COMPOSITIO MATHEMATICA

J. E. VAUGHAN B. R. WENNER Note on a theorem of J. Nagata

Compositio Mathematica, tome 21, nº 1 (1969), p. 4-6 <http://www.numdam.org/item?id=CM 1969 21 1 4 0>

© Foundation Compositio Mathematica, 1969, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (http: //http://www.compositio.nl/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

\mathcal{N} umdam

Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques http://www.numdam.org/

Note on a theorem of J. Nagata

J. E. Vaughan and B. R. Wenner

In a 1963 issue of this journal, J. Nagata proved the following Theorem:

THEOREM. A metric space R has dim $\leq n$ if and only if we can introduce in R a topology-preserving metric ρ such that the spherical neighborhoods $S_{\varepsilon}(p), \varepsilon > 0$ of every point p of R have boundaries of dim $\leq n-1$ and such that $\{S_{\varepsilon}(p) : p \in R\}$ is closure-preserving for every $\varepsilon > 0$. [2, Theorem 1].

Subsequently in the same article the author used the metric constructed in this Theorem to give proofs of the following two Corollaries:

COROLLARY 2. A metric space R has dim $\leq n$ if and only if we can introduce a topology-preserving metric ρ into R such that dim $C_{\varepsilon}(p) \leq n-1$ for any irrational (or for almost all) $\varepsilon > 0$ and for any point p of R and such that $\{C_{\varepsilon}(p) : p \in R\}$ is closurepreserving for any irrational (or for almost all) $\varepsilon > 0$, where $C_{\varepsilon}(p) = \{q : \rho(p, q) = \varepsilon\}$.

COROLLARY 3. A metric space R has dim $\leq n$ if and only if we can introduce a topology-preserving metric ρ into R such that for all irrational (or for almost all) positive numbers ε and for any closed set F of R, dim $C_{\varepsilon}(F) \leq n-1$, where

$$C_{\varepsilon}(F) = \{p : \rho(p, F) = \varepsilon\}.$$

The purpose of this communication is two-fold: first, to show why the proofs of these two Corollaries are invalid, and second, to show that in general no such result can be obtained.

1

1. The first objective will be obtained by using Nagata's procedure to construct an equivalent metric on the real line R; we shall use the notation of [2] throughout. We define the follow-

by

ing sequence of open covers of $R: \mathfrak{U}_0 = \{R\}$, and for all i > 0 we let

$$\mathfrak{U}_i = \{(k \cdot 2^{-5(i-1)} - (\frac{3}{4}) \cdot 2^{-5(i-1)}, k \cdot 2^{-5(i-1)} + (\frac{3}{4}) \cdot 2^{-5(i-1)}): k = 0, \pm 1, \pm 2, \cdots \}$$

It is immediate that $\{\mathfrak{U}_i : i = 0, 1, 2, \cdots\}$ satisfies conditions (1), (2), and (3) in the proof of Theorem 1, and we define the metric ρ as in that proof.

The proof of Corollary 2 now purports to show that for any metric defined in this manner, any irrational $\varepsilon > 0$, and for any point p, $C_{\varepsilon}(p) = B[S_{\varepsilon}(p)]$. This is done by showing that $q \notin \overline{S_{\varepsilon}(p)}$ implies $q \notin C_{\varepsilon}(p)$. In our example (\mathbf{R}, ρ) , let us choose the irrational number $\varepsilon = 2^{-m_1} + 2^{-m_2} + \cdots$, where $m_i = \sum_{j=1}^{i} j$ for all $i = 1, 2, \cdots$; then choose $p = \frac{3}{4} + (\frac{7}{4}) \sum_{i=2}^{\infty} 2^{-5(m_i-1)}$, and q = 0. A routine calculation shows that

$$\mathfrak{S}_{m_1m_2}\ldots = \{(k-p, k+p): k = 0, \pm 1, \pm 2, \cdots\},$$

hence $S_{\varepsilon}(p) = S(p, \mathfrak{S}_{m_1m_2}...) = (1-p, 1+p)$; also, $q \notin \overline{S_{\varepsilon}(p)}$ as $p < \frac{3}{4} + 2\sum_{i=2}^{\infty} 2^{-5(m_i-1)} < \frac{3}{4} + 2^{-8} < 1.$

Now for i = 2 we see that $S(q, \mathfrak{U}_{m_i}) \cap S(p, \mathfrak{S}_{m_1m_2}...) = \emptyset$ and $m_{i+1} \ge m_i + 2$, but the following statement, "Then it is easily seen that $q \notin S(p, \mathfrak{S}_{m_1...m_im_i+1})$ " [2, p. 232, top] is false. For let $t = \frac{3}{4} + (\frac{7}{4}) \cdot 2^{-10} + (\frac{7}{4}) \cdot 2^{-15}$; then $q = 0 \in (-t, t) \in \mathfrak{S}_{m_1m_2m_2+1}$. Moreover,

$$\begin{split} t-p &= \frac{3}{4} + \left(\frac{7}{4}\right) \cdot 2^{-10} + \left(\frac{7}{4}\right) \cdot 2^{-15} - \left(\frac{3}{4} + \left(\frac{7}{4}\right) \cdot 2^{-10} + \left(\frac{7}{4}\right) \sum_{i=3}^{\infty} 2^{-5(m_i-1)}\right) \\ &= \left(\frac{7}{4}\right) \left(2^{-15} - \sum_{i=3}^{\infty} 2^{-5(m_i-1)}\right) \\ &> \left(\frac{7}{4}\right) \left(2^{-15} - 2^{-24}\right) > 0, \end{split}$$

so $p \in (-t, t)$, hence $q \in S(p, \mathfrak{S}_{m_1m_2m_2+1})$.

In the case of (\mathbf{R}, ρ) there is no possibility of avoiding this roadblock, as it is by no means true that $C_{\varepsilon}(p) = B[S_{\varepsilon}(p)]$ for irrational $\varepsilon > 0$. This can be seen by consideration of the ε and p used above. For all $r \in [-1+p, 1-p]$ we see that $S_{\varepsilon}(r) = (-p, p)$, so $p \in B[S_{\varepsilon}(r)]$, which implies $\rho(p, r) = \varepsilon$; thus

$$[-1+p, 1-p] \subset C_{\varepsilon}(p).$$

But

$$B[S_{\varepsilon}(p)] = B[(1-p, 1+p)] = \{1-p, 1+p\},\$$

so $C_{\varepsilon}(p) \neq B[S_{\varepsilon}(p)]$. We note finally that

$$\dim C_{\varepsilon}(p) = 1 > 0 = \dim \mathbf{R} - 1;$$

hence a metric constructed as in Theorem 1 does not necessarily have the property described in Corollary 2, nor that in Corollary 3.

2

Corollary 2 asserts the existence of a topology-preserving metric for any *n*-dimensional metric space R which satisfies the following two properties for all irrational $\varepsilon > 0$:

- (i) dim $C_{\varepsilon}(p) \leq n-1$ for all $p \in R$, and
- (ii) $\{C_{\varepsilon}(p) : p \in R\}$ is closure-preserving.

Although the space (\mathbf{R}, ρ) can be shown to satisfy (ii), we have seen that it does not fulfill (i). On the other hand, \mathbf{R} with the usual metric satisfies (i) but not (ii). The following Theorem demonstrates that a connected metric space of dimension greater than zero cannot simultaneously satisfy (i) and (ii) for small ε :

THEOREM. Let (R, d) be a connected space with at least two points, n > 0, and $0 < \varepsilon < \frac{1}{2}$ diam (R). If (i) and (ii) are satisfied for this n and ε , then dim $R \leq n-1$.

PROOF. Let $p \in R$; the set $B = \{z : d(p, z) > \varepsilon\} \neq \emptyset$ (if not, then for all $x, y \in R$ we have $d(x, y) \leq d(x, p) + d(y, p) \leq 2\varepsilon$, so diam $(R) \leq 2\varepsilon$, which contradicts the hypothesis). Hence there exists a point $q \in R$ such that $d(p, q) = \varepsilon$, for otherwise the two nonempty sets $S_{\varepsilon}(p)$ and B would yield a separation of the connected space R. Hence $p \in R$ implies $p \in C_{\varepsilon}(q)$ for some $q \in R$, so $R = \bigcup \{C_{\varepsilon}(q) : q \in R\}$. By a Theorem of Nagami [1, Theorem 1], conditions (i) and (ii) imply dim $R \leq n-1$.

This Theorem demonstrates that for the above class of spaces Corollary 2 is invalid. It remains an open question as to whether or not Corollary 3 is invalid for a similar class of spaces.

BIBLIOGRAPHY

- K. NAGAMI
- Some theorems in dimension theory for non-separable spaces, Journal of the Math. Soc. of Japan 9 (1957), pp. 80-92.
- J. NAGATA
- Two theorems for the n-dimensionality of metric spaces, Comp. Math. 15 (1963), pp. 227-237.

(Oblatum 29-4-68)