Compositio Mathematica

R. D. Anderson
David W. Henderson
James E. West

Negligible subsets of infinite-dimensional manifolds

Compositio Mathematica, tome 21, nº 2 (1969), p. 143-150
<http://www.numdam.org/item?id=CM_1969__21_2_143_0>

COMPOSITIO MATHEMATICA, Vol. 21, Fasc. 2, 1969, pag. 143-150
Wolters-Noordhoff Publishing
Printed in the Netherlands

Negligible subsets of infinite-dimensional manifolds

by

R. D. Anderson, David W. Henderson, and James E. West *

We say that a subset A of a space X is *negligible in* X if $X \setminus A$ is homeomorphic to X. In [1], [3], [4], [5], [6], [11], and [12], various authors have shown that certain subsets of certain infinite-dimensional topological linear spaces are negligible. We extend these results to infinite-dimensional metrizable manifolds modelled on separable Fréchet spaces. (We call such manifolds, *F-manifolds.*) In fact, we give a characterization of strongly negligible closed subsets of F-manifolds, where we say that A is *strongly negligible* in X if, for each open cover G of X, there is a homeomorphism of X onto $X \setminus A$ which is limited by G. Theorem 5 asserts that a closed subset A is strongly negligible in an F-manifold X if and only if A has *Property Z*, i.e. (following Anderson in [4]) for each non-empty, homotopically trivial, open set U in X, it is true that $U \setminus A$ is non-empty and homotopically trivial. In Lemma 1 Property Z is shown to be equivalent, in F-manifolds, to several other properties, one of which is the property shown by Eells and Kuiper (in [8]) to imply homotopy-negligibility. In the same paper, Eells and Kuiper also show how their results, together with recent results of Burghelea, Kuiper and Moulis, can be used to imply some of our results.

The authors feel that the methods of this paper offer simple and convenient frameworks for the reduction of many point-set topological problems in F-manifolds (and thus in many Banach manifolds) to problems in the Hilbert cube Q and in s (the countably infinite product of real lines). The reduction is effected by an iterated use of star-finite coverings, the second use involving open sets homeomorphic to s and thus admitting canonical compactifications to Q. It has been shown (see [7], [9], and [2] or [5]) that all separable, infinite-dimensional, Fréchet spaces are homeomorphic to s.

* Supported in part by National Science Foundation grant GP 7952X.

Lemma 1. *The following four conditions are equivalent for a closed subset A of an F-manifold X:*

1. *A has Property Z in X.*

2. *A has Property Z locally in X (i.e. each point of A lies in an open set 0 for which $A \cap 0$ has Property Z in 0.)*

3. *Each point of A has a fundamental system of neighborhoods U in X for which the inclusion $U \setminus A \to U$ is a homotopy equivalence.*

4. *Each point of A lies in an open set 0 of which there is a homeomorphism onto s carrying $A \cap 0$ into a closed linear subspace which has an infinite-dimensional closed complementary linear subspace (i.e., one for which s is the direct product (sum) of it with the first).*

Proof. It is trivial that 1. implies 2. implies 3. That 3. implies 1. is essentially the content of the main lemma of [8]. That 2. and 4. are equivalent is Theorems 8.4 and 9.1 of [4].

Lemma 2. *If A is a subset of the F-manifold X which has Property Z in X and if A' is a closed subset of A, then $A \setminus A'$ has Property Z in $X \setminus A'$ and A' has Property Z in X.*

Proof. The first statement follows immediately from Lemma 1; the second may be shown from the definition of F-manifold and Theorems 8.4 and 9.1 of [4] to show that A' has Property Z locally in X.

For G an open cover of a space X, a function f of X into itself is said to be *limited by* G if for each x in X there is an element of G containing both x and $f(x)$.

An open cover G of an open subset U of a space X is said to be a *normal cover* provided that whenever f is a homeomorphism of U into itself which is limited by G, there is an extension of f to a homeomorphism of X into itself which is the identity on $X \setminus U$. This extension is said to be the *normal extension* of f.

Lemma 3. *For any open set U in a metric space X, there is a normal cover of U.*

Proof. Let d be a metric for X and for each x in U, set $U_x = \{y \in U | d(x, y) < \frac{1}{2} d(x, X \setminus U)\}$. Let $G = \{U_x\}_{x \in U}$. This cover suffices, as is easily seen.

Lemma 4. *Each component of an F-manifold is separable.*

Proof. This is immediate from a result due to Sierpinski [14; p. 111] which asserts that a connected, locally separable, metric space is separable

THEOREM 1. *Let U be an open subset of s. For each open cover G of U there is a triple (G_1, G_2, G_3) of star-finite covers of U refining G such that, denoting G_i by $\{g_j^i\}_{j \in N}$,*

1. *G_1 and G_2 are open covers by sets homeomorphic to s, while G_3 is a cover by closed sets of s;*

2. *G_1 is a normal cover of U; and*

3. *for each $j \in N$, $g_j^3 \subset g_j^2 \subset \bar{g}_j^2 \subset g_j^1$.*

PROOF. Let s be regarded as lying in the Hilbert cube Q as the "pseudo-interior", i.e., let $Q = \prod_{i \in N} I_i$, where for each i, $I_i = [-1, 1]$, and let $s = \prod_{i \in N} I_i^0$, with $I^0 = (-1, 1)$. Let G_0 be a normal cover of U refining G, and set $V = \bigcup_{g^0 \in G_0} Q \setminus \overline{s \setminus g^0}$. Since V is a separable, locally compact metric space, it may be written as the union of an increasing sequence $\{X_i\}_{i \in N}$ of compacta, each lying in the interior of its successor. Let D be the set of all triples (A, B, C) of basic connected open subsets of V such that $A \subset \bar{A} \subset B \subset \bar{B} \subset C$ and there is a g^0 in G^0 such that $C \subset Q \setminus \overline{s \setminus g^0}$, and let E be the subset of D composed of all triples (A, B, C) for which $\bar{C} \cap X_i \neq \emptyset$ implies that $\bar{C} \subset X_{i+1}^0$, for each i. The collection of all initial elements of members of E is an open cover of V, and each X_i is compact; so for each i, there exists a finite subset H_i of E for which the initial elements of members of H_i cover $X_i \setminus X_{i-1}^0$. Set

$$G_1 = \{C \cap s \mid \text{there are an } i, \text{ a } B, \text{ and an } A \text{ for which } (A, B, C) \text{ is in } H_i\},$$

$$G_2 = \{B \cap s \mid \text{there are an } i, \text{ an } A, \text{ and a } C \text{ such that } (A, B, C) \text{ is in } H_i\}, \text{ and}$$

$$G_3 = \{\bar{A} \cap s \mid \text{there are an } i, \text{ a } B, \text{ and a } C \text{ for which } (A, B, C) \text{ is in } H_i\}.$$

Index $\bigcup_{i \in N} H_i$ by N and induce the same indexing on G_1, G_2, and G_3.

LEMMA 5. *Each open cover of an F-manifold has a star-finite open refinement.*

PROOF. This is immediate from Lemma 4 and from Theorem 1 of [10], which uses the same idea as our Theorem 1 above.

THEOREM 2. *For any countable, star-finite, open cover G of a space X, there exists an ordering $\{g_i\}_{i \in N}$ of the elements of G such that if $\{f_i\}_{i \in N}$ is a sequence of homeomorphisms with f_1 a homeo-*

morphism of X into itself which is the identity off g_1 and, for each $i > 1$, with f_i a homeomorphism of $f_{i-1} \cdots f_1(X)$ into itself which is the identity off g_i, then it follows that $f(x) = \text{limit}_{i\to\infty} f_i \cdots f_1(x)$ defines a homeomorphism of X into itself.

PROOF. Let $\{G_i\}_{i\in N}$ be any ordering of the elements of G; let $H_1 = \{G_1\}$, and (inductively for $i > 1$) let H_i be the set of all elements of $G \setminus \bigcup_{j<i} H_j$ which intersect members of H_{i-1} together with the least-indexed element of $G \setminus \bigcup_{j<i} H_j$. Since G is star-finite, each H_i is finite. Let $\{g_i\}_{i\in N}$ be an ordering of G listing the elements of H_1, then those of H_3, followed by those of H_2 and then of H_5, and so forth. Now suppose $\{f_i\}_{i\in N}$ is a sequence of functions as in the hypothesis with respect to the cover $\{g_i\}_{i\in N}$. Let $h_k = f_k \cdots f_1$. Set $j(k) = \max\{n | g_n \in H_k\}$, and let $H_k^* = \bigcup_{j\leq k} H_j$. If m is an odd integer, then

$$j(m-2) < j(m) < j(m-1) < j(m+2) < j(m+1) < j(m+4).$$

Now for $\quad k \leqq j(m-1),\ h_k(H_m^*) \subset H_m^*,$

and for $\quad j(m-1) < k \leqq j(m+2),\ f_k | H_m^* \cap h_{k-1}(X)$

is the identity; therefore, $h_k(H_m^*) \subset H_m^*$ for all $k \leqq j(m+2)$. Thus, for

$$k \leqq j(m+1),\ h_k(H_m^*) \subset H_{m+1}^*.$$

Now, since $k > j(m+1)$ implies that $f_k | H_{m+1}^* \cap h_{k-1}(X)$ is the identity, it follows that for

$$k > j(m+1),\ h_k | H_m^* = h_{j(m+1)} | H_m^*,$$

and the theorem follows.

THEOREM 3. *If U is open in s, A is a relatively closed subset of U with Property Z in U, and G is an open cover of U, there is a homeomorphism of s onto $s \setminus A$ which is the identity off U and is limited on U by G.*

PROOF. Let G_1, G_2, and G_3 be as per Theorem 1, and assume that G_1 is indexed in accordance with Theorem 2.

Let H_1 be a normal open cover of g_1^2. Since g_1^2 is homeomorphic to s and (by Lemma 2) $A \cap g_1^3$ is a closed subset of g_1^2 with Property Z in it, there exists, by Theorem 8.4 of [4] [1] and Lemma

[1] Theorem 8.4 of [4] asserts that if X is a closed subset of s with Property Z, then there is a homeomorphism of s onto itself carrying X into a closed linear subspace of s which has an infinite-dimensional complementary closed linear subspace.

9.1 of [5] [2], a homeomorphism of g_1^2 onto $g_1^2 \setminus (A \cap g_1^3)$ which is limited by H_1. Let f_1 be the normal extension of this homeomorphism to s.

For each $i > 0$, $A \cap (\bigcup_{j<i} g_j^3) \cap g_i^2$ is a relatively closed set in g_i^2 and has Property Z in it (Lemma 2), so $g_i^2 (\setminus (A \cap (\bigcup_{j<i} g_j^3))$ is homeomorphic to s (Theorem 8.4 of [4] and Lemma 9.1 of [5], together with the choice of g_i^2 as being homeomorphic to s). Let a_i be such a homeomorphism.

Suppose, now, that $f_1, \cdots, f_{i-1}$ are homeomorphisms of $s, \cdots, s \setminus (A \cap (\bigcup_{j<i-1} g_j^3))$ onto

$$s \setminus (A \cap g_1^3), \cdots, s \setminus (A \cap (\bigcup_{j<i-1} g_j^3))$$

which are the identity on $s \setminus g_1^2, \cdots, s \setminus g_{i-1}^2$, respectively, and for which $f_j \cdots f_1(\bar{g}_m^2 \setminus \bigcup_{n<m} g_n^2)$ lies in g_m^1 for each $j \leqq i-1$ and $m \leqq j$. Let H_i be a normal open cover of $g_i^2 \setminus (A \cap (\bigcup_{j<i} g_j^3))$ for which no element meets both $f_{i-1} \cdots f_1(\bar{g}_m^2 \setminus \bigcup_{n<m} g_n^2)$ and $s \setminus g_m^1$ for any $m < i$. Now apply Theorem 8.4 of [4] to obtain a homeomorphism b_i of s onto itself carrying $a_i(A \cap (g_i^3 \setminus \bigcup_{j<i} g_j^3))$ into a closed linear subspace which has an infinite-dimensional complementary closed linear subspace. Let c_i be a homeomorphism of s onto $s \setminus a_i b_i (A \cap (g_i^3 \setminus \bigcup_{j<i} g_j^3))$ limited by the image under $b_i a_i$ of H_i, as guaranteed by Lemma 9.1 of [5]. The homeomorphism $a_i^{-1} b_i^{-1} c_i b_i a_i$ of

$$g_i^2 \setminus (A \cap (\bigcup_{j<i} g_j^3)) \quad \text{onto} \quad g_i^2 \setminus (A \cap (\bigcup_{j \leqq i} g_j^3))$$

is now limited by H_i; let f_i be the normal extension of it to $s \setminus (A \cap (\bigcup_{j<i} g_j^3))$.

By induction, there exists a sequence $\{f_i\}_{i \in N}$ of homeomorphisms of the sort supposed in the preceding paragraph. By Theorem 2, there is a homeomorphism f, defined by the formula $f(x) = \text{limit}_{i \to \infty} f_i \cdots f_1(x)$ for each x in X, of s onto $s \setminus A$ which is the identity off U. This homeomorphism is limited on U by G because if x is in U and i is the integer for which $g_i^2 \setminus \bigcup_{j<i} g_j^2$ contains x, then $f_m \cdots f_1(x)$ is in g_i^1 for each m. Thus, $f(x)$ is in g_i^1 because there is an l for which $f_{l+k} \cdots f_1(x) = f_l \cdots f_1(x)$ for all $k > 0$ by the nature of the ordering of G_1. Since G_1 limits f and refines G, G also limits f.

[2] Lemma 9.1 of [5] asserts that if X is a closed subset of s lying in a closed linear subspace with a complementary closed linear subspace of infinite dimension and if G is an open cover of s, then there is a homeomorphism of s onto $s \setminus X$ limited by G which may be required to be the identity off any open set containing X.

THEOREM 4. *If X is an F-manifold and A is a closed subset of X which has Property Z, then X is homeomorphic to $X \setminus A$; furthermore, the homeomorphism may be required to be limited by any open cover of X and to be the identity on the complement of any open set of X containing A.*

PROOF. It suffices to prove the theorem without requiring the homeomorphism to be the identity on the complement of a given open set U containing A. This is true because then if H is any open cover of X, there is a normal open cover H_1 of U which refines the open cover of U composed of the intersections of elements of H with U, and the normal extension to X of a homeomorphism of U onto $U \setminus A$ which is limited by H_1 will be limited by H and will be the identity off U. Therefore, let H be any open cover of X. By Lemma 5 and the definition of X, there is a star-finite open refinement G of H by sets homeomorphic to open subsets of s. Because the homeomorphism may be defined on each component of X separately, there is no loss of generality in the supposition that X is separable and, hence, that G is countable. By Theorem 2, there is an ordering $\{g_i\}_{i \in N}$ of G such that if $\{f_i\}_{i \in N}$ is a sequence of homeomorphisms with f_1 carrying X into itself and moving no point not in g_1, and with, for $i > 1$, f_i a homeomorphism of $f_{i-1} \cdots f_1(X)$ into itself which moves no point not in g_i, then $f(x) = \text{limit}_{i \to \infty} f_i \cdots f_1(x)$ defines a homeomorphism of X onto $\bigcap_{i \in N} f_i \cdots f_1(X)$. Assume that G is so ordered.

For each $i \in N$, let H_i be a normal open cover of

$$g_i \setminus (A \cap (\bigcup_{j<i} g_j)),$$

and let a_i be a homeomorphism of g_i onto an open subset of s. Let c_1 be a homeomorphism of $a_1(g_1)$ onto $a_1(g_1 \setminus A)$ limited by the image under a_1 of H_1 (Theorem 3). Set f_1 to be the normal extension of $a_1^{-1} c_1 a_1$ to X. For each $i > 1$, let b_i be a homeomorphism of $s \setminus a_i(A \cap g_i \cap (\bigcup_{j<i} g_j))$ onto s which is the identity off $a_i(g_i \cap (\bigcup_{j<i} g_j))$, as guaranteed by Theorem 3. Since $b_i a_i(g_i \setminus (A \cap (\bigcup_{j<i} g_j))$ is open in s and $b_i a_i(A \cap (g_i \setminus (\bigcup_{j<i} g_j))$ is relatively closed in it and has Property Z in it, Theorem 3 yields a homeomorphism c_i of $a_i(g_i)$ onto

$$a_i(g_i) \setminus b_i a_i(A \cap (g_i \setminus \bigcup_{j<i} g_j))$$

which is limited by the image under $b_i a_i$ of H_i. Now, $a_i^{-1} b_i^{-1} c_i b_i a_i$ is a homeomorphism of $g_i \setminus (A \cap (\bigcup_{j<i} g_j))$ onto $g_i \setminus A$ limited by

H_i. Let f_i be the normal extension of it to $X \setminus (A \cap (\bigcup_{j<i} g_j))$, and define $f : X \to X \setminus A$ by $f(x) = \text{limit}_{i \to \infty} f_i \cdots f_1(x)$. By Theorem 2, f is the desired homeomorphism.

THEOREM 5. *If A is a closed subset of an F-manifold X, then A is strongly negligible in X if and only if A has Property Z.*

PROOF. Only the necessity remains to be proved. Let A be strongly negligible and let $f : B^{n+1} \to U$ be a map of the $(n+1)$-ball into an homotopically trivial open subset U of X such that $f(S^n)$ is contained in $U \setminus A$. We need to show that $f|S^n$ can be extended to a map $F : B^{n+1} \to U \setminus A$. Let h be a homeomorphism of X onto $X \setminus A$ which is limited by $\{U, X \setminus f(B^{n+1})\}$ and which is fixed on the closed set $f(S^n)$. (See the proof of Theorem 4.) $F = h \circ f$ is the desired extension.

Added in proof. Extensions of our results to non-separable spaces have recently been given by W. H. Cutler (*Negligible subsets of non-separable Hilbert manifolds* (to appear)). In addition, Cutler shows that our Theorem 4 may be strengthened to require the homeomorphism to be ambient isotopic to the identity by a small isotopy.

REFERENCES

R. D. ANDERSON
[1] On a theorem of Klee, Proc. AMS 17 (1966), 1401–4.

R. D. ANDERSON
[2] Hilbert space is homeomorphic to the countable infinite product of open intervals, Trans. AMS 72 (1966), 516–519.

R. D. ANDERSON
[3] Topological properties of the Hilbert cube and the infinite product of open intervals, Trans. AMS 126 (1967), 200–216.

R. D. ANDERSON
[4] On topological infinite deficiency, Mich. Math. J. 14 (1967), 365–383.

R. D. ANDERSON and R. H. BING
[5] A complete, elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines, Bul. AMS 74 (1968), 771–792.

Cz. BESSAGA and V. L. KLEE, Jr.
[6] Two topological properties of topological linear spaces, Israel J. Math. 2 (1964), 211–220.

Cz. BESSAGA and A. PELCZYNSKI
[7] Some remarks on homeomorphisms of F-spaces, Bull. Acad. Polon. Sci. Ser. Sci Math. Astron. Phys. 10 (1962), 265–270.

J. EELLS, JR. and N. H. KUIPER
[8] Homotopy negligible subsets.

M. I. Kadec

[9] On topological equivalence of all separable Banach spaces (Russian), Dokl. Akad. Nauk. SSSR 167 (1966), 23—25.

S. Kaplan

[10] Homology properties of arbitrary subsets of Euclidean spaces, Trans. AMS 62 (1947), 248—271.

V. L. Klee, Jr.

[11] A note on topological properties of normed linear spaces, Proc. AMS 7 (1956), 673—4.

Bor—Luh Lin

[12] Two topological properties concerning infinite-dimensional normed linear spaces, Trans. AMS 114 (1965), 156—175.

R. S. Palais

[13] Homotopy theory of infinite-dimensional manifolds, Topology 5 (1966), 1—16.

W. Sierpinski

[14] Sur les espaces métriques localement séparables, Fund. Math. 21 (1933), 107—113.

(Oblatum 31–5–68)

Louisiana State University,
Cornell University,
and
The Institute for Advanced Study