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by

J. E. Vaughan and B. R. Wenner

In a 1963 issue of this journal, J. Nagata proved the following Theorem:

THEOREM. *A metric space R has* dim $\leqq n$ *if and only if we can introduce in R a topology-preserving metric ρ such that the spherical neighborhoods $S_\varepsilon(p)$, $\varepsilon > 0$ of every point p of R have boundaries of* dim $\leqq n-1$ *and such that $\{S_\varepsilon(p) : p \in R\}$ is closure-preserving for every $\varepsilon > 0$.* [2, *Theorem* 1].

Subsequently in the same article the author used the metric constructed in this Theorem to give proofs of the following two Corollaries:

COROLLARY 2. *A metric space R has* dim $\leqq n$ *if and only if we can introduce a topology-preserving metric ρ into R such that* $\dim C_\varepsilon(p) \leqq n-1$ *for any irrational* (*or for almost all*) $\varepsilon > 0$ *and for any point p of R and such that $\{C_\varepsilon(p) : p \in R\}$ is closure-preserving for any irrational* (*or for almost all*) $\varepsilon > 0$, *where* $C_\varepsilon(p) = \{q : \rho(p, q) = \varepsilon\}$.

COROLLARY 3. *A metric space R has* dim $\leqq n$ *if and only if we can introduce a topology-preserving metric ρ into R such that for all irrational* (*or for almost all*) *positive numbers ε and for any closed set F of R,* $\dim C_\varepsilon(F) \leqq n-1$, *where*

$$C_\varepsilon(F) = \{p : \rho(p, F) = \varepsilon\}.$$

The purpose of this communication is two-fold: first, to show why the proofs of these two Corollaries are invalid, and second, to show that in general no such result can be obtained.

1

1. The first objective will be obtained by using Nagata's procedure to construct an equivalent metric on the real line $\boldsymbol{R}$; we shall use the notation of [2] throughout. We define the follow-

ing sequence of open covers of $\boldsymbol{R}$: $\mathfrak{U}_0 = \{\boldsymbol{R}\}$, and for all $i > 0$ we let

$$\mathfrak{U}_i = \{(k \cdot 2^{-5(i-1)} - (\tfrac{3}{4}) \cdot 2^{-5(i-1)}, k \cdot 2^{-5(i-1)} + (\tfrac{3}{4}) \cdot 2^{-5(i-1)}) : k = 0, \pm 1, \pm 2, \cdots\}.$$

It is immediate that $\{\mathfrak{U}_i : i = 0, 1, 2, \cdots\}$ satisfies conditions (1), (2), and (3) in the proof of Theorem 1, and we define the metric ρ as in that proof.

The proof of Corollary 2 now purports to show that for any metric defined in this manner, any irrational $\varepsilon > 0$, and for any point p, $C_\varepsilon(p) = B[S_\varepsilon(p)]$. This is done by showing that $q \notin \overline{S_\varepsilon(p)}$ implies $q \notin C_\varepsilon(p)$. In our example $(\boldsymbol{R}, \rho)$, let us choose the irrational number $\varepsilon = 2^{-m_1} + 2^{-m_2} + \cdots$, where $m_i = \sum_{j=1}^{i} j$ for all $i = 1, 2, \cdots$; then choose $p = \frac{3}{4} + (\frac{7}{4}) \sum_{i=2}^{\infty} 2^{-5(m_i-1)}$, and $q = 0$. A routine calculation shows that

$$\mathfrak{S}_{m_1 m_2} \ldots = \{(k-p, k+p) : k = 0, \pm 1, \pm 2, \cdots\},$$

hence $S_\varepsilon(p) = S(p, \mathfrak{S}_{m_1 m_2} \ldots) = (1-p, 1+p)$; also, $q \notin \overline{S_\varepsilon(p)}$ as

$$p < \tfrac{3}{4} + 2 \sum_{i=2}^{\infty} 2^{-5(m_i-1)} < \tfrac{3}{4} + 2^{-8} < 1.$$

Now for $i = 2$ we see that $S(q, \mathfrak{U}_{m_i}) \cap S(p, \mathfrak{S}_{m_1 m_2} \ldots) = \emptyset$ and $m_{i+1} \geqq m_i + 2$, but the following statement, "Then it is easily seen that $q \notin S(p, \mathfrak{S}_{m_1 \ldots m_i m_i+1})$" [2, p. 232, top] is false. For let $t = \frac{3}{4} + (\frac{7}{4}) \cdot 2^{-10} + (\frac{7}{4}) \cdot 2^{-15}$; then $q = 0 \in (-t, t) \in \mathfrak{S}_{m_1 m_2 m_2+1}$. Moreover,

$$\begin{aligned} t - p &= \tfrac{3}{4} + (\tfrac{7}{4}) \cdot 2^{-10} + (\tfrac{7}{4}) \cdot 2^{-15} - \left(\tfrac{3}{4} + (\tfrac{7}{4}) \cdot 2^{-10} + (\tfrac{7}{4}) \sum_{i=3}^{\infty} 2^{-5(m_i-1)}\right) \\ &= (\tfrac{7}{4})(2^{-15} - \sum_{i=3}^{\infty} 2^{-5(m_i-1)}) \\ &> (\tfrac{7}{4})(2^{-15} - 2^{-24}) > 0, \end{aligned}$$

so $p \in (-t, t)$, hence $q \in S(p, \mathfrak{S}_{m_1 m_2 m_2+1})$.

In the case of $(\boldsymbol{R}, \rho)$ there is no possibility of avoiding this roadblock, as it is by no means true that $C_\varepsilon(p) = B[S_\varepsilon(p)]$ for irrational $\varepsilon > 0$. This can be seen by consideration of the ε and p used above. For all $r \in [-1+p, 1-p]$ we see that $S_\varepsilon(r) = (-p, p)$, so $p \in B[S_\varepsilon(r)]$, which implies $\rho(p, r) = \varepsilon$; thus

$$[-1+p, 1-p] \subset C_\varepsilon(p).$$

But

$$B[S_\varepsilon(p)] = B[(1-p, 1+p)] = \{1-p, 1+p\},$$

so $C_\varepsilon(p) \neq B[S_\varepsilon(p)]$. We note finally that

$$\dim C_\varepsilon(p) = 1 > 0 = \dim \boldsymbol{R}-1;$$

hence a metric constructed as in Theorem 1 does not necessarily have the property described in Corollary 2, nor that in Corollary 3.

2

Corollary 2 asserts the existence of a topology-preserving metric for any n-dimensional metric space R which satisfies the following two properties for all irrational $\varepsilon > 0$:

(i) $\dim C_\varepsilon(p) \leqq n-1$ for all $p \in R$, and

(ii) $\{C_\varepsilon(p) : p \in R\}$ is closure-preserving.

Although the space $(\boldsymbol{R}, \rho)$ can be shown to satisfy (ii), we have seen that it does not fulfill (i). On the other hand, $\boldsymbol{R}$ with the usual metric satisfies (i) but not (ii). The following Theorem demonstrates that a connected metric space of dimension greater than zero cannot simultaneously satisfy (i) and (ii) for small ε:

THEOREM. *Let (R, d) be a connected space with at least two points, $n > 0$, and $0 < \varepsilon < \frac{1}{2}\operatorname{diam}(R)$. If* (i) *and* (ii) *are satisfied for this n and ε, then* $\dim R \leqq n-1$.

PROOF. Let $p \in R$; the set $B = \{z : d(p, z) > \varepsilon\} \neq \emptyset$ (if not, then for all $x, y \in R$ we have $d(x, y) \leqq d(x, p)+d(y, p) \leqq 2\varepsilon$, so $\operatorname{diam}(R) \leqq 2\varepsilon$, which contradicts the hypothesis). Hence there exists a point $q \in R$ such that $d(p, q) = \varepsilon$, for otherwise the two nonempty sets $S_\varepsilon(p)$ and B would yield a separation of the connected space R. Hence $p \in R$ implies $p \in C_\varepsilon(q)$ for some $q \in R$, so $R = \cup \{C_\varepsilon(q) : q \in R\}$. By a Theorem of Nagami [1, Theorem 1], conditions (i) and (ii) imply $\dim R \leqq n-1$.

This Theorem demonstrates that for the above class of spaces Corollary 2 is invalid. It remains an open question as to whether or not Corollary 3 is invalid for a similar class of spaces.

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