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On Grothendieck universes

by

N. H. Williams

The purpose of this note is to observe that by using the full set theory of Gödel [2], i.e. including the axiom D (Foundation), Grothendieck universes may be characterized as follows: U is a Grothendieck universe if and only if for some inaccessible cardinal α, U is the collection of all sets of rank α.

Define a set U to be a Grothendieck universe, after Gabriel [1], if

UA1. For each x, $x \in U \Rightarrow x \subseteqq U$,

UA2. For each x, $x \in U \Rightarrow P(x) \in U$,

UA3. For each x, $x \in U \Rightarrow \{x\} \in U$,

UA4. For each family $\{x_i\}_{i \in I}$ such that $I \in U$ and such that $x_i \in U$ for each $i \in I$, $\bigcup \{x_i : i \in I\} \in U$,

UA5. U is non-empty.

A cardinal number is to be understood as an ordinal number which is equipollent to no smaller ordinal number.

Call a cardinal number α inaccessible in the narrower sense of Tarski [6] (henceforth referred to as inaccessible) if

IA1. card $(\bigcup \{x_i : i \in I\}) < \alpha$ for each family $\{x_i\}_{i \in I}$ of sets with card $(I) < \alpha$ and with card $(x_i) < \alpha$ for each $i \in I$,

IA2. For any two cardinals ξ and η, $\xi^\eta < \alpha$ whenever $\xi < \alpha$ and $\eta < \alpha$.

Use transfinite induction to define a function Ψ over the class of all ordinals (after von Neumann [4]) by

(1) $\Psi(0) = 0$,

(2) $\Psi(\beta+1) = P(\Psi(\beta))$,

(3) if λ is a limit ordinal, $\Psi(\lambda) = \bigcup \{\Psi(\beta) : \beta < \lambda\}$

Thus Ψ is the rank function: $\Psi(\beta)$ is the collection of all sets of rank $\leqq \beta$. Then the following results are well known (e.g. Shepherdson [5]):

(a) For each set x, there is an ordinal β such that

$$x \in \Psi(\beta),$$

(b) If α is an inaccessible cardinal, then

$$\beta < \alpha \Rightarrow \text{card}\,(\Psi(\beta)) < \alpha,$$

(c) If α is an inaccessible cardinal, then

$$x \in \Psi(\alpha) \Leftrightarrow x \subseteqq \Psi(\alpha) \,\&\, \text{card}\,(x) < \alpha.$$

It is shown in Kruse [3] that U is a Grothendieck universe to which an infinite set belongs if and only if U is a super-complete model in the sense of Shepherdson [5]. In Shepherdson [5], it is shown that all the super-complete models are of the form $\Psi(\alpha)$ for some uncountable inaccessible cardinal α. It is the purpose of this note to distinguish to which $\Psi(\alpha)$ a given universe U corresponds.

The following result is proved in Kruse [3]:

U is a Grothendieck universe if and only if there exists an inaccessible cardinal α such that $x \in U \Leftrightarrow x \subseteqq U \,\&\, \text{card}\,(x) < \alpha$.

For a given universe U, this inaccessible cardinal is clearly uniquely determined. Write it as $\alpha(U)$. Then the main result is:

Theorem 1. $U = \Psi(\alpha(U))$.

Proof. First, to show that $\Psi(\alpha(U)) \subseteqq U$. For this, use transfinite induction on the ordinal β to prove

$$A(\beta) : \beta < \alpha(U) \Rightarrow \Psi(\beta) \subseteqq U.$$

(i) Trivially $A(0)$.

(ii) Assume $A(\beta)$; to show $A(\beta+1)$. Let $\beta+1 < \alpha(U)$, then $\beta < \alpha(U)$ and so $\Psi(\beta) \subseteqq U$. But $\beta < \alpha(U) \Rightarrow \text{card}\,(\Psi(\beta)) < \alpha(U)$, by (b). Hence $\Psi(\beta) \subseteqq U \,\&\, \text{card}\,(\Psi(\beta)) < \alpha(U)$, thus $\Psi(\beta) \in U$. But then $\Psi(\beta+1) = P(\Psi(\beta)) \in U$, by UA2, and so $\Psi(\beta+1) \subseteqq U$, by UA1. Hence $A(\beta+1)$.

(iii) Let λ be a limit ordinal, and assume $\beta < \lambda \Rightarrow A(\beta)$; to show $A(\lambda)$. Let $\lambda < \alpha(U)$, and then $\beta < \lambda \Rightarrow \Psi(\beta) \subseteqq U$. Hence $\Psi(\lambda) = \bigcup \{\Psi(\beta) : \beta < \lambda\} \subseteqq U$; thus $A(\lambda)$.

Hence, for any ordinal β, $\beta < \alpha(U) \Rightarrow \Psi(\beta) \subseteqq U$. But $\Psi(\alpha(U)) = \bigcup \{\Psi(\beta) : \beta < \alpha(U)\}$, hence $\Psi(\alpha(U)) \subseteqq U$.

Now suppose that $U \setminus \Psi(\alpha(U)) \neq 0$.

Then, by the axiom of foundation, there is a set x such that

$$x \in U \setminus \Psi(\alpha(U)) \,\&\, x \cap [U \setminus \Psi(\alpha(U))] = 0.$$

Since $x \in U \Rightarrow \big(x \subseteqq U \;\&\; \text{card}\,(x) < \alpha(U)\big)$, and

$$\big(x \subseteqq U \;\&\; x \cap [U \setminus \Psi(\alpha(U))] = 0\big) \Rightarrow x \subseteqq \Psi\big(\alpha(U)\big),$$
$$x \subseteqq \Psi\big(\alpha(U)\big) \;\&\; \text{card}\,(x) < \alpha(U),$$

hence $x \in \Psi\big(\alpha(U)\big)$, by (c). But this contradicts $x \in U \setminus \Psi\big(\alpha(U)\big)$, and hence $U = \Psi\big(\alpha(U)\big)$.

THEOREM 2.

If U is a Grothendieck universe, then card $(U) = \alpha(U)$.

This follows from the fact that for inaccessible cardinals α, card $\big(\Psi(\alpha)\big) = \alpha$, (which may be shown by relativizing to $\Psi(\alpha)$ the proof of the existence of a one-to-one mapping from the class of all ordinals to the class of all sets).

Note that the equivalence of the two statements:

(A) For every cardinal β, there is an inaccessible cardinal α such that $\beta < \alpha$,

(B) For every set x, there is a Grothendieck universe U such that $x \in U$,

follows immediately from (a) and Theorem 1.

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