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The Fredholm theory of integral equations for special types of compact operators on a separable Hilbert space

by

H. J. Brascamp

1. Introduction

In the classical Fredholm theory (Fredholm [8], Plemelj [9]) one is concerned with the integral equation

$$f(x) = f_0(x)+\lambda \int K(x, y)f(y)dy. \tag{1.1}$$

The function $f_0(x)$ is continuous on a finite interval $[a, b]$, and $K(x, y)$ is continuous on $[a, b]\times[a, b]$; one seeks the continuous functions $f(x)$ satisfying the equation.

The theory has been made applicable to more general functional equations of the form

$$f = f_0+\lambda Kf. \tag{1.2}$$

For example, f_0 and f may be elements of some general Banach space (Ruston [10], [11], Leżański [13]); the Fredholm solution is then valid for K belonging to the trace-class of operators on this Banach space. The trace-class thus seems to be the natural domain for the Fredholm theory.

However, a modification of the formulae has been made by Carleman [15] and Smithies [1] to make them applicable to operators K belonging to the Schmidt-class of a separable Hilbert space. In the present paper a further modification is carried out in order to make it possible for K to belong to larger classes of operators. The urge for it was aroused by the author's work under the supervision of Prof. Clasine van Winter on finite systems of interacting particles [14].

In section 3 the equation (1.2) will be solved for operators K such, that $(K^*K)^2$ belongs to the trace-class. This immediately leads to the solution of the integral equation (1.1), where K is a special type of integral operator introduced in section 2.3.

All this can easily be carried over to operators K with $(K^*K)^m$ $(m = 1, 2, 3, \cdots)$ belonging to the trace-class. Some indications will be given in section 4.

2. Compact operators

2.1 The classes $\mathfrak{K}_p$

In this section we will repeat some properties of bounded linear operators on a separable Hilbert space $\mathfrak{H}$, as given by Schatten [3].

The set of bounded linear operators A, with the bound $||A||$ as a norm, is a Banach algebra $\mathfrak{B}$. A subset $\mathfrak{J} \subseteqq \mathfrak{B}$ is an *ideal* in $\mathfrak{B}$, if it follows from $A \in \mathfrak{J}$ and $X, Y \in \mathfrak{B}$ that $XAY \in \mathfrak{J}$. Moreover, $\mathfrak{J}$ is a *norm ideal* if the following conditions are fulfilled ([3], Ch. V. 1).

a) On I a norm α is defined, satisfying

$$\alpha(A) \geqq 0;\ \alpha(A) = 0 \text{ precisely if } A = 0$$

$$\alpha(cA) = |c|\alpha(A) \qquad \text{for any complex number } c \tag{2.1}$$

$$\alpha(A_1+A_2) \leqq \alpha(A_1)+\alpha(A_2)$$

$$\alpha(A) = ||A|| \text{ for operators } A \text{ of rank } 1 \tag{2.2}$$

$$\alpha(XAY) \leqq ||X||\alpha(A)||Y|| \text{ for } A \in \mathfrak{J} \text{ and } X, Y \in \mathfrak{B} \tag{2.3}$$

b) $\mathfrak{J}$ is complete with respect to the norm α.

An operator A is *compact* or *completely continuous* ([3], Ch. I) precisely if it has a polar decomposition

$$A = \sum_i \lambda_i \phi_i \otimes \psi_i^*, \tag{2.4}$$

with orthonormal sequences $\{\phi_i\}$ and $\{\psi_i\}$ and positive numbers λ_i; the sum may be finite or infinite, in the latter case λ_i tending to 0 for $i \to \infty$.

The bound of an operator of the form (2.4) is

$$||A|| = \max_i \lambda_i.$$

With this bound as a norm, the class $\mathfrak{C}$ of compact operators is a norm ideal in $\mathfrak{B}$.

Now introduce the subsets $\mathfrak{K}_p(1 \leqq p < \infty)$ of $\mathfrak{C}$. The class $\mathfrak{K}_p$ consists of the compact operators A for which the series $\sum_i \lambda_i^p$ converges. With the norm

$$||A||_p = (\sum_i \lambda_i^p)^{1/p} \tag{2.5}$$

$\mathfrak{K}_p$ is a norm ideal in $\mathfrak{B}$ ([3], Ch. V. 6—7). It is clear that A^* and A have the same p-norms, and that

$$(2.6) \qquad ||A||_p \geqq ||A||_q \geqq ||A||$$

for any $p \leqq q$.

Some special examples of classes $\mathfrak{K}_p$ are the following.

The Schmidt-class (σc) ([3], Ch. II) consists of the bounded operators with the property

$$(2.7) \quad (\sigma(A))^2 = \sum_i ||A\chi_i||^2 = \sum_{i,j} |(A\chi_i, \omega_j)|^2 = \sum_j ||A^*\omega_j||^2 < \infty.$$

The sets $\{\chi_i\}$ and $\{\omega_j\}$ are arbitrary complete orthonormal systems. An operator satisfying eq. (2.7) is compact. Hence, with eqs. (2.4) and (2.7), $\sigma(A) = ||A||_2$, so $(\sigma c) = \mathfrak{K}_2$.

The Schmidt-class is made a Hilbert space by the inner product

$$(2.8) \qquad (A, B) = \sum_i (A\chi_i, B\chi_i).$$

The trace-class (τc) ([3], Ch. III) consists of the operators which are the product of two Schmidt-operators. Let $A = BC$; the trace of A is then defined by

$$(2.9) \qquad \operatorname{tr} A = \sum_i (A\chi_i, \chi_i) = (C, B^*).$$

A norm τ is defined by

$$(2.10) \qquad \tau(A) = \operatorname{tr} (A^*A)^{\frac{1}{2}}$$

It is clear that $\tau(A) = ||A||_1$ and $(\tau c) = \mathfrak{K}_1$.

The norms σ and τ are connected by

$$(2.11) \qquad \begin{aligned} \tau(A^*A) = \tau(AA^*) &= [\sigma(A)]^2, \\ \tau(AB) &\leqq \sigma(A)\sigma(B). \end{aligned}$$

Finally, we will introduce the class (ρc) as the next step after $(\tau c) \to (\sigma c)$. The class (ρc) is defined to exist of the bounded operators A such that $A^*A \in (\sigma c)$. The corresponding norm is $\rho(A) = [\sigma(A^*A)]^{\frac{1}{2}}$.

Because $A^*A \in (\sigma c)$, it is compact; but then also $(A^*A)^{\frac{1}{2}}$ and A are compact ([3], Ch. I. 3—4). With eq. (2.4), $\rho(A) = ||A||_4$ and $(\rho c) = \mathfrak{K}_4$. The class (ρc) defined here is thus a norm ideal in $\mathfrak{B}$.

The norms ρ and σ are connected by formulae analogous to eq. (2.11), which will be derived here. Let $A, B \in (\rho c)$, $A = \sum_i \lambda_i \chi_i \otimes \omega_i^*$, $B = \sum_i \mu_i \phi_i \otimes \psi_i^*$.

Then

$$AB = \sum_{i,j} \lambda_i \mu_j \chi_i (\phi_j, \omega_i) \otimes \psi_j^*,$$

so

$$\begin{aligned}[\sigma(AB)]^2 = \sum_{k,l} |(\chi_k, AB\psi_l)|^2 &= \sum_{k,l} \lambda_k^2 \mu_l^2 |(\phi_l, \omega_k)|^2 \\ &\leqq [\sum_{k,l} \lambda_k^4 |(\phi_l, \omega_k)|^2]^{\frac{1}{2}} [\sum_{k,l} \mu_l^4 |(\phi_l, \omega_k)|^2]^{\frac{1}{2}} \\ &\leqq (\sum_k \lambda_k^4)^{\frac{1}{2}} (\sum_l \mu_l^4)^{\frac{1}{2}} = [\rho(A)]^2 [\rho(B)]^2.\end{aligned}$$

Hence

$$\begin{aligned}\sigma(A^*A) &= \sigma(AA^*) = [\rho(A)]^2, \\ \sigma(AB) &\leqq \rho(A)\rho(B).\end{aligned} \tag{2.12}$$

Combination of eqs. (2.11) and (2.12) gives

$$\tau(ABCD) \leqq \rho(A)\rho(B)\rho(C)\rho(D). \tag{2.13}$$

2.2 Operators on a finite dimensional Hilbert space

Let $\phi_1, \cdots, \phi_n$ be a basis for the n-dimensional Hilbert space $\mathfrak{H}_n$. An element f can be represented by n complex numbers (column vector) $f_i = (f, \phi_i)$. Every linear operator A can then be represented by an $n \times n$ matrix $A_{ij} = (A\phi_j, \phi_i)$. The operator A acts on an element f by $(Af)_i = \sum_j A_{ij} f_j$, the product of two operators is given by the matrix multiplication $(AB)_{ij} = \sum_k A_{ik} B_{kj}$.

In this case the classes $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{K}_p$, mentioned in the preceding section, coincide. The Schmidt-norm is found by (cf. eq. (2.7))

$$\sigma(A) = (\sum_{i,j} |A_{ij}|^2)^{\frac{1}{2}}, \tag{2.14}$$

the trace by (cf. eq. (2.9))

$$\operatorname{tr} A = \sum_i A_{ii}. \tag{2.15}$$

Every operator has precisely n eigenvalues α_i, viz. the solutions of the equation *det* $(A - \alpha) = 0$. An appropriate choice of the basis $\{\phi_i\}$ casts A_{ij} into the semi-diagonal form ([4], Ch. 9.13)

$$\begin{aligned}A_{ii} &= \alpha_i, \\ A_{ij} &= 0 \quad \text{for} \quad j < i.\end{aligned} \tag{2.16}$$

As a consequence, one has

$$\operatorname{tr} A^m = \sum_i \alpha_i^m, \qquad (m = 1, 2, \cdots) \tag{2.17}$$

$$\det A = \prod_i \alpha_i. \tag{2.18}$$

Finally, writing out $[\sigma(A^*A)]^2$ according to eq. (2.14),

$$\sum_i |\alpha_i|^4 \leqq [\rho(A)]^4. \tag{2.19}$$

2.3 Integral operators

The Hilbert space $L^2(X)$ consists of the (equivalence classes of) measurable and quadratic integrable functions on X. The inner product is

$$(f, g) = \int f(x)\bar{g}(x)dx.$$

For simplicity, X will be taken as a finite or infinite interval of $R^n \times Z^m$. That is, every point $x \in X$ is given by n real numbers x_i and m integer numbers s_i. The Lebesgue integral of an integrable function $h(x)$ is then

$$\int h(x)dx = \sum_{s_1 \cdots s_m} \int h(x_1, \cdots, x_n, s_1, \cdots, s_m)dx_1 \cdots dx_n.$$

For such X the Hilbert space $L^2(X)$ has denumerable infinite dimension (unless X is a finite interval of Z^m; then the dimension is finite).

In introducing the idea of integral operator, we will use the following strict definition. A linear operator A on $L^2(X)$ is called an integral operator, if there exists a measurable function $A(x, y)$ on $X \times X$ such that for every $f \in L^2(X)$

$$\int |A(x, y)f(y)|dy < \infty \tag{2.20}$$

and

$$(Af)(x) = \int A(x, y)f(y)dy \in L^2(X), \tag{2.21}$$

both for almost every x. The function $A(x, y)$ is the integral kernel of A.

A special class of integral operators has been introduced by Zaanen ([2], Ch. 9, § 7—8). It consists of the integral operators A whose kernels $A(x, y)$ satisfy

$$\int |A(x, y)f(y)|dy \in L^2(X) \tag{2.22}$$

for every $f \in L^2(X)$.

This class of operators has the following important properties. Let the operators A, A_1 and A_2 satisfy eq. (2.22). Then the adjoint operator A^* has the integral kernel

$$A^*(x, y) = \bar{A}(y, x), \tag{2.23}$$

and the product A_1A_2 has the integral kernel

$$(A_1A_2)(x, y) = \int A_1(x, z)A_2(z, y)dz. \tag{2.24}$$

The integral (2.24) converges absolutely for almost every (x, y). The operators A^* and A_1A_2 again have property (2.22). Finally, the operators with property (2.22) are bounded.

Let us now confine our attention to the classes of operators, defined in section 2.1. The Schmidt-class (σc) of operators on $L^2(X)$ precisely consists of the integral operators whose kernels are functions in $L^2(X \times X)$. (The proof given in [3], Ch. II.2 for $L^2[0, 1]$ also holds for $L^2(X)$). The inner product defined in eq. (2.8) corresponds to the inner product in $L^2(X \times X)$, that is

$$\begin{aligned} (A, B) &= \int A(x, y)\bar{B}(x, y)dxdy, \\ [\sigma(A)]^2 &= \int |A(x, y)|^2 dxdy. \end{aligned} \tag{2.25}$$

The Schmidt-operators clearly satisfy eq. (2.22), and therefore also eqs. (2.23) and (2.24). Let now A belong to the trace-class, so $A = BC$ with $B, C \in (\sigma c)$. The integral kernel of A is

$$A(x, y) = \int B(x, z)C(z, y)dz.$$

With eq. (2.9),

$$\operatorname{tr} A = (C, B^*) = \int C(x, y)B(y, x)dxdy = \int A(x, x)dx. \tag{2.26}$$

Unlike the Schmidt-class, the class (ρc) does not merely consist of integral operators. This will be shown by the following example.

The function $q(x)$ $(-\infty < x < \infty)$ has period 2, and

$$q(x) = |x|^{-\frac{3}{2}} \exp(-ix^{-5}), \quad -1 < x \leqq 1.$$

The operator Q on $L^2[-1, 1]$ is defined by

$$(Qf)(x) = \int_{-1}^{1} q(x-y)f(y)dy. \tag{2.27}$$

Since Q does not satisfy eq. (2.20) it is not an integral operator in the sense used here. Now choose on $L^2[-1, 1]$ the complete orthonormal set $\{\phi_n\}$, $-\infty < n < \infty$,

$$\phi_n(x) = \frac{1}{\sqrt{2}} \exp(in\pi x).$$

Then clearly

$$Q = \sum_{n=-\infty}^{\infty} \hat{q}_n \phi_n \otimes \phi_n^*,$$

where

$$\hat{q}_n = \frac{1}{\sqrt{2}} \int_{-1}^{1} q(x) \exp(-in\pi x) dx = \sqrt{2} \int_0^1 x^{-\frac{3}{2}} \cos(n\pi x + x^{-5}) dx.$$

Owing to Titchmarsh ([5], section 4.11) the numbers $\hat{q}_n$ are finite, and

$$\hat{q}_n = O(n^{-\frac{1}{3}}) \quad \text{for} \quad n \to \infty,$$
$$\hat{q}_n = O(|n|^{-\frac{3}{4}}) \quad \text{for} \quad n \to -\infty.$$

Hence

$$\sum_n |\hat{q}_n|^4 < \infty,$$

that is, the operator Q belongs to the class (ρc).

We will now introduce a subset of (ρc) which only contains integral operators. First consider a measurable function $A(x, y)$ satisfying

$$\begin{aligned} &\int |A(x, y)A(x, y')A(x', y)A(x', y')| dx dx' dy dy' \\ (2.28) \qquad &= \int dx dx' \left[\int |A(x, y)A(x', y)| dy\right]^2 \\ &= \int dy dy' \left[\int |A(x, y)A(x, y')| dx\right]^2 = \left[r(A)\right]^4 < \infty. \end{aligned}$$

According to the theorems by Fubini and Tonelli, the order of integration is arbitrary ([7], Ch. XII. 2).

Take any $f \in L^2(X)$. Then, with Schwarz's inequality,

$$\begin{aligned} &\int dx \left[\int |A(x,y)f(y)| dy\right]^2 = \int |A(x,y)A(x,y')f(y)f(y')| dx dy dy' \\ (2.29) \qquad &\leqq \left[|f(y)f(y')|^2 dy dy'\right]^{\frac{1}{2}} \left\{\int dy dy' \left[\int |A(x,y)A(x,y')| dx\right]^2\right\}^{\frac{1}{2}} \\ &= ||f||^2 [r(A)]^2. \end{aligned}$$

Hence, the function $A(x, y)$ may be considered as the kernel of an integral operator with property (2.22). It further follows from eq. (2.29) that, for every $f, g \in L^2(X)$,

$$(2.30) \qquad \int |g(x)A(x, y)f(y)| dx dy \leqq ||f|| \, ||g|| r(A).$$

The set of measurable functions $A(x, y)$ satisfying eq. (2.28) defines a class of integral operators, which will be denoted by (rc). We have seen that operators belonging to this class satisfy eq. (2.22), and therefore eqs. (2.23) and (2.24).

Particularly, eq. (2.23) yields

$$r(A^*) = r(A). \tag{2.31}$$

LEMMA 2.1. *An estimation of the size of the class* (rc) *is given by*

$$(\sigma c) \subseteqq (rc) \subseteqq (\rho c). \tag{2.32}$$

The equalities only hold in the case of a finite dimensional Hilbert space. If $A \in (rc)$, $B \in (\sigma c)$,

$$\rho(A) \leqq r(A); \quad r(B) \leqq \sigma(B). \tag{2.33}$$

PROOF. Let $A \in (rc)$. By eqs. (2.23) and (2.24), the integral kernel of A^*A is

$$(A^*A)(x, y) = \int \bar{A}(z, x)A(z, y)dz.$$

Since $\rho(A) = [\sigma(A^*A)]^{\frac{1}{2}}$ by definition, eqs. (2.25) and (2.28) give, that $\rho(A) \leqq r(A)$, and so $(rc) \subseteqq (\rho c)$. We find further that the ρ-norm of an operator $A \in (rc)$ is given by

$$\begin{aligned}[\rho(A)]^4 &= \int \bar{A}(x, y)A(x, y')A(x', y)\bar{A}(x', y')dx\,dx'\,dy\,dy' \\ &= \int dx\,dx' \left|\int A(x, y)\bar{A}(x', y)dy\right|^2 \\ &= \int dy\,dy' \left|\int \bar{A}(x, y)A(x, y')dx\right|^2.\end{aligned} \tag{2.34}$$

Let now $B \in (\sigma c)$. Denote by B_0 the operator with integral kernel $|B(x, y)|$. Then $r(B) = \rho(B_0)$ and $\sigma(B) = \sigma(B_0)$. Because $\rho(B_0) \leqq \sigma(B_0)$ (eq. (2.6)), it follows that $r(B) \leqq \sigma(B)$, hence $(rc) \supseteqq (\sigma c)$.

For a finite dimensional Hilbert space $(\rho c) = (\sigma c)$, so

$$(\sigma c) = (rc) = (\rho c).$$

If the dimension is infinite, the example (2.27) shows that (rc) is really smaller than (ρc). Now consider operators with integral kernels

$$g(x)h(x-y) \quad \text{or} \quad h(x-y)g(y),$$

such that $g \in L^4(X)$ and $h \in L^{\frac{4}{3}}(X)$. It immediately follows from eq. (2.28) that these operators belong to (rc). If, however, not both g and h belong to $L^2(X)$, they do not belong to (σc). This proves that (σc) is really smaller than (rc).

LEMMA 2.2. *With the quantity* $r(A)$ *considered as the norm of* $A \in (rc)$, *the class* (rc) *is a Banach algebra.*

PROOF. Let us first show that $r(A)$ is really a norm. The only non-trivial relations to be verified are

$$r(AB) \leqq r(A)r(B); \quad r(A+B) \leqq r(A)+r(B). \tag{2.35}$$

The first one follows from eqs. (2.12) and (2.33),

$$r(AB) \leqq \sigma(AB) \leqq \rho(A)\rho(B) \leqq r(A)r(B).$$

For the second one, remark that

$$|A(x, y)+B(x, y)| \leqq |A(x, y)|+|B(x, y)|.$$

Hence

$$\begin{aligned} r(A+B) &\leqq r(A_0+B_0) \\ &= \rho(A_0+B_0) \leqq \rho(A_0)+\rho(B_0) = r(A)+r(B). \end{aligned}$$

It only remains to prove that (rc) is complete. Choose a Cauchy sequence $\{A_n\} \in (rc)$. Then there exists a set of indices n_i such that $n_i < n_{i+1}$ and

$$r(A_{n_{i+1}}-A_{n_i}) \leqq 2^{-i}.$$

Define $B_1 = A_{n_1}$, $B_{i+1} = A_{n_{i+1}}-A_{n_i}$. Then, for every $f, g \in L^2(X)$

$$\sum_{i=1}^{\infty} \int |g(x)B_i(x, y)f(y)|dx dy \leqq \sum_i ||f||\,||g||r(B_i) < \infty.$$

As a consequence, $\sum_{i=1}^{\infty} |B_i(x, y)|$ converges for almost every (x, y). Define $A(x, y) = \sum_{i=1}^{\infty} B_i(x, y)$. Then

$$r(A) \leqq \sum_{i=1}^{\infty} r(B_i) < \infty,$$

so $A \in (rc)$. Further

$$r(A-A_{n_j}) \leqq \sum_{i=j+1}^{\infty} r(B_i) \leqq 2^{-j+1}.$$

Because $r(A-A_n) \leqq r(A-A_{n_j})+r(A_{n_j}-A_n)$, it follows that $\lim_{n\to\infty} r(A-A_n) = 0$. This proves the lemma.

During the proof, the following property has been found. For every sequence $\{A_n\}$ converging to A in the r-norm, there exists a subsequence $\{A_{n_i}\}$ such that $\{A_{n_i}(x, y)\}$ converges to $A(x, y)$ almost everywhere.

To conclude this section, remark that the class (rc) is not an ideal in $\mathfrak{B}$. For as the example (2.27) shows, (rc) is not unitarily invariant.

3. The functional equation $f = f_0+\lambda Kf$

3.1. The equation in the space C^p

The solution of the equation

$$f = f_0+\lambda Kf \tag{3.1}$$

in the p-dimensional Euclidian space C^p will be cast in a shape that remains valid for a Hilbert space of denumerable infinite dimension, with an operator K belonging to the class (ρc). For a smaller class of operators, the Schmidt-class (σc), this has been worked out by Smithies ([1], Ch. VI). See further Zaanen ([2], Ch. 9, § 16—17).

Eq. (3.1) has a unique solution if and only if λ^{-1} is not an eigenvalue of K; the matrix $1-\lambda K$ is then non-singular, its inverse being

$$(1-\lambda K)^{-1} = \frac{D(\lambda)}{d(\lambda)}, \tag{3.2}$$

with

$$\begin{aligned} d(\lambda) &= \det\,(1-\lambda K), \\ D(\lambda) &= \operatorname{adj}\,(1-\lambda K), \end{aligned} \tag{3.3}$$

that is

$$(D(\lambda)g, f) = -\det \begin{pmatrix} 0 & \bar{f}_1 \cdots \bar{f}_p \\ g_1 & \\ \cdots & 1-\lambda K \\ g_p & \end{pmatrix} \tag{3.4}$$

The quantities $d(\lambda)$ and $D(\lambda)$ are polynomials of degree p,

$$d(\lambda) = \sum_n d_n\lambda^n; \quad D(\lambda) = \sum_n D_n\lambda^n.$$

The coefficients follow from the recurrence relations

$$\begin{aligned} &D_0 = 1; \quad D_n = d_n+D_{n-1}K = d_n+KD_{n-1}\ (n \geqq 1); \\ &d_0 = 1; \quad d_n = -\frac{1}{n}\operatorname{tr} D_{n-1}K\ (n \geqq 1). \end{aligned}$$

Explicit expressions are (with $\sigma_j = tr\, K^j$)

$$d_n = \frac{(-1)^n}{n!}\det\begin{bmatrix} \sigma_1 & n-1 & 0 & \cdots & 0 & 0 \\ \sigma_2 & \sigma_1 & n-2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & \cdots & \sigma_1 & 1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_2 & \sigma_1 \end{bmatrix}, \tag{3.5}$$

$$D_n = \frac{(-1)^n}{n!} \det \begin{bmatrix} K & n & 0 & 0 & \cdots & 0 & 0 \\ K^2 & \sigma_1 & n-1 & 0 & \cdots & 0 & 0 \\ K^3 & \sigma_2 & \sigma_1 & n-2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ K^n & \sigma_{n-1} & \sigma_{n-2} & \sigma_{n-3} & \cdots & \sigma_1 & 1 \\ K^{n+1} & \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_2 & \sigma_1 \end{bmatrix}. \tag{3.5}$$

The preceding formulae are modified by Smithies as follows. Put $\delta(\lambda) = d(\lambda) \exp(\sigma_1\lambda)$ and $\Delta(\lambda) = D(\lambda) \exp(\sigma_1\lambda)$. The quantities $\delta(\lambda)$ and $\Delta(\lambda)$ are power series in λ. The coefficients δ_n and Δ_n are given by determinants of the form (3.5) with everywhere σ_1 replaced by 0 ([1], section 6.5 and [2], Ch. 9 § 16). The new formulae remain valid for a Hilbert space of denumerable infinite dimension with $K \in (\sigma c)$. The essential point is, that for $K \in (\sigma c)$ the traces $\sigma_j = tr\, K^j$ are finite for $j \geqq 2$ ([1], section 6.6 and [2], Ch. 9 § 17).

The theory will be extended here to an infinite dimensional Hilbert space with $K \in (\rho c)$. In that case, the traces σ_1, σ_2 and σ_3 do not necessarily exist; however, $\sigma_j < \infty$ for $j \geqq 4$ (eq. (2.13)). Therefore the formulae (3.3—5) are modified in such a way, that σ_1, σ_2 and σ_3 no longer occur. Define

$$\begin{aligned} \delta(\lambda) &= d(\lambda) \exp(\sigma_1\lambda+\tfrac{1}{2}\sigma_2\lambda^2+\tfrac{1}{3}\sigma_3\lambda^3), \\ \Delta(\lambda) &= D(\lambda) \exp(\sigma_1\lambda+\tfrac{1}{2}\sigma_2\lambda^2+\tfrac{1}{3}\sigma_3\lambda^3). \end{aligned} \tag{3.6}$$

The coefficients of the power series

$$\delta(\lambda) = \sum_n \delta_n\lambda^n; \quad \Delta(\lambda) = \sum_n \Delta_n\lambda^n \tag{3.7}$$

satisfy expressions like (3.5), with everywhere σ_1, σ_2 and σ_3 replaced by 0. The coefficients may as well be found from the recurrence relations

$$\begin{aligned} &\Delta_0 = 1; \;\; \Delta_n = \delta_n + \Delta_{n-1}K = \delta_n + K\Delta_{n-1} && (n \geqq 1); \\ &\delta_0 = 1; \;\; \delta_1 = \delta_2 = \delta_3 = 0; \;\; \delta_n = -\frac{1}{n}\operatorname{tr} \Delta_{n-4}K^4 && (n \geqq 4). \end{aligned} \tag{3.8}$$

These relations are proved in much the same way as Smithies's modified formulae.

3.2. Further investigation of the power series

In this section we will derive estimations for the quantities $\delta(\lambda)$, δ_n, $\|\Delta(\lambda)\|$ and $\|\Delta_n\|$.

Lemma 3.1. *For any complex number* z,

$$|(1-z)\exp(z+\tfrac{1}{2}z^2+\tfrac{1}{3}z^3)| \leqq \exp[\tfrac{3}{4}|z|^4].$$

Proof. Define $F(z) = |(1-z)\exp[z+\frac{1}{2}z^2+\frac{1}{3}z^3-\frac{3}{4}|z|^4]|^2$. With $z = x+iy$,

$$\begin{aligned} f(x, y) &= F(x+iy) \\ &= [(1-x)^2+y^2]\exp[2x+x^2-y^2+\tfrac{2}{3}x^3-2xy^2-\tfrac{3}{2}(x^2+y^2)^2]. \end{aligned}$$

Clearly, $f(x, y) \to 0$ for $x^2+y^2 \to \infty$. Straightforward calculations give, that $f(x, y)$ has maxima precisely for $(x, y) = (0, 0)$ and $(x, y) = (\frac{4}{3}, 0)$. The values in these points are $f(0, 0) = 1$ and $f(\frac{4}{3}, 0) = \frac{1}{9}\exp\frac{104}{81} < 1$. Hence, $f(x, y) \leqq 1$ for any (x, y), which concludes the proof.

Lemma 3.2. *Let* A *be any* $p \times p$ *matrix. Then*

$$|\det(1-A)\exp(\operatorname{tr} A+\tfrac{1}{2}\operatorname{tr} A^2+\tfrac{1}{3}\operatorname{tr} A^3)| \leqq \exp\{\tfrac{3}{4}[\rho(A)]^4\}.$$

Proof. Denote by α_i the eigenvalues of A. Then

$$\begin{aligned} &|\det(1-A)\exp(\operatorname{tr} A+\tfrac{1}{2}\operatorname{tr} A^2+\tfrac{1}{3}\operatorname{tr} A^3)| \\ &\quad = \prod_{i=1}^{p} |(1-\alpha_i)\exp(\alpha_i+\tfrac{1}{2}\alpha_i^2+\tfrac{1}{3}\alpha_i^3)| \\ &\quad \leqq \prod_{i=1}^{p}\exp[\tfrac{3}{4}|\alpha_i|^4] = \exp\tfrac{3}{4}[\sum_{i=1}^{p}|\alpha_i|^4] \leqq \exp\{\tfrac{3}{4}[\rho(A)]^4\}, \end{aligned}$$

with eqs. (2.17—19) and lemma 3.1.

Lemma 3.3. *The numbers* $\delta(\lambda)$ *and* δ_n *satisfy*

$$|\delta(\lambda)| \leqq \exp\{\tfrac{3}{4}|\lambda|^4[\rho(K)]^4\}, \tag{3.9}$$

$$|\delta_n| \leqq (3e/n)^{\frac{1}{4}n}[\rho(K)]^n. \tag{3.10}$$

Proof. Eqs. (3.3) and (3.6) yield

$$\delta(\lambda) = \det(1-\lambda K)\exp(\sigma_1\lambda+\tfrac{1}{2}\sigma_2\lambda^2+\tfrac{1}{3}\sigma_3\lambda^3).$$

Eq. (3.9) is now an immediate consequence of lemma 3.2. The function $\delta(\lambda)$ is analytic in the complex λ-plane, so Cauchy's inequality yields ([6], Section 2.5)

$$|\delta_n| \leqq r^{-n}\exp\{\tfrac{3}{4}r^4[\rho(K)]^4\}$$

for any $r > 0$. Taking $r = (n/3)^{\frac{1}{4}}[\rho(K)]^{-1}$, we find (3.10).

Lemma 3.4. *The operators* $\Delta(\lambda)$ *and* Δ_n *satisfy*

$$||\Delta(\lambda)|| \leqq \exp\{\tfrac{65}{12}+4|\lambda|\rho(K)+3|\lambda|^2[\rho(K)]^2+\tfrac{3}{4}|\lambda|^4[\rho(K)]^4\}, \tag{3.11}$$

$$||\Delta_n|| \leqq (3e/n)^{\frac{1}{4}n}[\rho(K)]^n\exp[\tfrac{65}{12}+4(n/3)^{\frac{1}{4}}+(3n)^{\frac{1}{2}}]. \tag{3.12}$$

PROOF. Eqs. (3.4) and (3.6) yield

$$(\Delta(\lambda)g, f) = -\det(1-A)\exp(\lambda\,\mathrm{tr}\,K+\tfrac{1}{2}\lambda^2\,\mathrm{tr}\,K^2+\tfrac{1}{3}\lambda^3\,\mathrm{tr}\,K^3), \tag{3.13}$$

with

$$A = \begin{pmatrix} 1 & \bar{f}_1 \cdots \bar{f}_p \\ g_1 & \\ \cdots & \lambda K \\ g_p & \end{pmatrix}$$

clearly,

$$\mathrm{tr}\,A = 1+\lambda\,\mathrm{tr}\,K. \tag{3.14}$$

Writing out the matrix multiplications A^2 and A^3, one finds

$$\mathrm{tr}\,A^2 = 1+2(g, f)+\lambda^2\,\mathrm{tr}\,K^2, \tag{3.15}$$

$$\mathrm{tr}\,A^3 = 1+3(g, f)+3\lambda(Kg, f)+\lambda^3\,\mathrm{tr}\,K^3. \tag{3.16}$$

Application of eqs. (3.14—16) and lemma 3.2 to eq. (3.13) yields

$$|(\Delta(\lambda)g, f)| \leqq |\exp[-\tfrac{11}{6}-2(g, f)-\lambda(Kg, f)]|\exp\{\tfrac{3}{4}[\rho(A)]^4\}.$$

Write out the matrix product A^*A. With eq. (2.14),

$$[\rho(A)]^4 = [\sigma(A^*A)]^2 = 1+2||f||^2+2||g||^2+||f||^4+||g||^4$$
$$+4\,\mathrm{Re}\,\lambda(Kf, g)+2|\lambda|^2||Kf||^2+2|\lambda|^2||K^*g||^2+|\lambda|^4[\rho(K)]^4.$$

Substitute this result, and take $||f|| = ||g|| = 1$. Then

$$|(\Delta(\lambda)g, f)| \leqq \exp\{\tfrac{65}{12}+4|\lambda|\,||K||+3|\lambda|^2||K||^2+\tfrac{3}{4}|\lambda|^4[\rho(K)]^4\}$$
$$\leqq \exp\{\tfrac{65}{12}+4|\lambda|\rho(K)+3|\lambda|^2[\rho(K)]^2+\tfrac{3}{4}|\lambda|^4[\rho(K)]^4\},$$

for any $f, g \in C^p$ with $||f|| = ||g|| = 1$. This inequality is equivalent with eq. (3.11).

Application of Cauchy's inequality to the analytic function $(\Delta(\lambda)g, f)$ yields

$$|(\Delta_n g, f)| \leqq r^{-n}\exp\{\tfrac{65}{12}+4r\rho(K)+3r^2[\rho(K)]^2+\tfrac{3}{4}r^4[\rho(K)]^4\}.$$

Eq. (3.12) follows by taking $r = (n/3)^{\frac{1}{4}}[\rho(K)]^{-1}$.

3.3. Approximation of an operator $K \in (\rho c)$ by operators of finite rank.

Let the operator K belong to the class (ρc) of an infinite dimensional Hilbert space $\mathfrak{H}$. Its polar decomposition be

$$K = \sum_i \lambda_i \phi_i \otimes \psi_i^*,$$

with

$$[\rho(K)]^4 = \sum_i \lambda_i^4 < \infty.$$

Define ([2], Ch. 9, § 17)

$$K_p = P_p K P_p,$$

where P_p is the orthogonal projection

$$P_p = \sum_{i=1}^{p} \phi_i \otimes \phi_i^*.$$

An immediate consequence of eq. (2.3) is that

$$\rho(K_p) \leqq \rho(K). \tag{3.17}$$

LEMMA 3.5. *With operators K and K_p as defined above, the following properties hold true.*

i) $\lim_{p\to\infty} \rho(K-K_p) = 0.$

ii) $\lim_{p\to\infty} \rho(K^n-K_p^n) = 0$ for any $n \geqq 1$

iii) $\lim_{p\to\infty} \rho(K_p^n) = \rho(K^n)$ for any $n \geqq 1$

iv) With $\sigma_{np} = \operatorname{tr} K_p^n$ and $\sigma_n = \operatorname{tr} K^n$,
$\lim_{p\to\infty} \sigma_{np} = \sigma_n$ for any $n \geqq 4$

v) $\lim_{p\to\infty} (K_p^n g, f) = (K^n g, f)$ for any $n \geqq 1$, $f, g \in \mathfrak{H}$.

PROOF.

i) $$\begin{aligned}[\rho(K-K_p)]^2 &= \sigma(K^*(1-P)K+(1-P)K^*PK(1-P)) \\ &\leqq \sigma(K^*(1-P)K)+\sigma(K^*PK(1-P)).\end{aligned}$$

The two terms in the right member will be treated separately.

a) $K^*(1-P)K = \sum_{i>p} \lambda_i^2 \psi_i \otimes \psi_i$, so

$$\sigma(K^*(1-P)K) = \Big(\sum_{i>p} \lambda_i^4\Big)^{\frac{1}{2}} \leqq \varepsilon^2/2 \text{ for } p \geqq N_1.$$

b) $K^*PK(1-P) = \sum_{i\leqq p} \sum_{j>p} \lambda_i^2 \psi_i(\phi_j, \psi_i) \otimes \phi_j^*$, so

$$\begin{aligned}[\sigma(K^*PK(1-P))]^2 &= \sum_{j>p} \sum_{i\leqq p} \lambda_i^4 |(\phi_j, \psi_i)|^2 \\ &\leqq \sum_{j>p} \sum_{i\leqq Q} \lambda_i^4 |(\phi_j, \psi_i)|^2 + \sum_{i>Q} \lambda_i^4.\end{aligned}$$

First, choose Q so large that $\sum_{i>Q}\lambda_i^4 \leqq (\varepsilon^4/8)$. Now, with every fixed $i \leqq Q$ a number N_i exists such that

$$\sum_{j>p} |(\phi_j, \psi_i)|^2 \leqq \tfrac{1}{8}\varepsilon^4[\rho(K)]^{-4} \quad \text{for} \quad p \geqq N_i.$$

Then, for $p \geqq N_2 = \max_{i \leqq Q} N_i$,

$$\sum_{j>p} \sum_{i \leqq Q} \lambda_i^4 |(\phi_j, \psi_i)|^2 \leqq \varepsilon^4/8.$$

The result is, that

$$\sigma(K^* PK(1-P)) \leqq \varepsilon^2/2 \quad \text{for} \quad p \geqq N_2.$$

Combining a) and b), we find that

$$\rho(K-K_p) \leqq \varepsilon \quad \text{for} \quad p \geqq N = \max(N_1, N_2),$$

which concludes the proof of part i).

ii) $$\begin{aligned}\rho(K^n-K_p^n) &\leqq \rho\big(K^{n-1}(K-K_p)\big)+\rho\big((K^{n-1}-K_p^{n-1})K_p\big)\\ &\leqq ||K||^{n-1}\rho(K-K_p)+||K||\rho(K^{n-1}-K_p^{n-1})\\ &\leqq n||K||^{n-1}\rho(K-K_p).\end{aligned}$$

iii) $|\rho(K^n)-\rho(K_p^n)| \leqq \rho(K^n-K_p^n)$.

iv) With eqs. (2.3) and (2.13),

$$\begin{aligned}|\sigma_n-\sigma_{np}| &= |\mathrm{tr}\,(K^n-K_p^n)| \leqq \tau(K^n-K_p^n)\\ &\leqq \tau\big(K^{n-1}(K-K_p)\big)+\tau\big(K^{n-2}(K-K_p)K_p\big)+\cdots\\ &\quad +\tau\big((K-K_p)K_p^{n-1}\big)\\ &\leqq n||K||^{n-4}[\rho(K)]^3\rho(K-K_p).\end{aligned}$$

v) $|(K_p^n g,f)-(K^n g,f)| \leqq ||f||\,||g||\,||K_p^n-K^n|| \leqq ||f||\,||g||\rho(K_p^n-K^n)$.

The proof of lemma 3.5 has thus been concluded.

3.4. The equation in an infinite dimensional Hilbert space

Again the operator K is supposed to belong to the class (ρc) of the infinite dimensional Hilbert space $\mathfrak{H}$, so the traces $\sigma_n = \mathrm{tr}\, K^n$ are finite for $n \geqq 4$. It is therefore possible to define the quantities δ_n and Δ_n according to the recurrence relations (3.8).

Lemma 3.6. *The absolute value of δ_n and the bound of Δ_n satisfy eqs.* (3.10) *and* (3.12). *The power series*

$$\sum_{n=0}^{\infty} \delta_n \lambda^n \quad \text{and} \quad \sum_{n=0}^{\infty} (\Delta_n g, f)\lambda^n \tag{3.18}$$

define the analytic functions $\delta(\lambda)$ and $(\Delta(\lambda)g, f)$. The operator $\Delta(\lambda)$ thus defined is bounded.

Proof. For K_p instead of K (section 3.3), define by the recurrence relations (3.8) the quantities δ_{np} and Δ_{np}. The operators

K_p and Δ_{np} may then be considered as operators on a p-dimensional subspace of $\mathfrak{H}$, without difference for the traces and the norms. One may thus apply the lemmas 3.3 and 3.4 and eq. (3.17) to obtain

$$\begin{aligned} &|\delta_{np}| \leqq (3e/n)^{\frac{1}{4}n}[\rho(K)]^n, \\ (3.19) \qquad &|(\Delta_{np}g, f)| \leqq (3e/n)^{\frac{1}{4}n}[\rho(K)]^n \exp\left[\tfrac{65}{12}+4(n/3)^{\frac{1}{4}}+(3n)^{\frac{1}{2}}\right], \end{aligned}$$

for any p and $||f|| = ||g|| = 1$.

It follows now from the recurrence relations and lemma 3.5, that

$$\lim_{p\to\infty} \delta_{np} = \delta_n; \quad \lim_{p\to\infty} (\Delta_{np}g, f) = (\Delta_n g, f).$$

The required inequalities are obtained by letting $p \to \infty$ in eq. (3.19). The analyticity of $\delta(\lambda)$ and $(\Delta(\lambda)g, f)$ follows from the absolute convergence of the power series (3.18). Finally, note that the series $\sum||\Delta_n|| \, |\lambda|^n$ converges, so $\Delta(\lambda)$ is bounded and

$$(3.20) \qquad \lim_{N\to\infty} ||\Delta(\lambda) - \sum_{n=0}^{N} \Delta_n \lambda^n|| = 0.$$

Corollary. *The recurrence relations* (3.8) *yield*

$$\sum_{n=1}^{\infty} \Delta_n \lambda^n = \sum_{n=1}^{\infty} \delta_n \lambda^n + \sum_{n=1}^{\infty} \Delta_{n-1} K \lambda^n = \sum_{n=1}^{\infty} \delta_n \lambda^n + \sum_{n=1}^{\infty} K \Delta_{n-1} \lambda^n;$$

$$-\sum_{n=4}^{\infty} n\delta_n \lambda^{n-1} = \operatorname{tr}\left[\sum_{n=4}^{\infty} \Delta_{n-4} K^4 \lambda^{n-1}\right].$$

The series converge, so, with $\Delta_0 = 1$, $\delta_0 = 1$, $\delta_1 = \delta_2 = \delta_3 = 0$,

$$(3.21) \qquad \delta(\lambda) = \Delta(\lambda)(1-\lambda K) = (1-\lambda K)\Delta(\lambda);$$

$$(3.22) \qquad -\frac{d}{d\lambda}\delta(\lambda) = \lambda^3 \operatorname{tr} \Delta(\lambda) K^4.$$

The following theorems give the unique solution of the functional equation.

Theorem 3.7. *The function* $\delta(\lambda)$ *has a zero for* $\lambda = \lambda_i$, *precisely if the homogeneous equation* $f = \lambda_i K f$ *has a solution* $f \neq 0$; *that is, if* λ_i^{-1} *is an eigenvalue of* K.

Proof. Let $(1-\lambda_i K) f_i = 0$, with $f_i \neq 0$.
Eq. (3.21) yields then

$$\delta(\lambda_i) f_i = \Delta(\lambda_i)(1-\lambda_i K) f_i = 0,$$

that is, $\delta(\lambda_i) = 0$.

On the other hand, let $\delta(\lambda_i) = 0$. Because $\delta(\lambda)$ is an analytic function, and $\delta(\lambda) \not\equiv 0$ ($\delta(0) = \delta_0 = 1$), it can have only isolated zeros λ_i, in the neighbourhood of which

$$\delta(\lambda) = (\lambda-\lambda_i)^q \sum_{n=0}^{\infty} \alpha_n(\lambda-\lambda_i)^n \qquad (\alpha_0 \neq 0).$$

Then $\delta(\lambda)$ has a zero of order q, and

$$\delta(\lambda_i) = \delta'(\lambda_i) = \cdots = \delta^{(q-1)}(\lambda_i) = 0,\ \delta^{(q)}(\lambda_i) \neq 0.$$

Let us assume now that the equation $f = \lambda_i Kf$ has the solution $f = 0$ only. We will show that this leads to a contradiction.

Because $\delta(\lambda_i) = 0$,

$$(1-\lambda_i K)\Delta(\lambda_i)f = \delta(\lambda_i)f = 0$$

for any $f \in \mathfrak{H}$, so $\Delta(\lambda_i)f = 0$ for any $f \in \mathfrak{H}$, or $\Delta(\lambda_i) = 0$. Differentiation of eq. (3.21) yields, for $\lambda = \lambda_i$,

$$(1-\lambda_i K)\Delta'(\lambda_i)-K\Delta(\lambda_i) = \delta'(\lambda_i) = 0,$$

and hence $\Delta'(\lambda_i) = 0$. Repeating this procedure, one gets, after subsequently differentiating $q-1$ times,

$$\Delta(\lambda_i) = \Delta'(\lambda_i) = \cdots = \Delta^{(q-1)}(\lambda_i) = 0.$$

Now differentiate eq. (3.22) $q-1$ times, and put $\lambda = \lambda_i$.

$$-\delta^{(q)}(\lambda_i) = \operatorname{tr}\,[\text{lin. comb. of } \Delta(\lambda_i)K^4,\ \Delta'(\lambda_i)K^4, \cdots, \Delta^{(q-1)}(\lambda_i)K^4].$$

But then $\delta^{(q)}(\lambda_i) = 0$, which is the promised contradiction. The equation $f = \lambda_i Kf$ thus necessarily has a solution $f \neq 0$, which concludes the proof of this theorem.

Theorem 3.8. *If $\delta(\lambda) \neq 0$, the equation $f = f_0+\lambda Kf$ has at most one solution.*

Proof. Let f_1 and f_2 be two solutions. Then

$$f_1-f_2 = \lambda K(f_1-f_2).$$

Because $\delta(\lambda) \neq 0$, $f_1 = f_2$ by theorem 3.7.

Theorem 3.9. *If $\delta(\lambda) \neq 0$, the equation*

$$f = f_0+\lambda Kf$$

has the unique solution

$$f = \frac{\Delta(\lambda)}{\delta(\lambda)} f_0. \qquad (3.23)$$

Proof. This final result immediately follows from eq. (3.21) and the theorems 3.7 and 3.8.

3.5. Integral equations on $L^2(X)$

In order to make the solution of theorem 3.9 applicable to integral equations, we introduce the following quantities.

$$E_n = K\Delta_n;\quad E(\lambda) = K\Delta(\lambda);\quad Z_n = K^2\Delta_n;\quad Z(\lambda) = K^2\Delta(\lambda).$$

Since K belongs to (ρc) and Δ_n and $\Delta(\lambda)$ are bounded, the operators E_n and $E(\lambda)$ belong to (ρc) and Z_n and $Z(\lambda)$ belong to (σc). According to eq. (3.20), the series

$$E(\lambda) = \sum_{n=0}^{\infty} E_n\lambda^n \tag{3.24}$$

converges in the ρ-norm, and the series

$$Z(\lambda) = \sum_{n=0}^{\infty} Z_n\lambda^n \tag{3.25}$$

converges in the σ-norm. It follows from eq. (3.21), that

$$\frac{\Delta(\lambda)}{\delta(\lambda)} = 1+\lambda\,\frac{E(\lambda)}{\delta(\lambda)} = 1+\lambda K+\lambda^2\,\frac{Z(\lambda)}{\delta(\lambda)}. \tag{3.26}$$

Substituted in eq. (3.23), these expressions provide somewhat different forms of the solution. The quantities $E(\lambda)$ and $Z(\lambda)$ may be found directly from the recurrence relations (cf. eq. (3.8))

$$\begin{aligned} &E_0 = K;\quad E_n = \delta_n K+KE_{n-1} = \delta_n K+E_{n-1}K && (n\geqq 1);\\ &\delta_0 = 1;\quad \delta_1 = \delta_2 = \delta_3 = 0;\quad \delta_n = -\frac{1}{n}\operatorname{tr} E_{n-4}K^3 && (n\geqq 4), \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &Z_0 = K^2;\quad Z_n = \delta_n K^2+KZ_{n-1} = \delta_n K^2+Z_{n-1}K && (n\geqq 1);\\ &\delta_0 = 1;\quad \delta_1 = \delta_2 = \delta_3 = 0;\quad \delta_n = -\frac{1}{n}\operatorname{tr} Z_{n-4}K^2 && (n\geqq 4). \end{aligned} \tag{3.28}$$

Now suppose that the Hilbert space is $L^2(X)$, and let K be an operator belonging to (rc). Then it follows from

$$\begin{aligned} E(\lambda) &= \delta(\lambda)K+\lambda Z(\lambda),\\ E_n &= \delta_n K+Z_{n-1}, \end{aligned} \tag{3.29}$$

that the operators $E(\lambda)$ and E_n belong to the class (rc) and that the series (3.24) converges in the r-norm. Further it follows from the fact, that the series

$$\sum_{n=0}^{\infty} r(E_n) \quad \text{and} \quad \sum_{n=0}^{\infty} \sigma(Z_n)$$

converge and from arguments like those, given in the proof of lemma 2.2, that the series

$$E(x, y; \lambda) = \sum_{n=0}^{\infty} E_n(x, y)\lambda^n; \tag{3.30}$$

$$Z(x, y; \lambda) = \sum_{n=0}^{\infty} Z_n(x, y)\lambda^n \tag{3.31}$$

converge absolutely for almost every (x, y).

The solution of the equation

$$f(x) = f_0(x)+\lambda \int K(x, y)f(y)dy$$

now can be written as

$$\begin{aligned} f(x) &= f_0(x)+\frac{\lambda}{\delta(\lambda)}\int E(x, y; \lambda)f_0(y)dy \\ &= f_0(x)+\lambda\int K(x, y)f_0(y)dy+\frac{\lambda^2}{\delta(\lambda)}\int Z(x, y; \lambda)f_0(y)dy. \end{aligned} \tag{3.32}$$

The kernels $E(x, y; \lambda)$ and $Z(x, y; \lambda)$ are found from the series (3.30) and (3.31); the coefficients $E_n(x, y)$ and $Z_n(x, y)$ satisfy recurrence relations which follow from eqs. (3.27) and (3.28) by replacing the operators by their kernels and applying the rules (2.24) and (2.26) for the multiplication and the trace. The properties of the class (rc) yield the absolute convergence of all integrals occurring.

Let us conclude deriving inequalities for the kernels $E_n(x, y)$ and $Z_n(x, y)$. Eq. (3.28) yields

$Z_0 = K^2; \quad Z_1 = K^3;$

$Z_n = \delta_n K^2+\delta_{n-1}K^3+Z_{n-2}K^2 = \delta_n K^2+\delta_{n-1}K^2+K^2 \Delta_{n-2}K^2 \quad (n \geqq 2).$

That is,

$$\begin{aligned} Z_0(x, y) &= K_2(x, y) = \int K(x, z)K(z, y)dz \\ Z_1(x, y) &= K_3(x, y) = \int K(x, z)K(z, z')K(z', y)dz\,dz' \\ Z_n(x, y) &= \delta_n K_2(x, y)+\delta_{n-1}K_3(x, y)+M(x, y), \end{aligned}$$

where $M(x, y)$ is the integral kernel of $K^2\Delta_{n-2}K^2$.
Because $K^2 \in (\sigma c)$ and Δ_{n-2} is bounded, we have for almost every (x, y) ([2], Ch. 9, § 10)

$$|M(x, y)| \leqq ||\Delta_{n-2}||s(x)t(y),$$

where

$$s(x) = \left[\int |K_2(x, y)|^2 dy\right]^{\frac{1}{2}}$$

$$t(y) = \left[\int |K_2(x, y)|^2 dx\right]^{\frac{1}{2}}.$$

So almost everywhere

$$|Z_n(x, y)| \leqq |\delta_n| |K_2(x, y)| + |\delta_{n-1}| |K_3(x, y)| + ||\Delta_{n-2}|| s(x)t(y) \qquad (n \geqq 2).$$

Finally, it follows from eq. (3.29) that

$$|E_n(x, y)| \leqq |\delta_n| |K(x, y)| + |Z_{n-1}(x, y)|.$$

These inequalities once more prove the absolute convergence of the series (3.30) and (3.31).

4. Concluding remarks

1. Since the proof of theorem 3.9 only seems to depend on the first of the recurrence relations (3.8), one is lead to suppose some arbitrariness in the choice of $\delta(\lambda)$ and $\Delta(\lambda)$. In fact, Ruston [12] showed that any analytic function $\delta(\lambda)$ whose zeros are precisely the inverses of the eigenvalues of the compact operator K leads to a solution of the equation.

2. The arguments of sections 3.1—3.4 can be easily extended to operators K belonging to the class $\mathfrak{K}_{2m}$, $(m = 1, 2, 3, \cdots)$. The recurrence relations that define the solution become then

$$\Delta_0 = 1; \quad \Delta_n = \delta_n + K\Delta_{n-1} = \delta_n + \Delta_{n-1}K \qquad (n \geqq 1);$$

$$\delta_0 = 1; \quad \delta_1 = \delta_2 = \cdots \delta_{2m-1} = 0; \quad \delta_n = -\frac{1}{n} \operatorname{tr} \Delta_{n-2m} K^{2m} \qquad (n \geqq 2m).$$

In agreement with the first remark, the first recurrence relation is independent of m.

3. Finally, we will indicate how to define classes $\mathfrak{k}_{2m}$ of integral operators on $L^2(X)$ which are related to $\mathfrak{K}_{2m}$ in the same way as (rc) is related to (ρc). Given a function $A(x, y)$, define for notational convenience

$$A_0(x, y) = |A(x, y)|; \quad A_0^*(x, y) = |A(y, x)|.$$

Let now $A(x, y)$ be a measurable function such that

$$(|||A|||_{2m})^{2m} = \mathrm{tr}\,(A_0^* A_0)^m < \infty.$$

The functions $A(x, y)$ with these properties define the class $\mathfrak{k}_{2m}$ of integral operators. These integral operators have property (2.22). The class $\mathfrak{k}_{2m}$ with the norm $|||A|||_{2m}$ is a Banach algebra, and

$$\mathfrak{k}_{2m-2} \subseteqq \mathfrak{k}_{2m} \subseteqq \mathfrak{K}_{2m}.$$

We will only prove, for $m = 3$, the essential inequality

$$\int dx \left[\int |A(x, y)f(y)|dy\right]^2 \leqq ||f||^2(|||A|||_{2m})^2,$$

for any $f \in L^2(X)$. With $f_0(x) = |f(x)|$ we have

$$\begin{aligned} ||A_0 f_0||^4 &= (f_0, A_0^* A_0 f_0)^2 \leqq ||f||^2 ||A_0^* A_0 f_0||^2 \\ &= ||f||^2 (A_0 f_0, A_0 A_0^* A_0 f_0) \leqq ||f||^3 ||A_0 f_0|| \sigma(A_0 A_0^* A_0) \\ &= ||f||^3 ||A_0 f_0|| (|||A|||_6)^3. \end{aligned}$$

In fact, one should read integrals, like in eq. (2.29), for the norms and inner products. Going over to the adjoint operator then means changing the order of integration, which is justified because of the absolute convergence of the integrals.

The proofs for other numbers m proceed in the same way. The further properties of the classes $\mathfrak{k}_{2m}$ are proved by the methods given in the lemmas 2.1 and 2.2.

With the remarks 2 and 3 it will be clear, that the Fredholm theory for integral equations, given in section 3.5, can be extended to the case, that K is an integral operator belonging to $\mathfrak{k}_{2m}$.

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