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On the mean values of integral functions and their derivatives represented by Dirichlet series *

by

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Consider the Dirichlet series

$$(1.1)\quad f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \; (s = \sigma+it,\ \lambda_1 \geqq 0,\ \lambda_n < \lambda_{n+1} \to \infty \text{ with } n).$$

Let σ_c and σ_a be the abscissa of convergence and the abscissa of absolute convergence, respectively, of $f(s)$. If $\sigma_c = \sigma_a = \infty$, $f(s)$ represents an integral function.

Let the maximum modulus over a vertical line be as

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma+it)|$$

and the maximum term as

$$\mu(\sigma) = \max_{n\geqq 0} |a_n e^{\lambda_n(\sigma+it)}|.$$

If

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty,$$

we know ([1], p. 68)

$$(1.2)\qquad M(\sigma) < \mu(\sigma+D+\varepsilon), \qquad (\varepsilon > 0;\ \sigma > \sigma(\varepsilon)).$$

The mean values of $f(s)$ are defined as

$$(1.3)\qquad I_2(\sigma) = I_2(\sigma, f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(\sigma+it)|^2 dt$$

$$(1.4)\quad m_{2,k}(\sigma) = m_{2,k}(\sigma, f) = \lim_{T\to\infty} \frac{1}{Te^{k\sigma}}\int_0^{\sigma}\int_{-T}^{T} |f(x+it)|^2 e^{kx}\, dx\, dt,$$

where k is a positive number.

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In this paper we obtain a lower bound of $m_{2,k}(\sigma, f^{(1)})$ in terms of $m_{2,k}(\sigma)$ and σ, where $m_{2,k}(\sigma, f^{(1)})$ is the mean value of $f^{(1)}(s)$. the first derivative of $f(s)$, that is

$$(1.5)\quad m_{2,k}(\sigma, f^{(1)}) = \lim_{T\to\infty} \frac{1}{Te^{k\sigma}} \int_0^\sigma \int_{-T}^{T} |f^{(1)}(x+it)|^2 e^{kx}\, dx\, dt.$$

We also study some properties of $m_{2,k}(\sigma)$.

We first prove the following lemmas.

Lemma 1. *$m_{2,k}(\sigma)$ is a steadily increasing function of σ.*

Proof: We have

$$|f(s)|^2 = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} + \sum\sum_{m\neq n} a_m \bar{a}_n e^{\sigma(\lambda_m+\lambda_n)+it(\lambda_m-\lambda_n)},$$

the series on the right being absolutely and uniformly convergent in any finite t-range. Hence integrating term by term we obtain

$$\frac{1}{2T}\int_{-T}^{T} |f(s)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n} + \sum\sum_{m\neq n} a_m \bar{a}_n \frac{\sin T(\lambda_m-\lambda_n)}{T(\lambda_m-\lambda_n)}.$$

The term involving T is bounded for all T, m and n so that the double series converges uniformly with respect to T and each term tends to zero as $T \to \infty$. Thus we get

$$(1.6)\qquad I_2(\sigma) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(s)|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n}.$$

The series in (1.6) is absolutely and uniformly convergent and so again integrating term by term we obtain

$$(1.7)\qquad \begin{aligned} m_{2,k}(\sigma) &= \lim_{T\to\infty} \frac{1}{Te^{k\sigma}} \int_0^\sigma \int_{-T}^{T} |f(x+it)|^2 e^{kx}\, dx\, dt \\ &= 2\sum_{n=1}^{\infty} |a_n|^2 \frac{(e^{2\lambda_n\sigma} - e^{-k\sigma})}{2\lambda_n + k}. \end{aligned}$$

$m_{2,k}(\sigma)$ is steadily increasing follows from (1.7).

Lemma 2. $\log m_{2,k}(\sigma)$ *is a convex function of σ.*

Proof. From (1.3) and (1.4), we have

$$m_{2,k}(\sigma) = \frac{2}{e^{k\sigma}} \int_0^\sigma I_2(x) e^{kx}\, dx.$$

Therefore,

$$\frac{d(\log m_{2,k}(\sigma))}{d(\sigma)} = \left\{\frac{2I_2(\sigma)-km_{2,k}(\sigma)}{m_{2,k}(\sigma)}\right\}$$
$$= \left\{\frac{2I_2(\sigma)}{m_{2,k}(\sigma)} - k\right\},$$

which increases with $\sigma > \sigma_1$, since $e^{k\sigma}I_2(\sigma)$ is a convex function of $e^{k\sigma}m_{2,k}(\sigma)$ ([2], p. 135).

Hence,

$$\frac{d^2(\log m_{2,k}(\sigma))}{d\sigma^2} > 0 \quad \text{for} \quad \sigma > \sigma_1.$$

LEMMA 3. *If $f(s)$ is an integral function of Ritt-order ρ and lower order λ and $D < \infty$, then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} = \frac{\rho}{\lambda}.$$

PROOF. We have from (1.6)

$$\{\mu(\sigma)\}^2 \leqq I_2(\sigma) \leqq \{M(\sigma)\}^2. \tag{1.8}$$

If $D < \infty$, we get from (1.2) and (1.8)

$$\mu(\sigma) \leqq \{I_2(\sigma)\}^{\frac{1}{2}} \leqq M(\sigma) < \mu(\sigma+D+\varepsilon) \qquad \varepsilon > 0;\ \sigma > \sigma(\varepsilon).$$

Therefore,

$$\begin{aligned}\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log I_2(\sigma)}{\sigma} &= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\sigma}\\ &= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \frac{\rho}{\lambda}.\end{aligned} \tag{1.9}$$

Also, since $I_2(x)$ is an increasing function of x,

$$\begin{aligned} m_{2,k}(\sigma) &= \frac{2}{e^{k\sigma}}\int_0^\sigma I_2(x)e^{kx}dx\\ &\leqq \frac{2I_2(\sigma)}{e^{k\sigma}}\int_0^\sigma e^{kx}dx\\ &= \frac{2}{k}I_2(\sigma)(1-e^{-k\sigma})\end{aligned}$$

and we get, on using (1.9),

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_{2,k}(\sigma)}{\sigma} \leqq \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log I_2(\sigma)}{\sigma} = \frac{\rho}{\lambda}.$$

Further for $h > 0$

$$m_{2,k}(\sigma+h) = \frac{2}{e^{k(\sigma+h)}}\int_0^{\sigma+h} I_2(x)e^{kx}\,dx$$

$$\geqq \frac{2}{e^{k(\sigma+h)}}\int_\sigma^{\sigma+h} I_2(x)e^{kx}\,dx$$

$$\geqq \frac{2I_2(\sigma)}{k}\,(1-e^{-kh})$$

and we again get, on using (1.9),

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log m_{2,k}(\sigma)}{\sigma} \geqq \lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log I_2(\sigma)}{\sigma} = \frac{\rho}{\lambda},$$

and thus Lemma 3 follows.

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THEOREM 1. *If* $m_{2,k}(\sigma, f^{(1)})$ *is the mean value of* $f^{(1)}(s)$ *the first derivative of an integral function* $f(s)$ *other than an exponential polynomial, then*

$$m_{2,k}(\sigma, f^{(1)}) \geqq \frac{1}{2^2}\left\{\frac{\log m_{2,k}(\sigma)-1}{\sigma}\right\}^2 m_{2,k}(\sigma) \tag{2.1}$$

for $\sigma > \sigma_0$, *where* σ_0 *is a number depending on the function* f.

PROOF. We have

$$m_{2,k}(\sigma, f^{(1)}) = \lim_{T\to\infty}\frac{1}{Te^{k\sigma}}\int_0^\sigma\int_{-T}^{T}|f^{(1)}(x+it)|^2e^{kx}\,dx\,dt$$

$$= \lim_{T\to\infty}\frac{1}{Te^{k\sigma}}\int_0^\sigma\int_{-T}^{T}\left|\lim_{\varepsilon\to 0}\frac{f(x+it)-f(x(1-\varepsilon)+it)}{\varepsilon x}\right|^2 e^{kx}\,dx\,dt$$

$$\geqq \lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{1}{\varepsilon^2\sigma^2Te^{k\sigma}}\int_0^\sigma\int_{-T}^{T}\{|f(x+it)|-|f(x(1-\varepsilon)+it)|\}^2e^{kx}\,dx\,dt.$$

By Minkowski's inequality ([3], p. 384)

$$\left[\int_{-T}^{T}\{|f(x+it)|-|f(x(1-\varepsilon)+it)|\}^2dt\right]^{\frac12} \geqq \left\{\int_{-T}^{T}|f(x+it)|^2dt\right\}^{\frac12} - \left\{\int_{-T}^{T}|f(x(1-\varepsilon)+it)|^2dt\right\}^{\frac12}.$$

Hence,

$$m_{2,k}(\sigma, f^{(1)}) \geqq \lim_{\varepsilon\to 0}\lim_{T\to\infty} \frac{1}{\varepsilon^2\sigma^2 T e^{k\sigma}} \int_0^\sigma \left[\left\{\int_{-T}^{T} |f(x+it)|^2 dt\right\}^{\frac{1}{2}} - \left\{\int_{-T}^{T} |f(x(1-\varepsilon)+it)|^2 dt\right\}^{\frac{1}{2}}\right]^2 e^{kx} dx$$

$$= \lim_{\varepsilon\to 0}\lim_{T\to\infty} \frac{1}{\varepsilon^2\sigma^2 T e^{k\sigma}} \int_0^\sigma \left[\left\{e^{kx}\int_{-T}^{T} |f(x+it)|^2 dt\right\}^{\frac{1}{2}} - \left\{e^{kx}\int_{-T}^{T} |f(x(1-\varepsilon)+it)|^2 dt\right\}^{\frac{1}{2}}\right]^2 dx.$$

Again, using Minkowski's inequality, we obtain

$$m_{2,k}(\sigma, f^{(1)}) \geqq \lim_{\varepsilon\to 0}\lim_{T\to\infty} \frac{1}{\varepsilon^2\sigma^2 T e^{k\sigma}} \left[\left\{\int_0^\sigma e^{kx}\int_{-T}^{T} |f(x+it)|^2 dx\, dt\right\}^{\frac{1}{2}} - \left\{\int_0^\sigma e^{kx}\int_{-T}^{T} |f(x(1-\varepsilon)+it)|^2 dx\, dt\right\}^{\frac{1}{2}}\right]^2$$

$$\geqq \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2\sigma^2}\left[\{m_{2,k}(\sigma)\}^{\frac{1}{2}} - \left\{\frac{e^{-\varepsilon k\sigma}}{(1-\varepsilon)} e^{k((\sigma-\sigma\varepsilon)\varepsilon/(1-\varepsilon))} m_{2,k}(\sigma-\sigma\varepsilon)\right\}^{\frac{1}{2}}\right]^2$$

$$= \lim_{\varepsilon\to 0}\left[\frac{\{m_{2,k}(\sigma)\}^{\frac{1}{2}} - \{(1-\varepsilon)^{-1} m_{2,k}(\sigma-\sigma\varepsilon)\}^{\frac{1}{2}}}{\varepsilon\sigma}\right]^2.$$

Now take $g(\sigma) = \log m_{2,k}(\sigma)/\sigma$; $g(\sigma)$ is a positive indefinitely increasing function of σ for $\sigma > \sigma_0 = \sigma_0(f)$, in fact $\log m_{2,k}(\sigma)$ is a convex function of σ, and so we have

$$m_{2,k}(\sigma, f^{(1)}) \geqq \lim_{\varepsilon\to 0}\left\{\frac{e^{\sigma g(\sigma)/2} - (1-\varepsilon)^{-\frac{1}{2}} e^{(\sigma-\sigma\varepsilon)g(\sigma-\sigma\varepsilon)/2}}{\varepsilon\sigma}\right\}^2$$

$$\geqq \lim_{\varepsilon\to 0}\left\{\frac{e^{\sigma g(\sigma)/2} - (1+\varepsilon/2+\frac{3}{8}\varepsilon^2+\cdots)e^{(\sigma-\sigma\varepsilon)g(\sigma)/2}}{\varepsilon\sigma}\right\}^2$$

$$= \left\{\frac{g(\sigma)}{2} e^{\sigma g(\sigma)/2} - \frac{1}{2\sigma} e^{\sigma g(\sigma)/2}\right\}^2$$

$$= \frac{1}{2^2}\left\{\frac{\log m_{2,k}(\sigma)-1}{\sigma}\right\}^2 m_{2,k}(\sigma).$$

Corollary 1. If $m_{2,k}(\sigma, f^{(1)})$ is the mean value of $f^{(1)}(s)$, the first derivative of an integral function $f(s)$ other than an exponential polynomial, then

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\left\{\dfrac{m_{2,k}(\sigma, f^{(1)})}{m_{2,k}(\sigma)}\right\}^{\frac{1}{2}}}{\sigma} \geqq \frac{\rho}{\lambda},$$

where ρ and λ are the Ritt-order and lower order of $f(s)$ respectively, and $D < \infty$.

This follows from Theorem 1 and Lemma 3.

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THEOREM 2. *Let* $m_{2,k}(\sigma, f^{(r)})$, $(r = 1, 2, \cdots, p)$, *be the mean value of* $f^{(r)}(s)$, *the* r*-th derivative of an integral function* $f(s)$ *other than an exponential polynomial. If* $\lambda \geqq \delta > 0$ *and* $D < \infty$, *then*

$$m_{2,k}(\sigma) < m_{2,k}(\sigma, f^{(1)}) < \cdots < m_{2,k}(\sigma, f^{(p)})$$

for $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \cdots, \sigma_p)$.

PROOF: Writing the above corollary for the r-th derivative, we have

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \left\{\dfrac{m_{2,k}(\sigma, f^{(r)})}{m_{2,k}(\sigma, f^{(r-1)})}\right\}^{\frac{1}{2}}}{\sigma} \geqq \begin{matrix}\rho\\ \lambda\end{matrix}.$$

Therefore,

$$m_{2,k}(\sigma, f^{(r)}) > e^{2\sigma(\lambda-\varepsilon)} m_{2,k}(\sigma, f^{(r-1)})$$

for $\sigma > \sigma_r$.

If $\lambda \geqq \delta > 0$

$$m_{2,k}(\sigma, f^{(r)}) > m_{2,k}(\sigma, f^{(r-1)})$$

for $\sigma > \sigma_r$.

Giving r the values $r = 1, 2, \cdots, p$, we get

$$m_{2,k}(\sigma) < m_{2,k}(\sigma, f^{(1)}) < \cdots < m_{2,k}(\sigma, f^{(p)})$$

for $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \cdots, \sigma_p)$.

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THEOREM 3. *Let* $m_{2,k}(\sigma, f^{(p)})$ *be the mean value of* $f^{(p)}(s)$ *the* p*-th derivative of an integral function* $f(s)$ *other than an exponential polynomial. If* $\lambda \geqq \delta > 0$ *and* $D < \infty$, *then*

$$m_{2,k}(\sigma, f^{(p)}) > \frac{1}{2^{2p}} \left\{\frac{\log m_{2,k}(\sigma) - 1}{\sigma}\right\}^{2p} m_{2,k}(\sigma) \tag{4.1}$$

for $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \cdots, \sigma_{p-1}, \sigma_1^1, \sigma_2^1, \cdots, \sigma_p^1)$.

PROOF: Writing (2.1) for the r-th derivative, we have

$$\frac{m_{2,k}(\sigma, f^{(r)})}{m_{2,k}(\sigma, f^{(r-1)})} \geqq \frac{1}{2^2}\left\{\frac{\log m_{2,k}(\sigma, f^{(r-1)})-1}{\sigma}\right\}^2$$

for $\sigma > \sigma_r^1$.

Giving r the values $r = 1, 2, \cdots, p$ and multiplying them, we get

$$\frac{m_{2,k}(\sigma, f^{(p)})}{m_{2,k}(\sigma)} \geqq \frac{1}{2^{2p}}\left\{\frac{\log m_{2,k}(\sigma, f^{(p-1)})-1}{\sigma}\right\}^2$$

$$\left\{\frac{\log m_{2,k}(\sigma, f^{(p-2)})-1}{\sigma}\right\}^2 \cdots \left\{\frac{\log m_{2,k}(\sigma)-1}{\sigma}\right\}^2$$

for $\sigma > \sigma_0^1 = \max(\sigma_1^1, \sigma_2^1, \cdots, \sigma_p^1)$.

Using Theorem 2, we get

$$\frac{m_{2,k}(\sigma, f^{(p)})}{m_{2,k}(\sigma)} > \frac{1}{2^{2p}}\left\{\frac{\log m_{2,k}(\sigma)-1}{\sigma}\right\}^{2p}$$

for $\sigma > \sigma_0 = \max(\sigma_1, \sigma_2, \cdots, \sigma_{p-1}, \sigma_1^1, \sigma_2^1, \cdots, \sigma_p^1)$.

Corollary 1. If $m_{2,k}(\sigma, f^{(p)})$ is the mean value of $f^{(p)}(s)$, the p-th the derivative of an integral function $f(s)$ other than an exponential polynomial, then

$$\lim_{\sigma\to\infty} \begin{array}{c}\sup\\ \inf\end{array} \frac{\log\left\{\dfrac{m_{2,k}(\sigma, f^{(p)})}{m_{2,k}(\sigma)}\right\}^{1/2p}}{\sigma} \geqq \frac{\rho}{\lambda},$$

where ρ and λ are the Ritt-order and lower order of $f(s)$ respectively, and $D < \infty$.

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