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A polynomial approach to topological analysis

by

Kenneth O. Leland

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In this paper we show how the basic results of topological analysis including power series expansions may be obtained for twice continuously complex differentiable functions without the use of any type of integral, measure theory, topological indexes [6], or algebraic topology, by making use of elementary methods and the Stone-Weierstrass Theorem. As a byproduct we obtain the theory of harmonic functions in the two dimensional case, including the existence of conjugate harmonic functions and the resolution of the Dirichlet problem for the circle. To extend our results to complex differentiable functions in general, we need only make a single application of Whyburn's Maximum Modulus Theorem [6].

Our basic tools are the complex polynomials. In Theorem 1, we prove the Maximum Modulus Theorem for the elements of the family A of all continuous functions on the closure $\overline{U}$ of the unit disc U, which are twice continuously differentiable on U. In Theorem 2, applying Theorem 1, we adapt a theorem of Porcelli and Connell [5] to show that all functions which are uniform limits on $\overline{U}$ of sequences of polynomials lie in A and are infinitely differentiable on U.

The key to the paper, Theorem 4, makes use of a simple auxiliary function to obtain growth rate estimates for polynomials which depend only on the magnitude of their real parts.

Employing the Stone-Weierstrass Theorem we show that every real valued continuous function on the boundary B of U may be extended to a function on $\overline{U}$ which is the uniform limit on $\overline{U}$ of the real parts of a sequence of polynomials. Applying Theorem 4, we show that these sequences must converge on U to complex differentiable functions.

Given a twice continuously differentiable function f on U, we then readily obtain a polynomial sequence approximating f on U.

Simple arguments of the author [2] and of Porcelli and Connell [1] are then used to convert this sequence into a power series expansion for f on U.

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Let K denote the complex plane and ω the positive integers. For $x \in K$ and $\delta > 0$, set $U(\delta) = \{z \in K; |z| < \delta\}$ and set $B(\delta) = \bar{U}(\delta) - U(\delta)$. For real numbers a and b, set $R(a+bi) = a$ and $I(a+bi) = b$. Set $U = U(1)$ and $B = B(1)$.

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THEOREM 1. *Let $f \in A$, and set $u = Rf$ and $v = If$. Then for $x \in U$:*

(1) $|u(x)| \leqq \sup \{|u(t)|;\ t \in B\}$,

(2) $|v(x)| \leqq \sup \{|v(t)|;\ t \in B\}$,

(3) $|f(x)| \leqq M = \sup \{|f(t)|;\ t \in B\}$.

PROOF. By direct computation $u_x = v_y$, $u_y = -v_x$ (Cauchy-Riemann equations), and $u_{xx}+u_{yy} = v_{yx}-v_{xy} = v_{xy}-v_{xy} = 0$ on U, where $u_x(z) = \partial u/\partial x|_z$ for $z \in U$. Let $\varepsilon > 0$, set $r(z) = 4^{-1}\varepsilon|z|^2$ for $z \in \bar{U}$, and set $w = u+r$. Then by direct computation $w_{xx}+w_{yy} = 0+r_{xx}+r_{yy} = \varepsilon$ on U.

Assume that for some $x \in U$, $w(x) \geqq N = \sup \{w(t);\ t \in B\}$. Then there exists $x_0 \in U$, such that $w(x_0) = \sup \{w(t);\ t \in \bar{U}\}$. Now $w_x(x_0) = w_y(x_0) = 0$ and $w_{xx}(x_0)$ or $w_{yy}(x_0) > 0$. But then either w restricted to the real axis or w restricted to the imaginary axis must have a minimum at x_0. Since ε is arbitrary, we have $u(x) \leqq \sup \{u(t);\ t \in B\}$ for all $x \in U$. A similar argument holds for minimum values and thus (1) is proven. The argument for (2) is similar.

Assume there exists $x_0 \in U$, such that $|f(x_0)| > M$. Set $g = f/M$. Then $|g(x_0)| > 1$ and $|g(x)| \leqq 1$ for all $x \in B$. Now there exists $n \in \omega$, such that $|g(x_0)^n| > 2$. Clearly $g^n \in A$. But then from (1) and (2),

$$\begin{aligned}|g(x_0)^n| &\leqq |R[g(x_0)^n]|+|I[g(x_0)^n]| \\ &\leqq \sup \{|R[g(t)^n]|;\ t \in B\}+\sup\{|I[g(t)^n]|;\ t \in B\} \\ &\leqq 2 \sup \{|g(t)^n|;\ t \varepsilon B\} \\ &\leqq 2.\end{aligned}$$

Theorem 2. *Let $P_1, P_2, \ldots$ be a sequence of polynomials, which converges uniformly on $\overline{U}$ to a limit function f. Then $f \in A$ and all derivatives of f on U exist.*

Proof. Without loss of generality we take $P_i(0) = 0$ for $i \in \omega$. For $i \in \omega$ and $x \in K$, set $Q_i(x) = P_i(x)/x$ for $x \neq 0$, and set $Q_i(x) = P_i'(0)$ for $x = 0$. Then $Q_1, Q_2, \ldots$ is a sequence of polynomials which converges at least pointwise to $f(x)/x$ for all $x \in \overline{U}$, $x \neq 0$, and which converges uniformly on B.

From Theorem 1, the sequence $Q_1, Q_2, \ldots$ must converge uniformly on $\overline{U}$ to a limit function Q_0. Hence $f'(0)$ exists and is equal to $Q_0(0) = \lim_{n\to\infty} Q_n(0) = \lim_{n\to\infty} P_n'(0)$. Thus f is differentiable on U.

Let $x \in U(1/2)$ and $m, n \in \omega$. Then from Theorem 1,

$$\begin{aligned} |P_n'(x)-P_m'(x)| &\leqq \sup \{|[P_n(t)-P_n(x)](t-x)^{-1} \\ &\qquad -[P_m(t)-P_m(x)](t-x)^{-1}|;\ t \in B\} \\ &\leqq (1/2)^{-1} \sup \{|[P_n(t)-P_m(t)] \\ &\qquad +[P_m(x)-P_n(x)]|;\ t \in B\} \\ &\leqq 4 \sup \{|P_n(s)-P_m(s)|;\ s \in B\}. \end{aligned}$$

Thus the sequence of polynomials $P_1', P_2', \ldots$ converges uniformly on compact subsets of U to f'.

The first part of our argument is adapted from a theorem of Porcelli and Connell [5].

Theorem 3. *Let f be an element of A such that $f(0) = 0$ and such that exists a sequence of polynomials $P_1, P_2, \ldots$ which converges uniformly on $\overline{U}(\delta)$ for some $0 < \delta < 1$. Then for $x \in U$, $|f(x)| \leqq M \cdot |x|$, where $M = \sup \{|f(t)|;\ t \in B\}$.*

Proof. For $x \in \overline{U}$, set $g(x) = f(x)/x$ for $x \neq 0$, and set $g(x) = f'(0)$ for $x = 0$. g is clearly twice differentiable at x for $x \in U$, $x \neq 0$. Defining $Q_1, Q_2, \ldots$ as in the proof of Theorem 2, we have that $Q_1, Q_2, \ldots$ converges uniformly on $\overline{U}(\delta)$ to g, and thus g is twice continuously differentiable at 0. Then from Theorem 1, for $x \in \overline{U}$, $x \neq 0$, $|f(x)/x| = |g(x)| \leqq \sup \{|g(t)|;\ t \in B\} = \sup \{|f(t)/t|;\ t \in B\} = M$, and thus $|f(x)| \leqq M \cdot |x|$.

Theorem 4. *Let $0 < r < 1$, and let P be a polynomial such that $P(0) = 0$. Then for $x \in U(r)$, $|P(x)| \leqq 2M|x|(1-r)^{-1}$, where $M = \sup \{|RP(t)|;\ t \in B\}$.*

Proof. For $z \in \overline{U}$, set $g(z) = P(z)[2M+P(z)]^{-1}$. Let $z \in \overline{U}$, and set $A = P(z)$ and $a = RP(z)$. Then $-a \leqq |a| \leqq M$, and hence $0 \leqq M+a$ and $0 < 4M^2+4Ma$; and thus $|A|^2 \leqq 4M^2+4Ma+|A|^2 = (2M+A)(2M+\bar{A}) = |2M+A|^2$. Then $|g(z)|^2 = |A|^2|2M+A|^{-2} \leqq 1$, and thus $|g(z)| \leqq 1$.

For $z \in K$, $n \in \omega$, set $Q_n(z) = -\sum_{p=1}^{n} [-P(z)/2M]^p$. There exists $0 < \delta < 1$, such that $|-P(z)/2M| \leqq 1/2$ for all $z \in \bar{U}(\delta)$. Then the sequence of polynomials $Q_1, Q_2, \ldots$ converges uniformly on $\bar{U}(\delta)$ to $-(-P/2M)[1-(-P/2M)]^{-1} = P(2M+P)^{-1} = g$. Let $z \in U(r)$. The from Theorem 3, $|g(z)| \leqq |z|$ and thus

$$\begin{aligned}|P(z)| &\leqq |2M+P(z)| \cdot |z| \\ &\leqq 2M|z|+|P(z)| \cdot |z|.\end{aligned}$$

Thus $|P(z)|(1-|z|) = |P(z)|-|P(z)| \cdot |z| \leqq 2M|z|$ and $|P(z)| \leqq 2M|z|(1-|z|)^{-1} \leqq 2M|z|(1-r)^{-1}$.

THEOREM 5. *If ϕ is a continuous real valued function on B, then there exists a continuous real valued function h on $\bar{U}$, and a complex valued function w on U, such that $h(x) = \phi(x)$ for all $x \in B$, all derivatives of w on U exist, and such that $h(x) = Rw(x)$ for all $x \in U$. Moreover if $f \in A$, there exists a sequence of polynomials $P_1, P_2, \ldots$ which converges uniformly on $\bar{U}$ to f.*

PROOF. Let $C(B)$ be the Banach algebra of continuous complex valued functions on B, and let T be the closed subalgebra of $C(B)$ generated by functions of the form $P(z)$ and $\overline{P(z)}$, where $z \in B$ and P is a complex polynomial. Clearly for $f \in T$, Rf and If lie in T, and hence we may readily verify for $a, b \in B$, $a \neq b$, that there exists a real valued element g of T, such that $g(a) \neq g(b)$. Then from the Stone-Weierstrass Theorem T must contain all continuous real valued functions on B, and thus $T = C(B)$.

For $n, m \in \omega$, $n > m$, and $z \in B$, $z^n \bar{z}^m = z^{n-m}$ and $\bar{z}^n z^m = \bar{z}^{n-m}$, and thus there exist sequences of polynomials $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots$, such that $IP_i(0) = IQ_i(0) = 0$, such that the sequence $P_1+\bar{Q}_1, P_2+\bar{Q}_2, \ldots$ converges uniformly on B to ϕ. Now for $i \in \omega$, $R(P_i+\bar{Q}_i) = R(P_i+Q_i)$. Hence setting $h_i = R(P_i+Q_i)$ for $i \in \omega$, we have from Theorem 1, that the sequence $h_1, h_2, \ldots$ converges uniformly on $\bar{U}$ to a limit function h such that $h(x) = \phi(x)$ for $x \in B$.

From Theorem 4, the sequence $P_1+Q_1, P_2+Q_2, \ldots$ must converge uniformly on compact subsets of U to a limit function w. From Theorem 2, all derivatives of w on U exist.

Set $\phi(x) = Rf(x)$ for $x \in B$. Then from Theorem 1, $|Rf(x)-h(x)| \leqq \sup_{0<r<1} \{|R[f(t)-w(t)]|; t \in B(r)\} = 0$. Set $u = R(f-w)$ and $v = I(f-w)$. Then $0 = u_x = v_y$ and $0 = u_y = -v_x$ on U, and hence from the mean value theorem for real valued functions we readily obtain $f-w \equiv If(0)$.

Let $\varepsilon > 0$. Then there exists $0 < \delta < 1$, such that $|f(x)-f(y)| < \varepsilon/2$ for all $x, y \in \overline{U}$ such that $|x-y| \leqq 1-\delta$, and there exists $n \in \omega$, such that $|f(x)-P_n(x)| < \varepsilon/2$ for all $x \in \overline{U}(\delta)$. Set $A_n(x) = P_n(\delta x)$ for all $x \in K$. Then $\delta x \in \overline{U}(\delta)$ for all $x \in \overline{U}$, and $|x-\delta x| = (1-\delta)|x| \leqq 1-\delta$, and hence

$$\begin{aligned} |f(x)-A_n(x)| &\leqq |f(x)-f(\delta x)|+|f(\delta x)-P_n(\delta x)| \\ &\leqq \varepsilon/2+\varepsilon/2 = \varepsilon. \end{aligned}$$

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We observe that since $h_{xx}+h_{yy} = 0$ on U, that we have solved the Dirichlet problem for the circle. Iw is called the conjugate harmonic function of h.

We also observe that since f is a uniform limit of polynomials on $\overline{U}$, that Theorems 1, 2, 3, and 4 apply to f.

If we weaken the differentiability requirements on f, we are no longer able to apply Theorem 1 to show that $f-w \equiv If(0)$. Instead we must use the topological methods of Whyburn [6].

Remark. Let P be a polynomial and assume that P has no roots. Then $1/P$ is a bounded twice continuously differentiable function on K. Then from Theorem 3, for $z \in K$ and $r > |z|$,

$$\begin{aligned} |P(z)-P(0)| &\leqq \sup \{|P(t)|;\ t \in B(r)\}|z|r^{-1} \\ &\leqq \sup \{|P(t)|;\ t \in K\}|z|r^{-1}. \end{aligned}$$

Letting r increase without limit, we obtain $|P(z)-P(0)| = 0$. Thus P is constant and thus we have verified the Fundamental Theorem of Algebra.

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Theorem 6. *If* $P(z) = \sum_0^n a_p z^p$ *for* $z \in K$, *and* $|P(z)| \leqq 1$ *for* $z \in \overline{U}$, *then* $|a_i| \leqq 1$ *for* $i = 0, 1, \ldots, n$.

Proof. This theorem and proof are due to Porcelli and Connell [1]. Trivially the theorem holds for polynomials of degree zero. Suppose for $n \in \omega$, it holds for polynomials of degree n or less and $P(z) = \sum_0^{n+1} a_p z^p$ is a polynomial of degree $n+1$, such that $|P(z)| \leqq 1$ for $z \in \overline{U}$.

Let $\theta \in K$, $|\theta| = 1$ and set $Q(z) = 2^{-1}[P(z)-P(\theta z)]$. Then

$Q(0) = 0$. Set $Q_0(x) = Q(x)/x$ for $x \neq 0$ and set $Q_0(x) = Q'(0)$ for $x = 0$. Then Q_0 is a polynomial and from Theorem 1, $|Q_0(x)| \leqq \sup\{|Q_0(t)|;\ t \in B\} = \sup\{|Q(t)|;\ t \in B\} \leqq 1$. By the induction hypothesis, $|2^{-1}a_{p+1}(1-\theta^{p+1})| \leqq 1$ for $p = 0, 1, \ldots, n$. Taking θ such that $\theta^p = -1$, we have $|a_p| \leqq 1$ for $p = 1, 2, \ldots, n+1$ Finally $|a_0| = |P(0)| \leqq 1$.

Theorem 7. *Let f be a twice continuously differentiable function on U. Then there exists a power series $T(z) = \sum_0^\infty a_p z^p$, which converges uniformly on compact subsets of U to f.*

Proof. Let $n \in \omega$. Then from Theorem 5, there exists a sequence of polynomials $P_1, P_2, \ldots$, which converges uniformly on $\bar{U}(1-1/n)$ to f. Let p be a positive integer such that $|P_p(x)-f(x)| < 1/2^{n-1}$ for $x \in \bar{U}(1-1/n)$ and set $Q_n = P_p$. Then for $n \in \omega$ and $i, j \in \omega$, $i, j \geqq n$, $|Q_i(x)-Q_j(x)| \leqq 1/2^n$ for all $x \in \bar{U}(1-1/n)$, and hence from Theorem 6, $|a_{ip}-a_{jp}| \leqq [2^n(1-1/n)^p]^{-1}$ for all $p \in \omega$, where $\{a_{ij}\}_{i,j\in\omega}$ is a sequence in K such that $Q_j(z) = \sum_0^\infty a_{jp} z^p$ for $j \in \omega$.

Thus for $p \in \omega$, there exists $a_p \in K$, such that for $n \in \omega$, $|a_p - a_{ip}| \leqq 2[2^n(1-1/n)^p]^{-1}$ for all $i \geqq n$, $i \in \omega$. Let $p \in \omega$. Then for all $n \in \omega$, $|P_n(x)| \leqq 1+1/2^{n+1}$ for all $x \in \bar{U}(1-1/n)$. Whence from Theorem 6, $|a_{np}| \leqq (1+1/2^{n+1})\ (1-1/n)^{-p}$ for all $n \in \omega$, and hence $|a_p| = \lim_{n\to\infty} |a_{np}| \leqq 1$.

Thus the power series $T(z) = \sum_0^\infty a_p z^p$ converges uniformly on compact subsets of U. Let $z \in U$. Then for $n \in \omega$ such that $n > 2(1-|z|)^{-1}$, we have $|z|(1-1/n)^{-1} < (n-2)/(n-1) < 1$ and

$$\begin{aligned} |T(z)-Q_n(z)| &= |\textstyle\sum_{p=0}^\infty (a_p - a_{np})z^p| \\ &\leqq \textstyle\sum_{p=0}^\infty 2[2^n(1-1/n)^p]^{-1}|z|^p \\ &= 2^{1-n}[1-|z|(1-1/n)^{-1}]^{-1} \\ &\leqq (n-1)/2^{n-1}. \end{aligned}$$

Thus $f(z) = \lim_{n\to\infty} Q_n(z) = T(z)$ for all $z \in U$.

This argument may be found in [2].

Theorem 8. (*Open Mapping Theorem*) *Let f be a non-constant twice continuously differentiable function on U. Then $f(U)$ is an open set.*

Proof. From Theorem 7, f can be expanded in a power series $\sum_0^\infty a_p z^p$. There exists $n \in \omega$, such that $a_n \neq 0$ and $f(z) = a_0 + \sum_n^\infty a_p z^p$ for $z \in U$. Set $g(z) = \sum_0^\infty a_{n+p} z^p$ for $z \in U$. Then $f(z) = a_0 + z^n g(z)$ for $z \in U$, and $g(0) = a_n \neq 0$. Now there exists

$0 < \delta < 1$, such that $g(x) \neq 0$ for all $x \in \overline{U}(\delta)$. Hence $f(x)-a_0 = x^n g(x) \neq 0$ for all $x \in \overline{U}(\delta)$, $x \neq 0$, and thus $f(0) \notin f[B(\delta)]$.

Assume that $f(0)$ is a boundary point of $M = f[\overline{U}(\delta)]$, and set $r = \inf\{|t-f(0)|;\ t \in f[B(\delta)]\}$. Let w be a point of $K-M$ such that $|w-f(0)| < r/2$ and set $r' = \inf\{|t-w|;\ t \in M\}$. Then $0 < r' < r/2$ and there must exist $w' \in M$ such that $|w-w'|=r'$. Then $|f(x)-w| \geqq r'$ for all $x \in \overline{U}(\delta)$ and $|f(x)-w'| > r-r' > r'$ for all $x \in B(\delta)$.

For $x \in \overline{U}(\delta)$, set $g(x)=[(f(x)-w)/r']^{-1}$. Then g is twice continuously differentiable on $U(\delta)$, $\sup\{|g(t)|;\ t \in B(\delta)\} < 1$, and there exists $x_0 \in U(\delta)$ such that $f(x_0) = w'$ and hence $|g(x_0)| = 1$. But this contradicts Theorem 1.

Theorem 9. *Let h be a continuous function on $\overline{U}$ such that h is harmonic on U and $h(0) = 0$. Let $0 < r < 1$. Then for $x \in U(r)$ and $n \in \omega$,*

(1) $|h(x)| \leqq 2M|x|(1-r)^{-1}$,
where $M = \sup\{|h(t)|;\ t \in B\}$, and

(2) $\|A_0^{(n)}\| \leqq 2M(1-r)^{-1}r^{1-n}$,
where $A_0^{(n)}$ is the n-th derivative of h at 0.

Moreover if for some $n \in \omega$, $\lim_{x\to 0} h(x)/|x|^n = 0$, then

(3) $|h(x)| \leqq 2M|x|^{n+1}r^n(1-r)^{-1}$.

Let F be the family of harmonic functions on open subsets of K into R. Then from (1) in the terminology of [3] F is an L_N family for some $N > 0$. From [3] the elements of L_N families are expandable at least locally in power series. (2) and (3) insure the full radius of convergence of expansions of elements of F.

Proof. Let k be the conjugate harmonic function and set $w = h+ik$. Then from Theorem 4, for $x \in \overline{U}(r)$,

$$\begin{aligned}|h(x)| \leqq |w(x)| &\leqq \sup_{r<s<1} 2\sup\{|Rw(t)|;\ t \in B(s)\}|x|s^{-1}(1-r/s)^{-1}\\ &\leqq \sup_{r<s<1} 2\sup\{|h(t)|;\ t \in B(s)\}|x|s^{-1}(1-r/s)^{-1}\\ &\leqq 2M|x|(1-r)^{-1}.\end{aligned}$$

From Theorem 7, w can be expanded in a power series $\sum_0^\infty a_n z^n$. From Theorem 6, for $n \in \omega$, and $z \in U(r)$,

$$
\begin{aligned}
|A_0^{(n)}(z)| &= |R(a_n z^n)| \\
&\leqq |a_n| \cdot |z|^n \\
&\leqq \sup \{|w(t)|;\ t \in B(r)\} r^{-n} |z|^n \\
&\leqq [2Mr(1-r)^{-1}] r^{-n} |z|^n \\
&\leqq 2M|z|^n (1-r)^{-1} r^{1-n},
\end{aligned}
$$

and thus $||A_0^{(n)}|| \leqq 2M(1-r)^{-1} r^{1-n}$.

For $n \in \omega$ and $x \in \overline{U}(r/2)$,

$$
\begin{aligned}
|w(x)| &\leqq |x| \sup \{|h(t);\ t \in B(r)\} r^{-1}(1-1/2)^{-1} \\
&\leqq \sup \{|h(t)|;\ t \in B(r)\}.
\end{aligned}
$$

Assume (3) holds. Then for $x \in B(r/2)$,

$$
|w(x)| \cdot |x|^{-n} \leqq 2^n \sup \{|h(t)| \cdot |t|^{-n};\ t \in B(r)\}
$$

and thus $\lim_{x \to \infty} w(x)/|x|^n = 0$. It then readily follows that $a_i = 0$ for $i = 0, 1, \ldots, n$, and hence for $z \in U(r)$,

$$
\begin{aligned}
|h(z)| &\leqq \textstyle\sum_{n+1}^{\infty} |A_0^{(p)}(z)| \\
&\leqq 2M(1-r)^{-1} r^{-1} [(|z| r^{-1})(1-|z| r^{-1})^{-1}] \\
&\leqq 2M|z|^{n+1} r^n (1-r)^{-1}.
\end{aligned}
$$

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