

# COMPOSITIO MATHEMATICA

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*Compositio Mathematica*, tome 17 (1965-1966), p. 36-39

[http://www.numdam.org/item?id=CM\\_1965-1966\\_\\_17\\_\\_36\\_0](http://www.numdam.org/item?id=CM_1965-1966__17__36_0)

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## Note on approximation theorems

by

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This note is concerned with three approximation theorems for a class of definite integrals and sine series with monotone decreasing coefficients. S. Aljančić, R. Bojanić and M. Tomić [1] have considered previously the asymptotic value of the definite integral

$$\Phi(\lambda) = \int_a^b f(t)L(\lambda t)dt, \quad 0 \leq a < b < \infty, \quad \lambda \rightarrow \infty,$$

where  $f(t) \in L(a, b)$  and  $L(t)$  is a slowly increasing function in the sense of Karamata. In this note, we shall consider the asymptotic behaviour of definite integrals of a more general type:

$$\Phi(\lambda) = \int_0^\infty f(t)\phi(\lambda t)dt,$$

where  $\phi(t)$  is such that  $\phi(t)/t^{p_1}$  is non-decreasing and  $\phi(t)/t^{p_2}$  is non-increasing as  $t$  is increasing in  $(0, \infty)$  for  $0 \leq p_1 \leq p_2 < \infty$ . In another paper [2], Aljančić, Bojanić and Tomić have also considered the asymptotic behaviour of  $g(x) = \sum_1^\infty \lambda_n \sin nx$ ,  $\lambda_n \searrow 0$ , as  $x \rightarrow +0$ , under the condition that  $\lambda_n \sim An^{-\alpha}L(n)$ , as  $n \rightarrow \infty$ . We shall now consider the asymptotic behaviour of the same series when  $\lambda_n$  is asymptotically defined by a more general class of functions.

**NOTATION.** By  $\phi(x) \sim [p_1, p_2]$ ,  $0 \leq p_1 \leq p_2 < \infty$  or  $-\infty < p_1 \leq p_2 \leq 0$ , we denote the non-negative even function  $\phi(x)$ , such that  $\phi(x)x^{-p_1}$  is non-decreasing and  $\phi(x)x^{-p_2}$  is non-increasing, as  $x$  is increasing in  $(0, \infty)$ . By  $\phi(x) \sim \langle p_1, p_2 \rangle$ , we mean  $\phi(x) \sim [p_1 + \varepsilon, p_2 - \varepsilon]$  for some  $\varepsilon > 0$ . We define  $\phi(x) \sim [p_1, p_2 \rangle$ ,  $\phi(x) \sim \langle p_1, p_2]$  in a similar way. By  $a(x) \simeq b(x)$  and  $a(x) \succ b(x)$ ,  $x \rightarrow c$ , we mean  $a(x)/b(x) \rightarrow 1$  and  $K_1 < a(x)/b(x) < K_2$ , respectively, as  $x \rightarrow c$ .

**THEOREM 1.** If  $\phi(x) \sim [p_1, p_2]$ ,  $0 \leq p_1 \leq p_2 < \infty$ , and if  $f(t) \geq 0$ ,  $t^{p_1}f(t) \in L(0, 1)$ ,  $t^{p_2}f(t) \in L(1, \infty)$ , then  $\Phi(\lambda) \asymp \phi(\lambda)$  as  $\lambda \rightarrow \infty$ . More precisely, we have

$$(1) \quad K_1(f, p_1, p_2)\phi(\lambda) \leq \Phi(\lambda) = \int_0^\infty f(t)\phi(\lambda t)dt \leq K_2(f, p_1, p_2)\phi(\lambda),$$

uniformly for  $0 < \lambda < \infty$ , where

$$(2) \quad K_1 = \int_0^1 t^{p_2} f(t) dt + \int_1^\infty t^{p_1} f(t) dt, \quad K_2 = \int_0^1 t^{p_1} f(t) dt + \int_1^\infty t^{p_2} f(t) dt.$$

In fact, if  $t > 1$ , then  $\phi(t\lambda)(t\lambda)^{-p_2} \leq \phi(\lambda)\lambda^{-p_2}$ , and  $\phi(\lambda)\lambda^{-p_1} \leq \phi(t\lambda)(t\lambda)^{-p_1}$ , i.e.  $t^{p_1}\phi(\lambda) \leq \phi(t\lambda) \leq t^{p_2}\phi(\lambda)$ . Similarly, if  $0 < t < 1$ , then  $t^{p_2}\phi(\lambda) \leq \phi(t\lambda) \leq t^{p_1}\phi(\lambda)$ . It follows that

$$(3) \quad \begin{aligned} \phi(\lambda) \left\{ \int_0^1 t^{p_2} f(t) dt + \int_1^\infty t^{p_1} f(t) dt \right\} &\leq \int_0^\infty f(t) \phi(\lambda t) dt \\ &\leq \phi(\lambda) \left\{ \int_0^1 t^{p_1} f(t) dt + \int_1^\infty t^{p_2} f(t) dt \right\}. \end{aligned}$$

**THEOREM 2.** Let

$$(4) \quad g(x) = \sum_1^\infty \lambda_n \sin nx,$$

where  $\lambda_n$  decreases steadily to zero. If  $\phi(x) \sim \langle -1, 0 \rangle$  and if  $\lambda_n \simeq \phi(n)$ , as  $n \rightarrow \infty$ , then  $g(x) \asymp 1/x \phi(1/x)$ , as  $x \rightarrow +0$ .

**PROOF.** Since  $\phi(x) \sim \langle -1, 0 \rangle$ ,  $\varepsilon \phi(x)/x \leq -\phi'(x) \leq (1-\varepsilon)\phi(x)/x$  for some  $\varepsilon > 0$ . It follows that

$$g(x) = \sum_{n=1}^\infty \lambda_n \sin nx = \sum_{1 \leq n \leq [1/x]} + \sum_{n > 1/x} = T_1 + T_2,$$

where

$$(5) \quad \begin{cases} |T_1| \leq K \int_1^{1/x} \phi(t) dt \leq K \left(\frac{1}{x}\right) \phi\left(\frac{1}{x}\right) - K \int_1^{1/x} t \phi'(t) dt \\ \leq K \left(\frac{1}{x}\right) \phi\left(\frac{1}{x}\right) + (1-\varepsilon) \int_1^{1/x} \phi(t) dt \leq K \left(\frac{1}{x}\right) \phi\left(\frac{1}{x}\right), \end{cases}$$

as  $x \rightarrow +0$ .

From Abel's transformation, it is easy to see that

$$(6) \quad |T_2| = \left| \sum_{n > 1/x} \lambda_n \sin nx \right| \leq K \phi\left(\frac{1}{x}\right) x^{-1}.$$

It remains to show that  $|g(x)| > K(1/x)\phi(1/x)$ , as  $x \rightarrow +0$ . In fact,

$$(7) \quad \begin{cases} g(x) = \sum_1^\infty \Delta \lambda_n \frac{\sin^2(n + \frac{1}{2}) \frac{x}{2}}{\sin \frac{x}{2}} - \frac{\lambda_1}{2} \operatorname{tg} \frac{x}{4} \\ = \sum_1^\infty \Delta \lambda_n \frac{\sin^2(n + \frac{1}{2}) \frac{x}{2}}{\sin \frac{x}{2}} + o\left(\frac{1}{x} \phi\left(\frac{1}{x}\right)\right) = I + II, \end{cases}$$

where

$$(8) \quad \left\{ \begin{aligned} I &> \frac{K}{x} \sum_{\pi/2x \leq n \leq 3\pi/2x} \Delta\phi(n) \geq \frac{K}{x} \left\{ \phi\left(\frac{\pi}{2x}\right) - \phi\left(\frac{3\pi}{2x}\right) \right\} \\ &= \frac{K}{x} \left\{ \phi\left(\frac{\pi}{2x}\right) - \phi\left(\frac{\lambda\pi}{2x}\right) \right\} \quad (\lambda = \frac{3}{2}) \\ &\geq \frac{K}{x} (1 - \lambda^{-\varepsilon}) \phi\left(\frac{\pi}{2x}\right) \geq K \left(\frac{1}{x}\right) \phi\left(\frac{1}{x}\right), \end{aligned} \right.$$

and  $II = o\{1/x \phi(1/x)\}$ , as  $x \rightarrow +0$ . Hence  $|g(x)| > K(1/x)\phi(1/x)$ , as  $x \rightarrow +0$ .

**THEOREM 3.** If  $\phi(x) \sim \langle -2, -1 \rangle$ ,  $\lambda_n \searrow 0$ ,  $\lambda_n \cong \phi(n)$ , as  $n \rightarrow \infty$ , then  $g(x) \asymp (1/x)\phi(1/x)$ , as  $x \rightarrow +0$ .

**PROOF.** From (7), together with  $\phi(x) \sim \langle -2, -1 \rangle$ , when  $x \simeq 1/n$  we also have

$$(9) \quad \left\{ \begin{aligned} |g(x)| &= \left| \sum_1^\infty \Delta\lambda_n \frac{\sin^2(n+1) \frac{x}{2}}{\sin \frac{x}{2}} \right| + o\left\{ \frac{1}{x} \phi\left(\frac{1}{x}\right) \right\} \\ &\leq \frac{1}{x} \sum_1^n \Delta\lambda_n n^2 x^2 + \frac{\lambda_n}{x} \\ &\leq x \sum_1^n k^2 \Delta\lambda_k + \frac{K}{x} \phi\left(\frac{1}{x}\right) \\ &\leq x \sum_1^n (2k-1)\lambda_k + \frac{K}{x} \phi\left(\frac{1}{x}\right) \\ &\leq Kx \sum_1^n (2k-1)\phi(k) + \frac{K}{x} \phi\left(\frac{1}{x}\right). \end{aligned} \right.$$

Write  $\tau(x) = x\phi(x)$ , then  $\tau(x) \sim \langle -1, 0 \rangle$ , since  $\tau(x)x^{1-\varepsilon}$  is non-decreasing. It follows that

$$(10) \quad \int_1^n \tau(x) dx = n\tau(n) - \tau(1) - \int_1^n x\tau'(x) dx.$$

But since  $x^{1-\varepsilon}\tau(x)$  is non-decreasing,  $(1-\varepsilon)x^{-\varepsilon}\tau(x) + x^{1-\varepsilon}\tau'(x) \geq 0$ , and  $-x\tau'(x) \leq (1-\varepsilon)\tau(x)$ . Substituting this in (10) and it follows then from (9) that

$$(11) \quad \int_1^n \tau(x) dx \leq Kn\tau(n), \quad |g(x)| \leq Kx \cdot n^2 \phi(n) \leq K \left(\frac{1}{x}\right) \phi\left(\frac{1}{x}\right),$$

as  $x \simeq 1/n \rightarrow 0$ ,  $n \rightarrow \infty$ . By slight modification in (7) and (8), it may be readily shown that

$$(12) \quad |g(x)| > K \left(\frac{1}{x}\right) \phi\left(\frac{1}{x}\right).$$

Hence  $g(x) \asymp (1/x)\phi(1/x)$ , as  $x \rightarrow +0$ .

*Added in proof.* The author has just observed that the argument in the proof of Theorem 3 actually gives a more general result, viz. with  $\phi(x) \sim \langle -2, -1 \rangle$  replaced by  $\phi(x) \sim \langle -2, 0 \rangle$ . This includes both Theorem 2 and Theorem 3 as particular cases.

#### REFERENCES

S. ALJANČIĆ, R. BOJANIĆ and M. TOMIĆ,

[1] Sur la valeur asymptotique d'une classe des intégrales définies. Acad. Serbe Sci. Publ. Inst. Math. 7, 81–94 (1954).

S. ALJANČIĆ, R. BOJANIĆ and M. TOMIĆ,

[2] Sur le comportement asymptotique au voisinage de zéro des séries trigonométriques de sinus à coefficients monotones. Acad. Serbe Sci. Publ. Inst. Math. 10, 101–120 (1956).

(Oblatum 4-1-62).