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A note on spread-cardinals

Dedicated to A. Heyting on the occasion of his 70th birthday

by

Dirk Van Dalen

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Brouwer has defined the concept “gleichmächtig” in his early papers [1], [2] and analyzed some of its properties. It was defined quite generally for species. Knowing the pathological behaviour of species we cannot expect to prove much about cardinals. We propose therefore to restrict our attention for the time being to spreads (and, since our results will mainly be negative, to fans).

The problem we discuss in this paper is the existence of a natural partial ordering among this class of cardinals. There is no doubt that the equivalence (gleichmächtigkeits) relation as defined by Brouwer (which by the way coincides with Cantor’s definition) is the most natural one. There appear to be more possibilities to define the analogon of “equivalent to a subset of”. As we will show none of those give rise to a partial ordering.

This is rather disappointing as in another approach to constructive cardinals, the RET’s of Dekker and Myhill, there is a natural partial ordering [6].

The treatment below has been influenced by Tarski’s “Cardinal Algebras” en Dekker and Myhill’s “Recursive Equivalence Types”.

1. Definitions

For the intuitionistic terms the reader is referred to Heyting [7]. Capitals will denote spreads and fans, lower cast letters denote mappings from spreads to spreads. $\alpha, \beta, \gamma \dots$ will denote cardinals and ξ, η, ζ will denote elements of spreads. We will restrict ourselves to *undressed spreads* [4].

DEFINITION 1.1. $S \sim T$ iff there exists a full bijection $f : S \rightarrow T$.

From now on we will delete the adjective “full”, every mapping $f : S \rightarrow T$ will be supposed to be defined on all of S .

Clearly \sim is an equivalence relation.

DEFINITION 1.2. $[S] = \{T | T \sim S\}$.

We call the species $[S]$ the cardinal of S .

For convenience the underlying species of nodes of a fan F (the *spread-direction* [4]) is denoted by \tilde{F} , $\langle x_1 \cdots x_n \rangle \in \tilde{F}$ means that $\langle x_1 \cdots x_n \rangle$ is admissible under the spread law of F .

DEFINITION 1.3. $S+T$ is a spread W defined by $\langle x_1, \dots, x_n \rangle \in \tilde{W}$ iff

- (i) x_i even ($i = 1, \dots, n$) and $\langle \frac{1}{2}x_1, \dots, \frac{1}{2}x_n \rangle \in \tilde{S}$
- (ii) x_i odd ($i = 1, \dots, n$) and $\langle \frac{1}{2}(x_1-1), \dots, \frac{1}{2}(x_n-1) \rangle \in \tilde{T}$.

$S+T$ is the *disjoint union* of S and T . One can visualize the disjoint sum of two fans by placing them side by side.

DEFINITION 1.4. $[S]+[T] = [S+T]$.

It is easy to see that this sum of cardinals is well-defined. We could define a countable sum of cardinals. As finite sums only are used here we will refrain from doing so.

DEFINITION 1.5. $S \times T$ is a spread W defined by $\langle x_1, \dots, x_n \rangle \in \tilde{W}$ iff

- (i) $\langle K(x_1), \dots, K(x_n) \rangle \in \tilde{S}$ and
- (ii) $\langle L(x_1), \dots, L(x_n) \rangle \in \tilde{T}$ ¹

DEFINITION 1.6. $[S] \times [T] = [S \times T]$.

Again we have a well defined operation on cardinals.

DEFINITION 1.7. $[S]\rho[T]$ iff there is an injection $f: S \rightarrow T$

$[S]\sigma[T]$ iff there is a surjection $g: T \rightarrow S$

$[S]\tau[T]$ iff there is an injection $f: S \rightarrow T$
and $f(S)$ is removable in T

$[S]\tau'[T]$ iff there is an R such that

$$[S]+[R] = [T]$$

LEMMA 1.8. τ and τ' coincide.

PROOF: (i) Let $[S]\tau[T]$ and put $R = f(S)$.

Define $h: T \rightarrow \{0, 1\}$ by $h(\xi) = \begin{cases} 0 & \text{if } \xi \in R \\ 1 & \text{if } \xi \notin R. \end{cases}$

By assumption h is well-defined on T and hence by Brouwer's Bar Theorem there exists, a thin bar (removable subspecies of the species of nodes) such that all ξ passing through a node of the bar have the same value under h (the species μ_1 in [3]).

Define a new spread having as admissible segments the se-

¹ "inverses" of the pairing function J , cf. Davis, *Computability and Unsolvability*, p. 45.

quences of T starting at the bar. By reenumerating the initial segments we obtain a spread where the “even” segments belong to R and the “odd” segments to the complement of R . This shows that $T \sim R' + R''$ where $R' \sim S$. The construction of the required bijections is straightforward.

- (ii) Let $[S] + [R] = [T]$, then
 $T = S' + R'$ and $S' = f(S)$ where f is bijective.

Clearly $f(S)$ is removable in T .

LEMMA 1.9. ρ , σ and τ are reflexive and transitive.

PROOF: immediate.

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We will show that for none of the relations ρ , σ , τ the Cantor-Bernstein theorem holds. This is fairly evident as far as ρ and σ are concerned. In the case of τ it is considerably harder. I even entertained for some time the hope of proving τ a partial ordering, after failing to do so I tried to construct a counterexample. The one produced below is an inessential modification of the counterexample presented at a seminar in Nijmegen. The main characteristic of the demonstration is the repeated application of the fan theorem. The absence of otherwise sophisticated intuitionistic machinery seems rather surprising.

The construction itself is not too complicated, noting the basic idea of finding a fan that has auto-injections but not too many, it does not seem plausible to expect an easier construction (and proof). Our result will be slightly stronger than announced since we will deduce the statements even for fans.

2.1. There are fans F and G such that $[F]\rho[G]$ and $[G]\rho[F]$ and not $[F] = [G]$.

PROOF: Let F be the binary fan and $G = F + E$ where E is a fan consisting of one element.

2.2. There are fans F and G such that $[F]\sigma[G]$ and $[G]\sigma[F]$ and not $[F] = [G]$.

PROOF: as above.

2.3. A counterexample will show that τ is not a partial ordering of the cardinals. As the construction is rather involved we will develop some ad hoc techniques.

First we need a countable species of fans which are uncomparable under τ . For convenience we use a regular expression² to define the admissible sequences: $(0 \cup 1)^* \cup 1^*2^*1^*2^*\cdots 1^*$ (or 2^*). Employment of a string $1^*2^*\cdots$ of length n defines the fan H_n ($n \geq 1$). Using the fan-theorem one shows that $[H_n]\tau[H_m]$ does not hold for $n \neq m$.

As an example we will show that H_2 and H_3 are incomparable under τ . Assume there is a bijective mapping f from H_2 onto a removable subfan F of H_3 .

In \tilde{H}_2 and \tilde{H}_3 we distinguish four types of nodes:

type **a**: a node with at least one 0.

type **b**: $\langle 11 \cdots 1 \rangle$

type **c**: $\langle 11 \cdots 12 \cdots 2 \rangle$

type **d**: $\langle 11 \cdots 12 \cdots 21 \cdots 1 \rangle$

As F is a removable subfan of H_3 there is a finite species of nodes $\{a_1, \cdots, a_n\}$ such that F consists of all choice sequences passing through at least one of a_1, \cdots, a_n .

We claim that no node c of type **c** is preceded by any of a_1, \cdots, a_n . If that were the case we could find a finite number of nodes b_1, \cdots, b_k of \tilde{H}_2 such that the subfan determined by b_1, \cdots, b_k is mapped by f onto the subfan determined by c . No node of type **a** or **b** can occur among b_1, \cdots, b_k , hence all b_1, \cdots, b_k are of type **c** (in \tilde{H}_2). This leads us into a contradiction since b_1, \cdots, b_k , determine a subfan of exactly k elements and c determines an infinite subfan.

It follows that \tilde{F} does not contain nodes of type **b**, i.e. a_1, \cdots, a_n are of type **a** or **d**.

By a similar kind of reasoning we find out that the choice sequences $\langle 11 \cdots 122 \cdots \rangle$ of H_2 are mapped onto choice sequences $\langle 11 \cdots 12 \cdots 211 \cdots \rangle$ of F by f .

This again leads us into contradiction since F contains finitely

² Regular expressions (cf [5], [9]) using the symbols 0, 1, 2 are defined inductively:

- (i) 0, 1, 2 are regular expressions,
- (ii) if E and F are regular expressions then EF is a regular expression (concatenation)
- (iii) if E and F are regular expressions then $E \cup F$ is a regular expression
- (iv) if E is a regular expression then E^* is a regular expression.

The regular expressions can be used to denote sets of finite sequences of the symbols 0, 1, 2. 0, 1, 2 denote the sets $\{\langle 0 \rangle\}$, $\{\langle 1 \rangle\}$, $\{\langle 2 \rangle\}$, if E and F denote the sets W_E and W_F then $E \cup F$ denotes $W_E \cup W_F$, EF denotes $\{ab \mid a \in W_E \text{ and } b \in W_F\}$ and E^* denotes $\bigcup_{n < \omega} W_{E^n}$, where $E_n = EE \cdots E$ (n times).

many elements of this kind and H_2 has infinitely many of them.

Conclusion: $[H_2]\tau[H_3]$ does not hold. An analogous but easier reasoning shows that $[H_3]\tau[H_2]$ fails too.

2.4. Let an (ordered) sequence $\{F_i\}$ of fans be given. The fan-sum F of this sequence is defined as follows:

Form a new sequence F'_i : such that

$$\langle x_1, \dots, x_n \rangle \in \tilde{F}'_i \Leftrightarrow \langle x_1+1, \dots, x_n+1 \rangle \in \tilde{F}'_i,$$

then

$$\langle a_1, \dots, a_n \rangle \in \tilde{F} \Leftrightarrow a_1 = \dots = a_n = 0$$

and

$$\langle a_{k+1}, \dots, a_n \rangle \in \tilde{F}'_k \quad (1 \leq k \leq n).$$

Notation: $F = FS(F_1 F_2 F_3 \dots)$.

For convenience let us call $\lambda x[0]$ the *spine* of F .

The fan-sum is an immediate generalization of the finite sum.

2.5. Define the following fans:

$$A_p = FS(H_p, H_{p+1}, H_{p+2}, \dots)$$

$$B_p = FS(H_p, H_{p+2}, H_{p+4}, \dots)$$

$$C = FS(A_1, A_1, A_1, \dots)$$

$$D_0 = FS(B_1, B_2, A_3, B_3, B_4, A_5, \dots, B_{2k-1}, B_{2k}, A_{2k+1}, \dots)$$

$$D_1 = FS(A_3, B_3, B_4, A_5, \dots, B_{2k-1}, B_{2k}, A_{2k+1}, \dots)$$

$$D_2 = FS(B_2, A_3, B_4, B_5, A_5, \dots, B_{2k-1}, B_{2k}, A_{2k+1}, \dots)$$

$$\text{Finally } F = C + D_0$$

$$K = C + D_2$$

$$G = C + D_1.$$

Clearly G is a removable subfan of K and K is a removable subfan of F . Hence $[K]\tau[F]$, moreover we will show $F \sim G$ and therefore $[F]\tau[K]$.

Using the facts that a finite segment of a fan sum can be replaced by a sum and that

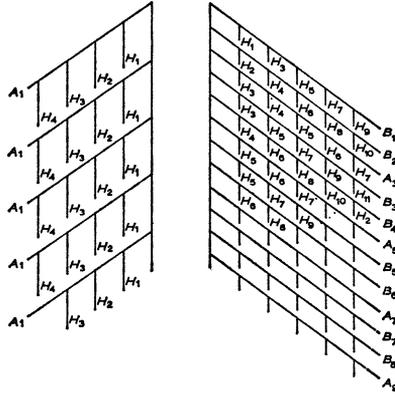
$$\begin{aligned} B_{2k-1} + B_{2k} + A_{2k+1} &\sim H_{2k-1} + B_{2k+1} + H_{2k} + B_{2k+2} + A_{2k+1} \\ &\sim A_{2k-1} + B_{2k+1} + B_{2k+2} \end{aligned}$$

we establish the following relation

$$\begin{aligned} F &= FS(B_1, B_2, A_3, B_3, B_4, A_5, B_5, B_6, A_7 \dots) + FS(A_1, A_1, \dots) \\ &\sim FS(A_1, B_3, B_4, B_3, B_4, A_5, B_5, B_6, A_7 \dots) + FS(A_1, A_1, \dots) \\ &\sim FS(B_3, B_4, A_3, B_5, B_6, A_5, B_7, B_8 \dots) + FS(A_1, A_1, \dots) \\ &\sim FS(A_3, B_3, B_4, A_5, B_5, B_6, \dots) + FS(A_1, A_1, \dots) = G \end{aligned}$$

2.6. It remains to be shown that $F \sim K$ does not hold.

The figure below visualizes the fan F . The short single lines represent the fans H_k . To simplify the proceedings we have omitted primes.



We will derive a contradiction from the existence of a bijec- tivity f from F to K . In the proof we will often apply the fan theorem. The actual working out of the details is rather tedious and uninteresting, therefore we will omit it. Any reader familiar with the fan-theorem can readily supply the missing details.

We state the following facts (due to the fan-theorem):

- (i) the spine of a subfan A_i is mapped onto the spine of another subfan A_j .
- (ii) the spine of a subfan B_i is mapped onto the spine of another subfan B_j .
- (iii) the spine ξ of D_0 is mapped onto itself.
- (iv) On the strength of (iii) we can find a node b of ξ (initial segment) such that every ips through b is mapped onto an ips through the topnode a of ξ and likewise there is a node c of ξ such that every ips through c is mapped onto an ips through b . It is no restriction to suppose that b dominates c .
- (v) By (i) we can find for every subfan A_i a node a_i such that every ips through a_i is mapped on an ips through a fixed node of the corresponding subfan both by f and f^{-1} .

Likewise we find with each subfan B_i a node b_i with the same properties.

Now consider all B and A spines through nodes of ξ that dominate c . On each of the B spines locate the first subfan H_{k_i} that passes through b_i and on each of the A spines locate the first

subfan H_{k_i} that passes through a_i . Thus we effectively determine a finite species of indices k_i with the required property. Let k be the maximum of this species, suppose k is odd, otherwise take $k+1$. Consider the species S of subfans H_k that are dominated by b and the subfans H_k of the subfans B_i . This clearly is a finite species and we can determine the images of the concerned subfans under f . Since f maps the B -subfans onto B -subfans and K does not contain the subfan B_1 S cannot be mapped onto itself. Owing to the fact that f is bijective there are less subfans H_k mapped into S from the complements³ of S than there are mapped from S into its complement.

By (iv) and (v) the only possible subfans to be mapped from S into its complement are those dominated by b and not by c . Call this species of subfans S_c^b . The members of S_c^b which are mapped into the complement of S must by (v) be subfans of A_i -subfans.

Also the members of S_c^b which occur as images of subfans from the complement of S are subfans of A_i -subfans. Hence the analogous species T_c^b of subfans H_{k+1} has the same properties as S_c^b , i.e. there are more elements of T (the species of H_{k+1} analogous to S) mapped into its complement than from the complement of T into T .

This clearly contradicts the fact that *all* members of T must occur as images under f .

Hence we showed that no bijective f exists such that $f : F \rightarrow K$.

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Even though τ is not a partial ordering we can still study the arithmetic of fan-cardinals (or spread-cardinals in general). Of course one cannot expect all the results of Tarski's cardinal algebra's, on the other hand most of the concepts introduced in [10] still make sense and deserve attention.

As regarding the question which of the relations ρ , σ or τ is most natural, an answer does not readily present itself. Since none of these is a partial ordering the species of fan cardinals is not as pleasing as the collection of R.E.T.'s. In one sense τ is to be preferred above ρ and σ , namely under ρ and σ there is a "maximal" cardinal (that of the binary fan). And the existence of such an object is presumably not attractive to most mathematicians.

³ relative to the species of all subfans H_k .

The usual procedure to obtain a partial ordering from a transitive, reflexive relation can be used, in my opinion the resulting equivalence relation is inferior to Brouwer's original definition.

Finally there is the possibility of Brouwer's notion of *congruence* [2] as a basis for cardinal-theory. However from Heyting's [8] it appears that congruence is too generous an equivalence relation.

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