} , whether or not \tilde{B} is bounded. Hence (1) is a class invariant condition, and Theorem 1 can be proved by choosing a representative of each equivalence class of B for which k_B assumes a particularly convenient form. We shall show that for certain realizations of the bounded symmetric domains the absolute value of the kernel function is bounded away from zero. This will certainly imply (1), and hence Theorem 1.

The specific realizations of interest are most conveniently expressed by means of Jordan algebras; see references [1] and

[5] for terminology and uncited results. Let \mathfrak{A} denote a simple compact real Jordan algebra of dimension d and rank r , unit element c , reduced trace σ , and reduced norm $|\cdot|$. Set $q = d/r$, and

$$\Gamma_{\mathfrak{A}}(s) = \int_{\exp \mathfrak{A}} e^{-\sigma(u)} |u|^{s-q} du$$

for positive real s and a fixed but arbitrarily normalized Euclidean measure du . Denote the Jordan product of two elements by juxtaposition, and define linear transformations $L(a)$ and $P(a)$ on the complexification $\mathfrak{A}(i)$ of \mathfrak{A} by $L(a)b = ab$ for all $a, b \in \mathfrak{A}(i)$, and $P(a) = 2L^2(a) - L(a^2)$. Then

$$Z(\mathfrak{A}) = \{z \in \mathfrak{A}(i) : Id - P(z)P(\bar{z}) > 0\}$$

is an irreducible bounded symmetric domain.

U. Hirzebruch [4] has shown that every equivalence class of irreducible bounded symmetric domains has a representative realized as $Z(\mathfrak{A})$, or as a totally geodesic submanifold of $Z(\mathfrak{A})$, for some simple compact real \mathfrak{A} . As Hirzebruch observed, the form of the embedding in the latter case shows that the Bergman kernel of a domain realized as such a totally geodesic submanifold of $Z(\mathfrak{A})$ is proportional to the restriction to that manifold of $k_{Z(\mathfrak{A})}$. Therefore Theorem 1 is a consequence of the following statement:

THEOREM 2. *If \mathfrak{A} denotes a simple compact real Jordan algebra, then*

$$abs k_{Z(\mathfrak{A})}(z, w) \geq (4\pi)^{-d} \Gamma_{\mathfrak{A}}(2q)/\Gamma_{\mathfrak{A}}(q)$$

for all $z, w \in Z(\mathfrak{A})$.

PROOF OF THEOREM 2. The Bergman kernel of $Z(\mathfrak{A})$ has the form [5]

$$(3) \quad k^{-1}(z, w) = \alpha \det (Id - 2L(z)L(\bar{w}) - 2L(\bar{w})L(z) - 2L(z\bar{w}) + P(z)P(\bar{w}))$$

with some real positive constant α . If z^{-1} exists, then [5]:

$$(4) \quad k^{-1}(z, w) = \alpha \det (P(z)P(z^{-1} - \bar{w})).$$

Eq. (3) shows that $k^{-1}(z, w)$ is a holomorphic function of (z, \bar{w}) on $Z(\mathfrak{A}) \times \overline{Z(\mathfrak{A})} = Z(\mathfrak{A}) \times Z(\mathfrak{A})$, and therefore the maximum modulus of k^{-1} occurs on the Šilov boundary of $Z(\mathfrak{A}) \times Z(\mathfrak{A})$, which is the product of the Šilov boundary $\check{S}(Z(\mathfrak{A}))$ of $Z(\mathfrak{A})$ with itself. In reference [7] it was shown that $\check{S}(Z(\mathfrak{A})) = \exp i\mathfrak{A}$, and, in particular, u^{-1} exists and also belongs to $\exp i\mathfrak{A}$ for every

$u \in \exp i\mathfrak{A}$. Thus we are assured that the simple expression in eq. (4) can be used in searching for the maximum modulus of $k^{-1}(z, w)$, with z and w running through $\exp i\mathfrak{A}$.

If $u \in \exp i\mathfrak{A}$, then $u = \exp ia$ for some $a \in \mathfrak{A}$, and

$$abs \det P(u) = abs \det P(\exp i\mathfrak{A}) = abs \det \exp 2iL(a) = 1;$$

hence

$$(5) \quad \begin{aligned} \alpha^{-1} abs k^{-1}(z, w) \\ = abs \det P(z)P(z^{-1}-\bar{w}) = abs \det P(z^{-1}-\bar{w}). \end{aligned}$$

Furthermore, $w \in \exp i\mathfrak{A}$ implies $\bar{w} = w^{-1}$. It has already been remarked that the latter also belongs to $\exp i\mathfrak{A}$; therefore

$$(6) \quad \max_{z, w \in \exp i\mathfrak{A}} abs \det P(z^{-1}-\bar{w}) = \max_{z, w \in \exp i\mathfrak{A}} abs \det P(z-w).$$

A further simplification can be effected by associating with each $u = \exp ia$ its square root $u^{\frac{1}{2}} = \exp (ia/2)$. Then

$$(7) \quad \begin{aligned} abs \det P(z-w) &= abs \det P(P(z^{\frac{1}{2}})(c-P^{-1}(z^{\frac{1}{2}})w)) \\ &= abs \det P(z^{\frac{1}{2}})P(c-P^{-1}(z^{\frac{1}{2}})w)P(z^{\frac{1}{2}}) \\ &= abs \det P(c-P^{-1}(z^{\frac{1}{2}})w), \end{aligned}$$

with the second equality following from application of the "fundamental formula" $P(P(a)b) = P(a)P(b)P(a)$, ref. [1], p. 91. Since $z \in \exp i\mathfrak{A}$, $(z^{\frac{1}{2}})^{-1} = z^{-\frac{1}{2}}$ exists, so the right hand side of eqs. (7) reduces to $abs \det P(c-P(z^{-\frac{1}{2}})P(w^{\frac{1}{2}})c)$. Theorem 1 of reference [5], chapter VII states that there exists $t \in \exp i\mathfrak{A}$ and $A \in Aut \mathfrak{A}$ (= the group of algebraic automorphisms of \mathfrak{A}) such that $P(z^{-\frac{1}{2}})P(w^{\frac{1}{2}}) = AP(t)$; therefore eqs. (7) reduce to

$$(8) \quad \begin{aligned} abs \det P(z-w) &= abs \det P(c-AP(t)c) \\ &= abs \det P(A(A^{-1}c-P(t)c) \\ &= abs \det P(c-t^2). \end{aligned}$$

Introduce the maximal decomposition of t : that is, find a complete set of primitive orthogonal idempotents $\{c_k\}$ and real numbers $\{\tau_k\}$ such that

$$t = \sum_{k=1}^r e^{i\tau_k} c_k.$$

The quantities $\{(1-e^{i\tau_k})\}$ are the eigenvalues (counted with their multiplicities) of $(c-t^2)$; therefore

$$(9) \quad abs \det P(c-t^2) = abs|c-t^2| = abs\left(\prod_{k=1}^r (1-e^{i\tau_k})\right)^{2a}.$$

The latter expression is maximal when $c^{i\tau_k} = -1$ for each k , and in this event the right side of eq. (9) is $(2^{\text{rank } \mathfrak{A}})^{2q} = 4^{\dim \mathfrak{A}} = 4^d$.

Recalling eq. (5), we have shown that

$$(10) \quad abs \, k^{-1}(z, w) \leq 4^d abs(\alpha).$$

The constant α has been evaluated for tube domains by Korányi [6]. His function $\Gamma^*(s)$ is easily shown to be the same as $\Gamma_{\mathfrak{A}}(qs)$. Then substitution of $z = w = 0$ into proposition 5.7 of [6] yields

$$k^{-1}(0, 0) = \pi^d \Gamma_{\mathfrak{A}}(q) / \Gamma_{\mathfrak{A}}(2q) = \alpha,$$

the last equality following from comparison with eq. (1). Combined with eq. (10), this completes the proof of Theorem 2.

REMARK. With \mathfrak{A} as above, choose a complete set $\{c_i\}$ of primitive orthogonal idempotents, put $d_k = \sum_{i=1}^k c_i$, and define $\mathfrak{A}_k = \{a \in \mathfrak{A} : ad_k = a\}$. Then \mathfrak{A}_k is a subalgebra of \mathfrak{A}_{k+1} , $\mathfrak{A}_r = \mathfrak{A}$, and $\text{rank } \mathfrak{A}_k = k$. Let $q_k = \dim \mathfrak{A}_k / \text{rank } \mathfrak{A}_k$, so $q_r = q$. Proposition 4 of reference [8] asserts that

$$\Gamma_{\mathfrak{A}}(s) = \pi^{(d-r)/2} \prod_{k=1}^r \Gamma(s + q_k - q_r),$$

with Γ denoting Euler's gamma function and an appropriately normalized measure on $\exp \mathfrak{A}$. Therefore

$$\Gamma_{\mathfrak{A}}(2q) / \Gamma_{\mathfrak{A}}(q) = \prod_{k=1}^r \Gamma(q + q_k) / \Gamma(q_k);$$

since $2q_k \in Z$ for each k and every simple compact real Jordan algebra, this ratio can be explicitly evaluated; it is always an integer.

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H. L. RESNIKOFF

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Rice University
Houston 77027, Texas