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by

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For a plane convex set of diameter d we shall show that the maximal distance between the centroid (of a uniform mass distribution) and the circumcentre is

$$\frac{1}{3}d \sin 2q \tag{1}$$

where the angle q is determined by

$$\sin 2q(2q+\sin 2q-\tfrac{1}{2}\pi)-1+2 \cos^4 q = 0. \tag{2}$$

Here $2q$ is approximately $140° \ 3 \cdot 3'$ and the maximal distance in (1) is $\cdot 2140 \cdots d$. This result is best possible and there exists an extremal configuration (see [1]).

Notation and conventions

An identity used to define a new symbol is written with the sign $= \cdot$. We shall generally use Capital Greek letters to denote plane sets and the frontier of such a set will be denoted by the corresponding lower case Greek letter. Points and real numbers will be denoted by capital and by lower case Roman letters respectively. Classes of sets will be denoted by script capital letters.

For any plane convex set Θ, let $d(\Theta)$, $C(\Theta)$, $G(\Theta)$, $\gamma(\Theta)$, $r(\Theta)$, $\Gamma(\Theta)$ denote respectively the diameter, circumcentre, centroid, circumcircle, circumradius of Θ, and the closed circular disc bounded by $\gamma(\Theta)$, respectively. (Centroid always refers to a uniform mass distribution).

The distance between two points P, Q is written $|P-Q|$. For any point P and any positive numbers r, d we define

$$\gamma(P, r) = \cdot \{Q; |Q-P| = r\}$$
$$\mathscr{K}(d) = \cdot \{\Theta; \Theta \text{ is a plane convex set and } d(\Theta) \leqq d\}.$$

We use $\Gamma(P, r)$ for the disc bounded by $\gamma(P, r)$ and, if $r = 0$ we use $\Gamma(P, r)$ to mean $\{P\}$, the set whose only member is P.

For any two points P, Q we use PQ to denote either the segment which has P, Q as end points, or the line of which that segment is part. If P and Q are diametrically opposite points of $\gamma(A, r)$ then $\alpha(\gamma(A, r); P, Q)$ denotes one of the two subarcs of $\gamma(A, r)$ bounded by P, Q. If the points P, Q are not diametrically opposite on $\gamma(A, r)$ then we use $\alpha(\gamma(A, r); P, Q)$ to denote the minor arc of $\gamma(A, r)$ bounded by P and Q.

For any two sets Θ, Φ

$$\Theta - \Phi = \cdot \{P;\ P \in \Theta \text{ and } P \notin \Phi\}$$

We use $\Xi(\Theta)$ to denote the convex cover of Θ.

$$t(\Theta) = \cdot\ |C(\Theta) - G(\Theta)|.$$
$$t \quad = \cdot \sup t(\Theta), \qquad \Theta \in \kappa(d).$$

Now if Θ is a semicircle and $d(\Theta) = d$ then $\Theta \in \mathscr{K}(d)$ and $t(\Theta) = 2d/3\pi$. We can find a convergent sequence of convex sets $\{\Theta_n\}$ such that

$$\Theta_n \in \kappa(d), \qquad t(\Theta_n) \to t \quad \text{as} \quad n \to \infty.$$

Denote the limit of the sequence of sets by Φ. Since $d(\Phi) = \lim d(\Theta_n) = d$, Φ is not a single point. Also $C(\Theta_n) \to C(\Phi)$ where $C(\Phi)$ is the circumcentre of Φ. If Φ is a segment then it can be shown that

$$\overline{\lim}\, t(\Theta_n) \leqq d/6.$$

But $t \geqq 2d/3\pi > d/6$. Hence Φ is not a segment. Thus Φ is a closed plane convex set with interior points and $t(\Phi) = t$, $d(\Phi) = d$.

In what follows we shall omit Φ and use C, G, γ, Γ, t, r in place of $C(\Phi)$, $G(\Phi)$, $\gamma(\Phi)$, $\Gamma(\Phi)$, $t(\Phi)$, $r(\Phi)$.

It is known that $C \in \Xi(\Phi \cap \gamma)$. If $P \in \Phi \cap \gamma$ is such that $C \notin \Xi(\Phi \cap \gamma - \{P\})$ then we say that P is *indispensible*. Otherwise any point P of Φ is called *dispensible*.

The line through G perpendicular to CG meets γ in two points, which we denote by K, L. The closed half-plane bounded by KL and containing C is denoted by Π^+. The other closed half plane bounded by KL is denoted by Π^-. Similarly the line through C perpendicular to CG meets γ in the points A, B and divides the plane into closed half planes Σ^+, Σ^- where $\Sigma^+ \subset \Pi^+$ and $\Sigma^- \supset \Pi^-$.

For any two points P, Q of γ that are not diametrically opposite, and any positive number x, we denote by $\gamma(P, Q; x)$ that circle which passes through P, Q and whose centre, distant x from PQ, lies on the same side of PQ as does C, the centre of γ. By $\gamma(P, P, x)$ we mean the limit of $\gamma(P, Q; x)$ as Q tends to P.

$$\Phi(P, Q; x) = \cdot\ \Phi \cap \Gamma(P, Q; x).$$

Properties of Φ

We next establish a number of properties of Φ, that will enable us eventually to identify it completely. We shall then be able to compute t.

PROPERTY (1). *Every point of $\Phi \cap \Pi^+ \cap \gamma$ is indispensible. $\Phi \cap \Pi^+ \cap \gamma$ is a set of k points and $\Phi \cap \Pi^+$ is bounded by $k+1$ segments which lie in ϕ and a sub-segment of KL, where $k = 1, 2$ or 3.*

If $\Theta \subset \Phi$, $\Theta \in \mathscr{K}(d)$ and $\Theta \cap \Pi^- = \Phi \cap \Pi^-$, then either $\Theta = \Phi$ or $|G(\Theta)-C| > |G-C|$. This last inequality and the extremal property of Φ imply that $C(\Theta) \neq C$. Thus there is no proper closed convex subset Θ of Φ sat'sfying both

$$\Theta \cap \Pi^- = \Phi \cap \Pi^- \text{ and } \gamma(\Theta) = \gamma. \tag{3}$$

Hence every point of $\Phi \cap \Pi^+ \cap \gamma$ is indispensible. There is certainly one point in this set and by Caratheodory's theorem there are at most three. The form of Φ is given above is the only one which excludes the existence of Θ satisfying (3).

PROPERTY (2). *$\Phi \cap \Pi^- \cap \gamma$ is of positive linear measure and consists of a finite set of points and a finite set of arcs.*

We need the following lemma

LEMMA. *If the closed set Λ meets the circle γ in a set of linear measure zero, then for $0 < x < r$ and $x \to r$, the plane measure of $\Lambda \cap \Gamma - \Gamma(P, P; x)$ is $o(r-x)$.*

The proof is omitted.

Firstly, Φ is not a right-angled triangle, since for such a triangle ψ, $t(\psi) = d/6 < t = t(\Phi)$.

Next we can assume that $\Phi \cap \gamma - \Sigma^-$ is not the empty set. For if it were, then $A, B \in \Phi \cap \gamma$, $r = \frac{1}{2}d$ and by the extremal property of Φ, Φ is bounded by AB, two segments and, possibly, an arc of γ. If this arc of γ degenerates into a single point then Φ is a right-angled triangle. This is false and therefore, in this case, $\Phi \cap \gamma$ contains a genuine arc of γ and thus property (2) is valid.

Suppose then that $\Phi \cap \gamma - \Sigma^-$ is not the empty set and P is a point of it. For x near r define

$$\nabla_x = \cdot\ \Phi(P, P; x).$$

Then

$$|C(\nabla_x)-G| = t-(r-x)\cos \angle GCP+O(r-x)^2,$$

and, if $\Phi \cap \gamma \cap \Pi^-$ is of zero linear measure (in which case $\Phi \cap \gamma$ is of zero linear measure), then by the lemma

$$|G(\nabla_x)-G| = o|r-x|.$$

Hence provided x is sufficiently near r,

$$|C(\nabla_x)-G(\nabla_x)| > t.$$

But $\nabla_x \in \mathscr{K}(d)$ and thus we have a contradiction with the extremal property of Φ. Hence $\Phi \cap \Pi^- \cap \gamma$ is not of zero linear measure.

To establish the remainder of property (2) suppose firstly that $r = \frac{1}{2}d$. Then every convex subset of Γ belongs to $\mathscr{K}(d)$. It follows from the extremal property of Φ and the part of property (2) already proved that $\Phi \cap \Pi^-$ is bounded by KL, possibly one or two other segments and, certainly, an arc of γ. Thus in this case $\Phi \cap \Pi^- \cap \gamma$ is simply a single arc of γ.

Suppose next that $r > \frac{1}{2}d$. Since $\Phi \cap \Pi^- \cap \gamma$ is a closed set there exist points $P, Q \in \Phi \cap \Pi^- \cap \gamma$ such that

$$\Phi \cap \Pi^- \cap \gamma \subset \boldsymbol{\alpha}(\gamma\colon P, Q). \tag{4}$$

Denote this last arc by $\boldsymbol{\alpha}$. Then for $T \in \boldsymbol{\alpha}$

$$\Gamma(P, d) \cap \Gamma(Q, d) \cap \Gamma - \Gamma(T, d) \subset \Sigma^+. \tag{5}$$

Suppose now that $R, S \in \Phi \cap \boldsymbol{\alpha}$ are such that

$$\Gamma - \Gamma(R, d) - \Gamma(S, d)$$

is not the empty set, then $\alpha(\gamma; R, S) \subset \Phi$. For otherwise let U be a point of $\alpha(\gamma; R, S)-\Phi$. Since $U \notin \Phi$ the extremal property of Φ implies that $d(\Phi \cup \{U\}) > d$ i.e. $\Phi - \Gamma(U, d)$ is not the empty set.

Let Y be one of the points of Φ most distant from U. Then $Y \in \Phi - \Gamma(U, d)$ and since

$$\Phi \subset \Gamma(P, d) \cap \Gamma(Q, d) \cap \Gamma$$

we deduce that $Y \in \Sigma^+ \cap \Phi$. Further $Y \in \Phi$ and Y is not a relative interior point of any segment of ϕ (by the definition of Y). From the nature of $\Sigma^+ \cap \Phi$ described in property (1) it follows that $Y \in \gamma$. But $\gamma \cap \Gamma(R, d) \cap \Gamma(S, d) \subset \Gamma(U, d)$ and

$$\Phi \cap \gamma \subset \Gamma(R, d) \cap \Gamma(S, d)$$

thus $\Phi \cap \gamma \subset \Gamma(U, d)$. This is a contradiction with the facts that

$Y \in \Phi \cap \gamma$ and $Y \notin \Gamma(U, d)$. Hence our assumption was false and $\alpha(\gamma; R, S) \subset \Phi$.

Hence any arc of $\gamma - \Phi$ in α has a length exceeding a certain positive number and this implies the second part of Property (2).

COROLLARY. *There are at most two indispensible points: for if there were three they would constitute the whole of $\Phi \cap \gamma$ and be linear measure zero.*

PROPERTY (3). *There are two indispensible points in $\Phi \cap \gamma$, of which at least one lies in Π^+, such that the segment joining them also lies in ϕ.*

This is obvious if k in property (1) is 2. Suppose then that $k = 1$ and denote the point $\Phi \cap \Pi^+ \cap \gamma$ by P. Let PT, PU be segments in ϕ with U, T on segment KL. The lines PT, PU meet ϕ in segments PT_1, PU_1. We shall show that at least one of T_1, U_1 lies on γ and is indispensible.

Suppose then that either $T_1 \notin \gamma$ or that $T_1 \in \gamma$ and T_1 is dispensible. Take a point of ϕ, T_2, close to T_1 so that PT_2 does not lie along PT_1. Let Φ_1 be the closed convex set formed by the intersection of Φ and that closed half plane bounded by PT_2 which contains U_1. Then for T_2 sufficiently close to T_1, $\gamma(\Phi_1) = \gamma$.

In any case $\Phi_1 \in \mathscr{K}(d)$ and if $G(\Phi - \Phi_1) \in \Pi^+ - \Pi^-$ then

$$|G(\Phi_1) - C| > |G - C|. \tag{6}$$

(6) is impossible by the extremal property of Φ. Hence $G(\Phi - \Phi_1) \in \Pi^-$.

As $T_2 \to T_1$, $G(\Phi - \Phi_1) \to T^*$ where T^* is a point of segment PT_1 such that

$$|P - T^*| = \tfrac{2}{3}|P - T_1|. \tag{7}$$

But $T^* \in \Pi^-$ thus $|P - T| \leqq |P - T^*|$ and hence

$$|P - T \leqq \tfrac{2}{3}|P - T_1|. \tag{8}$$

Similarly if either $U_1 \notin \gamma$ or $U_1 \in \gamma$ and U_1 is dispensible then

$$|P - U| \leqq \tfrac{2}{3}|P - U_1|. \tag{9}$$

If both (8) and (9) hold then the centroid of triangle PT_1U_1 lies in Π^-. But then the part of Φ not in this triangle is a convex set with interior points and lies in Π^-. Hence $G \in \Pi^- - \Pi^+$. But this is not so since $G \in \Pi^- \cap \Pi^+$. Thus at least one of (8), (9) is false and at least one of T_1, U_1 both lies on γ and is indispensible.

Since segments PU_1, PT_1 both belong to ϕ and P is an indispensible point of $\Phi \cap \gamma$ in Π^+, property (3) has been established.

PROPERTY (4). *If P and Q are two indispensible points of $\Phi \cap \gamma$ such that the segment PQ is contained in ϕ, then C is the mid point of segment.*

We need four auxiliary lemmas.

LEMMA 1. *If $f(x)$, $g(x)$ are non-negative continuous functions defined over $a \leqq x \leqq b$, then*

$$(13) \qquad \left(\int_a^b fg\right)\left(\int_a^b f\right)\left(\int_a^b g\right) \leqq \tfrac{1}{2}\left(\left(\int_a^b g\right)^2 \int_a^b f^2 + \left(\int_a^b f\right)^2 \int_a^b g^2\right).$$

For by the arithemetic-geometric mean inequality and Hölders inequality

$$\tfrac{1}{2}\left(\left(\int_a^b g\right)^2 \int_a^b f^2 + \left(\int_a^b f\right)^2 \int_a^b g^2\right) \geqq \left(\int_a^b g\right)\left(\int_a^b f\right)\left(\int_a^b f^2\right)^{\frac{1}{2}}\left(\int_a^b g^2\right)^{\frac{1}{2}}$$
$$\geqq \left(\int_a^b fg\right)\left(\int_a^b f\right)\left(\int_a^b g\right).$$

COROLLARY 1. *Under the conditions stated in the lemma if $M = \sup g(x)$ and $g(x) \not\equiv 0$, $g(x) \not\equiv M$, $f(x) \not\equiv 0$ in $a \leqq x \leqq b$ then*

$$(14) \qquad \left(\int_a^b fg\right)\int_a^b f - \tfrac{1}{2}\left(\int_a^b g\right)\left(\int_a^b f^2\right) < \tfrac{1}{2}M\left(\int_a^b f\right)^2$$

For $$\left(\int_a^b f\right)^2\left(\int_a^b g^2\right) < M\left(\int_a^b f\right)^2 \int_a^b g$$

and this combined with (13) gives (14)

COROLLARY 2. *If in Corollary 1 $f(x)$ is replaced by a function of two variables $f_t(x)$ and $g(x)$ by $\partial f_t(x)/\partial t$, then if $\partial f_t(x)/\partial t$ is a function continuous in t and x we have*

$$(15) \qquad \frac{d}{dt}\left|\frac{\int_a^b |f_t(x)|^2 dx}{\int_a^b f_t(x) dx}\right| < \operatorname*{Max}_{a \leqq x \leqq b}\left(\frac{\partial f_t(x)}{\partial t}\right)$$

provided the conditions of Corollary 1 hold.

LEMMA 2. *If P and Q are two indispensible points of $\Phi \cap \gamma$ then the whole of $\Phi \cap \gamma$ lies inside the closed strip bounded by the two lines through P and Q perpendicular to PQ.*

Denote the closed strip by ∇. Then since $C \in \Xi(\Phi \cap \gamma)$ and P, Q are indispensible, either, P, Q are diametrically opposite on γ (in which case the lemma is trivially true) or there is a point R such that $R \in \Phi \cap \gamma$ and $C \in \Xi(\{P\} \cup \{Q\} \cup \{R\})$. Now if there was a point S belonging to $(\Phi \cap \gamma) - \nabla$ then either

(a) S lies on the same side of QC as P

or

(b) S lies on the same side of PC as Q.

Suppose that S lies on the same side of QC as P. Then $C \in \Xi(\{S\} \cup \{Q\} \cup \{R\})$. This is impossible since it contradicts the indispensibility of P. Similarly if (b) holds then Q cannot be indispensible. We conclude that there is no such point as S and the lemma is true.

Lemma 3. *If P, Q are two indispensible points of $\Phi \cap \gamma$ and if the distance of C from PQ is t, where $t > 0$, then the remainder of $\Phi \cap \gamma$ apart from P and Q is a single arc lieing in ∇ say $\alpha(\gamma;\, U, V)$.*

There are two arcs of γ in ∇ one of which is $\alpha(\gamma;\, P, Q)$. Suppose that the other is $\alpha(\gamma;\, L, M)$. There is an arc of γ in Π^- say $\alpha(\gamma;\, R, S)$ and meeting $\alpha(\gamma;\, L, M)$ in an arc say $\alpha(\gamma;\, T, W)$. This last arc is non-void since it contains $(\Phi \cap \gamma) - \{P\} - \{Q\}$; which set cannot be empty since $C \in \Xi(\Phi \cap \gamma)$ and $C \notin \Xi(\{P\} \cup \{Q\})$. There is a subarc of $\alpha(\gamma;\, T, W)$ lieing in both closed circles $\Gamma(P, d)$ and $\Gamma(Q, d)$. This last subarc we denote by $\alpha(\gamma;\, U, V)$. It contains $\Phi \cap \gamma - \{\mathrm{P}\} - \{Q\}$ and by the extremal property of Φ coincides with $\Phi \cap \gamma - \{P\} - \{Q\}$.

Let U', V' be the points diametrically opposite to U, V on γ where U, V are defined in lemma 3.

Lemma 4. *Let ∇_1 be the closed strip bounded by the two lines perpendicular to $U'V'$ and passing through U' and V' respectively. If the conditions of Lemma 3 are valid and the distance of C from $U'V'$ is q then, denoting the distance of $G(\nabla_1 \cap \Phi(U'V';\, z))$ from $U'V'$ by $k(z)$, we have*

$$O > k(z) - k(q) > z - q \quad \textit{for} \quad z < q$$

and $q - z$ sufficiently small.

Take the line $U'V'$ as x axis and the line through C perpendicular to $U'V'$ as y axis. Let U' be $(-a, O)$, V' be (a, O). $\Phi(U', V';\, z) \cap \nabla_1$ is bounded by a concave curve depending on z and straight lines independent of z. Let the concave curve be

$$y = f_z(x).$$

If $(x, f_q(x))$ does not lie on γ then $\partial f_z(x)/\partial z$, evaluated at q, is zero. If $(x, f_q(x))$ does lie on γ then

$$f_q(x) = q + \sqrt{(a^2 + q^2 - x^2)}$$

and for values of z less than q the one-sided derivative is

$$\frac{\partial f_z(x)}{\partial z} = 1 + \frac{z}{\sqrt{a^2+z^2-x^2}} \leqq 2$$

$\partial f_z(x)/\partial z$ exists and is continuous except for at most two values of x. For all x, $f_z(x)$ is a uniformly Lipschitz function of z in $|x| \leqq a$. Hence (15) can be applied.

Since

$$k(z) = \frac{\frac{1}{2}\int_{-a}^{a} (f_z(x)-p(x))^2 dx}{\int_{-a}^{a} (f(x)-p(x))\, dx},$$

where $p(x)$ is the linear function defined by line PQ, it follows that

$$(16) \qquad \frac{dk(z)}{dz} < \frac{1}{2} \operatorname*{Max}_{-a \leqq x \leqq a} \left(\frac{\partial}{\partial z} (f_z(x)-p(x)) \right) \text{ at } z = q$$

$$(17) \qquad \frac{dk(z)}{dz} < 1 \text{ at } z = q$$

and

$$(18) \qquad O > k(z)-k(q) > 1 \text{ for } z < q \text{ and } (q-z) \text{ sufficiently small.}$$

Proof of Property (4). Suppose that property (4) is false and that C is not the midpoint of segment PQ, that is to say C does not lie on PQ. Let $\Theta_z = \Phi \cap \Gamma(U', V'; z)$ be defined for values of z smaller than q where q is the perpendicular distance of C from $U'V'$ and U', V' are as in lemma 3. As z varies and approaches q, $G(\Theta)_z$ moves along a curve and approaches G from a definite direction. For it may be shown that the coordinates of $G(\Theta_z) = \cdot\, G_z$ are differentiable at $z < q$ and have a one-sided derivative at q. Also for $z < q$ $\Gamma(U', V'; z)$ is the circumcircle of Θ_z (because $U'V'$ belong to $\alpha(\gamma; P, Q)$); let its centre be C_z. Choose z so that $q-z = \delta > O$, and δ is small. Let H be the midpoint of arc $\alpha(\gamma; U, V)$. The curve described by G_z is tangential to the line HG at G. Also the centroid G_z is the centroid of a mass proportional to the area of $\nabla_1 \cap \Theta_z$ at the centroid of this area, say G'_z, and a fixed mass proportional to area of $\Theta_z - \nabla_1$ at its centroid a fixed point (where "fixed" means "fixed to the first order in δ". There may be second order changes). Since the distance of $G'_z - G$ projected perpendicular to $U'V'$ divided by δ tends to λ as $\delta \to O$, and $\lambda < 1$, the same must be true of the distance $G_z - G$. (For G'_z, G_z describe similar curves, of which that described by G_z is the smaller).

If $\angle CHG = \alpha$ then

$$CG^2 = CH^2+HG^2-2HG\cdot CH\cos\alpha$$
$$= (CH-HG\cos\alpha)^2+(HG\sin\alpha)^2$$

Write $\delta = CC_z$, $\eta = GG_z$ then

$$C_zH-HG_z\cos\alpha = CH-HG\cos\alpha+(\delta-\eta\cos\alpha)$$
$$HG_z\sin\alpha = HG\sin\alpha+\eta\sin\alpha$$

$$(19)\quad C_zG_z^2 = CG^2+2[\eta HG\sin^2\alpha+(\delta-\eta\cos\alpha)(CH-HG\cos\alpha)] +O(\delta^2)$$
$$= CG^2+2[\eta(HG-CH\cos\alpha)+\delta(CH-HG\cos\alpha)] +O(\delta^2) > CG^2$$

because $\eta < \delta/\cos\alpha$ and $\underline{|CHG} \leqq \frac{1}{2}\pi$. (The last inequality is true because otherwise we would have a point of $\alpha(\gamma, U, V)$ in Π^+. This point being distinct from P, Q is not indispensible and belongs to $\Pi^+\cap\gamma\cap\Phi$. But by property (1) all points of $\Phi\cap\Pi^+\cap\gamma$ are indispensible).

Now C_z is the circumcentre of Θ_z for $z < q$ and thus (19) contradicts the extremal property of Φ.

Hence property (4) is correct.

COROLLARY 1. *The circumradius of* Φ *is* $\frac{1}{2}d$.

COROLLARY 2. Φ *is bounded by a diameter* PQ *of* γ, *a segment* PS, *arc* $\alpha(\gamma;\ R,\ S)$ *of* γ, *and a segment* RQ.

PROPERTY 5. *The line through G perpendicular to CG meets PS in L and QR in M where,* $PL = \frac{2}{3}\ PS$ *and* $QM = \frac{2}{3}\ QR$.

Let S_1 lie on γ near S. As S_1 tends to S the centroid of the area bounded by PS, $\alpha(\gamma;\ S,\ S_1)$, S_1P, tends to L. Thus by adding or removing a small area of this type we can move G along a curve tangential to LG at G. By the extremal property of Φ this curve lies inside $\gamma(C, GC)$. Thus LG is perpendicular to CG. Similarly MG is perpendicular to CG. Property 5 is established

PROPERTY 6. *Denoting the angles* $\underline{|PQR} = a$, $\underline{|SPO} = b$ *and writing* $a+b = k$, $a-b = l$ *so that* $-\pi/2 \leqq k-\pi \leqq l \leqq O$ *then either* Φ *is symmetric,* $a = b$, *or*

$$(20)\qquad \cos k(\tfrac{1}{2}\cos^2 k-\tfrac{1}{4}) = (\cos^2 k-\tfrac{1}{4})\cos l.$$

We use the notation of property (5). By property (5) the line CG divides Φ into two sets each of which have the same moment about the line CG. For two non-diametrically opposite points F, K

of γ let $\Omega(F,K)$ and $\Sigma(F,K)$ be the sets bounded respectively by $\alpha(\gamma; F, K)$ and segment FK on the one hand and by $\alpha(\gamma; F, K)$ and segments CF, CK on the other hand. Denote the moment of the set X about the line CG by $M(X, CG)$. Let Q' be the point of γ obtained by reflecting Q in the line CG. Then the equality of moments of the two parts of Φ about CG implies that

(21) $M(\Sigma(P, Q'); CG)+M(\Omega(P, S); CG) = M(\Omega(Q, R); CG).$

Write $\underline{|PCG} = \frac{1}{2}\pi - u$ then (21) is

(22) $\frac{2}{3}r^3 \sin u+\frac{2}{3}r^3 \cos^3 b \sin (b-u) = \frac{2}{3}r^3 \cos^3 a \sin (a+u)$

i.e.

(23) $\sin u(1-\cos^4 a-\cos^4 b) = \cos u(\cos^3 a \sin a-\cos^3 b \sin b).$

Now by elementary calculations

$$\tan u = \frac{\cos a \sin a-\cos b \sin b}{\frac{3}{2}-\cos^2 a-\cos^2 b}.$$

Substituting for (u) in (23) and cancelling $\sin (a-b)$ gives

$$\cos k(\tfrac{1}{2}\cos^2 k-\tfrac{1}{4}) = (\cos^2 k-\tfrac{1}{4}) \cos l.$$

Property 7. *If Φ such that $a \neq b$ then*

(24) $$123^\circ 27' \leqq a+b \leqq 125^\circ 18'.$$

Since $\frac{1}{2}\pi \leqq k \leqq \pi$ $\cos k$ is negative but $\cos l$ is positive thus $\cos^2 k-\frac{1}{4}$ and $\frac{1}{2}\cos^2 k-\frac{1}{4}$ are of opposite sign. Since

$$\tfrac{1}{2}\cos^2 k-\tfrac{1}{4} \leqq \cos^2 k-\tfrac{1}{4}$$

we must have

$$\tfrac{1}{2}\cos^2 k-\tfrac{1}{4} \leqq O \leqq \cos^2 k-\tfrac{1}{4}.$$

i.e.

$$-\frac{1}{\sqrt{2}} \leqq \cos k \leqq -\tfrac{1}{2}.$$

Next the function $x(\frac{1}{2}x^2-\frac{1}{4})/(x^2-\frac{1}{4})$ is monotonic increasing. Replacing x by $\cos k$ (and remembering that this function is $\cos l$ and $k-\pi \leqq l \leqq O$) we see that it lies between $-\cos k$ and 1. Hence

$$-\cos k \leqq \frac{\cos k(\frac{1}{2}\cos^2 k-\frac{1}{4})}{\cos^2 k-\frac{1}{4}} \leqq 1.$$

The first of these two inequalities leads to $\cos^2 k \leqq \frac{1}{3}$ i.e.

$k \leqq 125°18'$. The second of these inequalities leads to $\cos k \leqq y$ where

$$y^3-2y^2-\tfrac{1}{2}y+\tfrac{1}{2}=0$$

i.e. $\cos k \leqq -\cdot 55138$, i.e. $123°27' \leqq k$.

Thus Property (7) is proved.

Property (8). *If Φ is such that $a \neq b$ then*

$$\text{(25)} \qquad CG = \frac{\tfrac{2}{3}r \sin k \cos k\,(\tfrac{1}{8}-\tfrac{3}{4}\cos^2 k)}{(\cos^2 k-\tfrac{1}{4})^{\frac{1}{2}}(\tfrac{1}{2}\cos^4 k+\tfrac{1}{4}\cos^2 k-\tfrac{1}{16})^{\frac{1}{2}}}.$$

Projecting both CP, PL and CQ, QM along CG we have

$$\begin{aligned} CG &= r\sin u+\tfrac{2}{3}r\cdot 2\cos b\cdot\cos(90+u-b),\\ &= -r\sin u+\tfrac{2}{3}r\cdot 2\cos a\cdot\cos(90-u-a). \end{aligned}$$

Hence adding, dividing by 2 and substituting for a, b into k, l gives

$$CG = \tfrac{2}{3}r\sin k\cos(l+u),$$

But

$$\tan u = \frac{\cos k\sin l}{\tfrac{1}{2}-\cos k\cos l},$$

thus

$$\sin k\cos(l+u) = \frac{\sin k(\tfrac{1}{2}\cos l-\cos k)}{(\tfrac{1}{4}-\cos k\cos l+\cos^2 k)^{\frac{1}{2}}}.$$

Substituting for $\cos l$ from 20 gives the equation (25).

We shall use $\tfrac{2}{3}r\cdot F(k)$ for the length CG given in (25).

Property (9). *If $\cos^2 k \leqq \tfrac{1}{3}$ then $F(k) \geqq \cdot 7385489$.*

Write $\cos^2 k = Y$, then

$$(F(K))^2 = \frac{Y-13Y^2+48Y^3-36Y^4}{1-8Y+8Y^2+32Y^3}.$$

Now for $\tfrac{1}{3} \geqq Y \geqq \tfrac{1}{4}$, $1-8Y+8Y^2+32Y^3 \geqq 0$ hence $(F(k))^2 \geqq x$ if

$$\text{(26)} \qquad Y-13Y^2+48Y^3-36Y^4-x(1-8Y+8Y^2+32Y^3) \geqq 0.$$

It can be shown that if $x > \cdot 625$ the expression above is an decreasing function of Y. At $Y = \tfrac{1}{3}$ it is positive, hence (26) is true for $\tfrac{1}{4} \leqq Y \leqq \tfrac{1}{3}$.

Property (9) is established.

Property (10). *If $a \neq b$ then $t \leqq \cdot 49r$.*

Let CG produced meet γ in W so that G lies between C and W. Let PS and QR produced meet the tangent to γ at W in X and Y

respectively and let PS meet QR in T. By drawing a line through P parallel to LM it can be shown that the centroid of the set bounded by the segments PQ, QY, YX, XP is distant not more than $\cdot 49r$ from the line through C parallel to LM if $CT \leqq 8r$. Since this centroid necessarily lies on the side of LM opposite to C it follows that property (10) is true if $CT \leqq 8r$. Now by property (7) $PTQ \geqq 55^\circ 42'$ which implies that $CT \leqq 2r$. Hence property (10) is proved.

PROPERTY (11). $a = b$.

Otherwise $t = \frac{2}{3}r\, F(\theta) \geqq \frac{2}{3}r. \cdot 7385489 > \cdot 49r$ by property (8), (9). But this contradicts property (10). Hence property (11) is proved.

Finally by elementary calculation we see that in the extremal case $a = q$ where q satisfies (2).

REFERENCES

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