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## H. I. BROWN Entire methods of summation

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### Entire methods of summation

by

H. I. Brown

#### Introduction

In this paper we consider matrix transformations on the set of entire sequences into itself. We call such methods entire. By adopting M. S. Macphail's technique of applying a theorem of K. Knopp and G. G. Lorentz we obtain necessary and sufficient conditions on the elements of a matrix in order that it be an entire method. After some examples and preliminary Lemmas we then prove a consistency type theorem for entire methods of summation.

#### 1. Entire methods of summation

Let s represent the set of all sequences of complex numbers. A member of s, say  $x = \{x_k\}, k = 0, 1, 2, \cdots$ , is called an *entire* sequence if  $\sum_{k=0}^{\infty} |x_k| p^k$  converges for every p > 0. Let  $\mathscr{E}$  designate the set of entire sequences and let  $A = (a_{nk})$   $(n, k = 0, 1, 2, \cdots)$ be an infinite matrix of complex numbers. The set of equations

(1) 
$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k$$
  $(n = 0, 1, \cdots)$ 

defines an entire method of summation if each series in (1) converges and  $y = \{y_n\} \in \mathscr{E}$  whenever  $x \in \mathscr{E}$ . If, in addition,

$$\sum_{k=0}^{\infty} y_n = \sum_{k=0}^{\infty} x_k$$

then A is called a *regular entire method*.

For each positive integer p, let  $\mathscr{E}_p$  represent the set of sequences  $\{x_k\}$  such that

$$\sum_{k=0}^{\infty} |x_k| p^k < \infty.$$

In [3; p. 389], M. S. Macphail designates this set by l(p) and observes that the mapping

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$$\{x_k\} \rightarrow \{x_k p^k\}$$

is a one-to-one correspondence between  $\mathscr{E}_{p}$  and l (the set of absolutely convergent series). It was shown by K. Knopp and G. G. Lorentz [2] that a necessary and sufficient condition for a matrix  $A = (a_{nk})$  to transform l into itself (that is, for A to be an l-l method) is that there exists a constant M such that

(2) 
$$\sum_{n=0}^{\infty} |a_{nk}| < M$$
  $(k = 0, 1, 2, \cdots),$ 

and a necessary and sufficient condition for A to be absolutely regular (that is,  $\sum y_n = \sum x_k$  whenever  $x \in l$ ) is that in addition to (2) the equations

$$\sum_{n=0}^{\infty} a_{nk} = 1$$
 (k = 0, 1, 2, · · · )

hold. Thus, the matrix  $(a_{nk})$  maps  $\mathscr{E}_p$  into l if and only if the matrix  $(a_{nk}p^{-k})$  is an l-l method. That is,  $(a_{nk})$  maps  $\mathscr{E}_p$  into l if and only if there exists a constant M(p) such that

$$\sum_{n=0}^{\infty} |a_{nk}| p^{-k} < M(p)$$
 (k = 0, 1, 2, · · ·).

Similarly, for each positive integer q, a matrix  $(b_{nk})$  maps l into  $\mathscr{E}_q$  if and only if the matrix  $(b_{nk}q^n)$  is an l-l method, that is, if and only if there exists a constant M(q) such that

$$\sum_{n=0}^{\infty} |b_{nk}| q^n < M(q)$$
 (k = 0, 1, 2, · · ·).

Now  $\mathscr{E} = \cap \{\mathscr{E}_q : q = 1, 2, \dots\}$ ; hence, a matrix  $A = (a_{nk})$  is an entire method if and only if to each positive integer q, there corresponds a positive number  $p = p(q) \ge q$  such that A transforms  $\mathscr{E}_p$  into  $\mathscr{E}_q$ . In other words, A is an entire method if and only if to each  $q = 1, 2, \cdots$ , there corresponds a  $p = p(q) \ge q$ such that the matrix  $(a_{nk}q^np^{-k})$  is an l-l method. By taking q = 1 we obtain necessary and sufficient conditions for A to be a regular entire method. We summarize these remarks in the following theorem.

THEOREM 1. A necessary and sufficient condition for A to be an entire method is that for each positive integer q there exist  $p(q) \ge q$ and a constant M(p, q) such that

(3) 
$$\sum_{n=0}^{\infty} |a_{nk}| q^n p^{-k} < M(p, q) \qquad (k = 0, 1, 2, \cdots),$$

 $[\mathbf{2}]$ 

and a necessary and sufficient condition for A to be a regular entire method is that in addition to (3) the equations

$$\sum_{n=0}^{\infty} a_{nk} = 1$$
 (k = 0, 1, 2, · · ·)

hold.

**REMARK.** In order that A be an entire method it is necessary that each column of A be an entire sequence. Also, by taking q = 1 and p = p(1), it is necessary that each row be analytic, that is, for each  $n = 0, 1, 2, \dots$ , the sequence

$$\{|a_{n0}|, \, |a_{nk}|^{1/k}: k = 1, \, 2, \, \cdots \}$$

be bounded. However, one may easily show that these conditions are not sufficient. Indeed, the matrix defined by the set of equations

$$egin{aligned} a_{nn}&=n\,!, &n&=0,\,1,\cdots,\ a_{nk}&=0, & ext{otherwise}, \end{aligned}$$

has both entire rows and entire columns. However, the entire sequence  $\{1/n!\}$  is transformed into the constant sequence  $\{1\}$ .

#### 2. Examples

For each complex number t, the Euler-Knopp series-to-series method is defined by the set of equations

$$egin{aligned} E_{nk}(t) &= \binom{n}{k} t^{k+1} (1\!-\!t)^{n-k}, & k \leq n, \ E_{nk}(t) &= 0, & k > n. \end{aligned}$$

The transformations E(0) and E(1) are, respectively, the zero matrix and the identity, both of which are entire methods. However, if t is any other complex number, then the  $k^{\text{th}}$  column of  $(E_{nk})$  is not an entire sequence and so E(t) cannot be an entire method. (See the Remark.) Contrary to this, the Taylor matrix [1] is always entire. For each complex number t, the Taylor matrix T(t) is defined by the set of equations

$$T_{nk}(t) = 0, \qquad n > k,$$

$$T_{nk}(t)=inom{k}{n}(1\!-\!t)^{n+1}t^{k-n}, \qquad n\,\leq k.$$

The trivial cases T(0) (identity) and T(1) (zero) are certainly entire methods. If t is any complex number other than 0 or 1, then

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(4)  
$$\sum_{n=0}^{k} \binom{k}{n} |1-t|^{n+1} |t|^{k-n} q^{n} p^{-k} = |1-t| p^{-k} (q|1-t|+|t|)^{k} \\ \leq |1-t| (q+(q+1)|t|)^{k} / p^{k} \\ \leq (1+R) (q+(q+1)R)^{k} / p^{k},$$

where R is chosen to be so large that  $|t| \leq R$ . We may now choose p = 2(q+(q+1)R). Then (4) is dominated by  $(1+R)(1/2)^k$ , which shows that (3) is satisfied with M = 1+R. Thus, T(t) is entire.

Notice also that

$$\sum_{n=0}^{k} \binom{k}{n} (1-t)^{n+1} t^{k-n} = 1-t,$$

so that T(t) is regular if and only if t = 0, that is, if and only if T is the identity matrix.

#### 3. Preliminary lemmas

It is well known that  $\mathscr{E}$  is a locally convex FK space with its FK topology being given by the family of seminorms  $\{h_n : n = 1, 2, \dots\}$ , where for each  $x \in \mathscr{E}$ ,

$$h_n(x) = \max_{|z|=n} |\sum_{i=0}^\infty x_i z^i|.$$

Also, if we define an analytic sequence x to mean that the sequence  $\{|x_0|, |x_k|^{1/k} : k = 1, 2, \dots\}$  is a bounded sequence, then every continuous linear functional f on  $\mathscr{E}$  has the representation

$$f(x) = \sum_{n=0}^{\infty} t_n x_n,$$

for some analytic sequence t. (For a discussion of  $\mathscr{E}$ , see, for example, C. Goffman and G. Pedrick, *First Course in Functional Analysis*, pp. 220-222, 224, Prentice-Hall, New Jersey.)

Now let A be an entire method of summation and let  $\mathscr{E}_A$  represent its summability field, that is,

$$\mathscr{E}_A = \{ x \in s : Ax \in \mathscr{E} \}.$$

An application of [4, Theorem 1, p. 226 and Theorem 5, p. 230] shows how  $\mathscr{E}_A$  may inherit a locally convex FK topology given by the following family of seminorms:

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$$p_n(x) = |x_n|,$$
 (n = 0, 1, 2, · · ·),

$$q_n(x) = \sup_m |\sum_{k=0}^m a_{nk} x_k|,$$
 (n = 0, 1, 2, · · ·),

$$h_n(x) = \max_{|z|=n} |\sum_{i=0}^\infty ig( \sum_{k=0}^\infty a_{nk} x_k ig) z^i |, \qquad (n=1,\,2,\,3,\,\cdots).$$

Also, every  $f \in \mathscr{E}'_A$  (the dual space of  $\mathscr{E}_A$ ) may be evaluated as

(5) 
$$f(x) = \sum_{n=0}^{\infty} t_n \sum_{k=0}^{\infty} a_{nk} x_k + \sum_{k=0}^{\infty} \alpha_k x_k$$

for some analytic sequences t and  $\alpha$ , and all  $x \in \mathscr{E}_A$ . ( $\alpha$  is analytic because  $\mathscr{E}_A \supseteq \mathscr{E}$  and so  $\sum \alpha_k x_k$  converges for every  $x \in \mathscr{E}$ .)

To each entire method A there corresponds the functional  $S_A$  given by  $\infty \infty$ 

$$S_A(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{nk} x_k.$$

Since every matrix map between FK spaces is continuous [4; p. 204], it follows that  $S_A \in \mathscr{E}'_A$ .

LEMMA 1. If  $f \in \mathscr{E}'_A$ , then there exists an entire method B such that  $\mathscr{E}_B \supseteq \mathscr{E}_A$  and  $S_B(x) = f(x)$  for every  $x \in \mathscr{E}_A$ .

**PROOF.** Given an entire method A define the matrix  $B = (b_{nk})$  by the set of equations

$$egin{aligned} b_{0k} &= lpha_k + t_0 a_{0k} & (k = 0, \, 1, \, 2, \, \cdots) \ b_{nk} &= t_n a_{nk} & (n = 1, \, 2, \, \cdots; \, k = 0, \, 1, \, 2, \, \cdots), \end{aligned}$$

where t and  $\alpha$  are the analytic sequences given by equation (5) in the representation of f.

Let N be the smallest integer greater than or equal to the number max  $(M(\alpha), M(t))$ , where

$$M(\alpha) = \max\left(\sup_{k} \left(|\alpha_0|, |\alpha_k|^{1/k}\right), 1\right)$$
$$M(t) = \max\left(\sup_{k} \left(|t| - |t|^{1/n}\right), 1\right)$$

and

$$M(t) = \max \left( \sup_{n} (|t_0|, |t_n|^{1/n}), 1 \right).$$

N depends only on f.

To show that B is an entire method we apply Theorem 1. Let q be any positive integer whatsoever. Choose  $p \ge N \cdot q$  so that

$$\sup_k \sum_{n=0}^\infty |a_{nk}| (N\cdot q)^n p^{-k} = M(p,q) < \infty.$$

(This is possible because A is an entire method.) For this choice

of p observe that for each  $k = 0, 1, 2, \cdots$ ,

$$|\alpha_k|^{1/k}/p \leq 1.$$

Thus, for each  $k = 0, 1, 2, \cdots$ ,

$$\sum_{n=0}^{\infty} |b_{nk}| q^n p^{-k} \leq rac{|lpha_k|}{p^k} + \sum_{n=0}^{\infty} |t_n a_{nk}| q^n p^{-k} \ = \left(rac{|lpha_k|^{1/k}}{p}
ight)^k + \sum_{n=0}^{\infty} |a_{nk}| (t_n|^{1/n} \cdot q)^n p^{-k} \ \leq \left(rac{|lpha_k|^{1/k}}{p}
ight)^k + \sum_{n=0}^{\infty} |a_{nk}| (N \cdot q)^n p^{-k}.$$

It follows that

$$\sup_k \sum_{n=0}^\infty |b_{nk}| q^n p^{-k} < \infty;$$

hence, B is an entire method.

That  $\mathscr{E}_B \supseteq \mathscr{E}_A$  follows immediately from the construction of B. Finally, if  $x \in \mathscr{E}_A$ , then

$$S_B(x) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{nk} x_k$$
  
=  $\sum_{k=0}^{\infty} (\alpha_k + t_0 a_{0k}) x_k + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} t_n a_{nk} x_k$   
=  $f(x)$ ,

which proves the Lemma.

Let now A and B be two entire methods. Define  $C = (C_{nk})$  by the set of equations

$$C_{2n,k} = a_{nk}$$
  $(n, k = 0, 1, 2, \cdots),$   
 $C_{2n+1,k} = -b_{nk}$   $(n, k = 0, 1, 2, \cdots).$ 

LEMMA 2. C is an entire method such that  $\mathscr{E}_C = \mathscr{E}_A \cap \mathscr{E}_B$  and  $S_C(x) = S_A(x) - S_B(x)$  for every  $x \in \mathscr{E}_C$ .

**PROOF.** Since A and B are entire methods, given any positive integer q we may choose  $p \ge q^2$  so that

$$\sup_k \sum_{n=0}^{\infty} |a_{nk}| (q^2)^n p^{-k} < \infty$$

and

$$q\cdot \sup_k \sum_{n=0}^\infty |b_{nk}| (q^2)^n p^{-k} < \infty.$$

Thus,

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$$\sup_{k} \sum_{n=0}^{\infty} |C_{nk}| q^{n} p^{-k} \leq \sup_{k} \sum_{n=0}^{\infty} |a_{nk}| q^{2n} p^{-k} + \sup_{k} \sum_{n=0}^{\infty} |b_{nk}| q^{2n+1} p^{-k} < \infty,$$

so that C is an entire method.

Next,  $x \in \mathscr{E}_C$  if and only if for every p > 0,

(6) 
$$\left| \sum_{k=0}^{\infty} a_{0k} x_{k} \right| p^{0} + \left| \sum_{k=0}^{\infty} b_{0k} x_{k} \right| p^{1} + \left| \sum_{k=0}^{\infty} a_{1k} x_{k} \right| p^{2} + \left| \sum_{k=0}^{\infty} b_{1k} x_{k} \right| p^{3} + \cdots < \infty.$$

Since this is a series of non-negative terms, it is satisfied for every p > 0 if and only if

$$\sum_{k=0}^{\infty}\left|\sum_{k=0}^{\infty}a_{nk}x_k
ight|(p^2)^n+p\cdot\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}b_{nk}x_k
ight|(p^2)^n<\infty
ight|$$

for every p > 0, that is, if and only if  $x \in \mathscr{E}_A \cap \mathscr{E}_B$ .

Finally, let  $x \in \mathscr{E}_C$ . Then

$$S_{C}(x) = \sum_{k=0}^{\infty} a_{0k} x_{k} - \sum_{k=0}^{\infty} b_{0k} x_{k} + \sum_{k=0}^{\infty} a_{1k} x_{k} - \sum_{k=0}^{\infty} b_{1k} x_{k} \pm \cdots$$

Since this is an absolutely convergent series [take p = 1 in equation (6)], we may rearrange its terms to obtain

$$\begin{split} S_C(x) &= \sum_n \sum_k a_{nk} x_k - \sum_n \sum_k b_{nk} x_k \\ &= S_A(x) - S_B(x). \end{split}$$

#### 4. Consistency of entire methods of summation

Two entire methods A and B will be called *consistent* (relative to the functionals  $S_A$  and  $S_B$ ) if  $S_A(x) = S_B(x)$  for every  $x \in \mathscr{E}_A \cap \mathscr{E}_B$ .

THEOREM 2. In order that an entire method A be consistent with every entire method B whenever  $S_A(x) = S_B(x)$  for  $x \in \mathcal{E}$ , it is necessary and sufficient that  $\mathcal{E}$  be dense in  $\mathcal{E}_A \cap \mathcal{E}_B$  whenever  $S_B(x) = S_A(x)$  for  $x \in \mathcal{E}$  (where the closure is taken in the FK topology of  $\mathcal{E}_A \cap \mathcal{E}_B$ ).

**PROOF.** Assume  $\mathscr{E}$  is dense in  $\mathscr{E}_A \cap \mathscr{E}_B$  and that  $S_A(x) = S_B(x)$  for every  $x \in \mathscr{E}$ . Then  $F(x) = S_A(x) - S_B(x)$  defines a continuous linear functional on  $\mathscr{E}_A \cap \mathscr{E}_B$  which vanishes on  $\mathscr{E}$ ; hence, it must vanish on  $\mathscr{E}_A \cap \mathscr{E}_B$ . Thus, A and B are consistent.

Conversely, assume that A is an entire method which is con-

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sistent with every entire method that agrees with A on  $\mathscr{E}$ . Suppose there exists an entire method B such that  $S_B(x) = S_A(x)$  for  $x \in \mathscr{E}$  and yet  $\mathscr{E}$  is not dense in  $\mathscr{E}_A \cap \mathscr{E}_B$ . Then there exists  $f \in \mathscr{E}'_C$ such that f vanishes on  $\mathscr{E}$  and  $f(y) \neq 0$  for some  $y \in \mathscr{E}_C$ , where C is the entire method constructed from A and B as in Lemma 2.

By Lemma 1, there exists an entire method D such that  $\mathscr{E}_D \supseteq \mathscr{E}_C$  and  $S_D(x) = f(x)$  for  $x \in \mathscr{E}_C$ .

Define  $E = (e_{nk})$  by the set of equations

$$e_{nk} = d_{nk} + a_{nk}$$
 (n, k = 0, 1, 2, · · ·).

Then E is an entire method because for every k,

$$\sum_{n=0}^{\infty} |e_{nk}| q^n p^{-k} \leq \sum_{n=0}^{\infty} |d_{nk}| q^n p^{-k} + \sum_{n=0}^{\infty} |a_{nk}| q^n p^{-k}.$$

Since  $\mathscr{E}_D \supseteq \mathscr{E}_C$  we have  $\mathscr{E}_E \supseteq \mathscr{E}_C$ . Moreover, for

$$x \in \mathscr{E}, S_E(x) = S_D(x) + S_A(x) = S_A(x)$$

since f vanishes on  $\mathscr{E}$ . However, E is not consistent with A since  $y \in \mathscr{E}_E \cap \mathscr{E}_A, f(y) \neq 0$ , and  $S_E(y) = f(y) + S_A(y)$ . This contradicts our assumption, and the Theorem is proved.

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