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On an extension of a theorem of O. Szász

by

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Suppose that $f(x)$ is integrable in Lebesgue's sense and periodic with period 2π . Let its Fourier series be

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

and let

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} \bar{A}_n(x)$$

be the conjugate series of the Fourier series of $f(x)$. Write

$$\bar{S}_n(x) = \sum_{\nu=1}^n (b_\nu \cos \nu x - a_\nu \sin \nu x) \equiv \sum_{\nu=1}^n \bar{A}_\nu(x).$$

Let $\bar{\sigma}_n^\alpha(x)$ be the n -th Cesàre mean of order α of the sequence $\{\bar{S}_n(x)\}$. O. Szász [3] has established the following

THEOREM A. *At a given point x , if there exists a number $D(x)$, such that*

$$\begin{aligned} \text{(i)} \quad \Psi(t) &= \int_0^t \psi(u) du \\ &\equiv \int_0^t \{f(x+u) - f(x-u) - D(x)\} du \\ &= o(t), \end{aligned}$$

$$\text{(ii)} \quad \Psi^*(t) = \int_0^t |\psi(u)| du = O(t)$$

as $t \rightarrow +0$, then

$$\lim_{n \rightarrow \infty} \{\bar{\sigma}_{2n}^1(x) - \bar{\sigma}_n^1(x)\} = \frac{1}{\pi} D(x) \log 2.$$

It should be noted that the corresponding conditions:

$$\begin{aligned}
 \text{(iii)} \quad \Phi(t) &= \int_0^t \varphi(u) du \\
 &= \int_0^t \{f(x+u) + f(x-u) - 2f(x)\} du \\
 &= o(t),
 \end{aligned}$$

$$\text{(iv)} \quad \Phi^*(t) = \int_0^t |\varphi(u)| du = O(t)$$

give Lebesgue's $(C, 1)$ summability criterion for the Fourier series of $f(x)$ at x .

On the other hand, we have a (C, α) summability criterion due to Hahn [1].

THEOREM B. (iii) is not sufficient for $(C, 1)$ summability of the Fourier series of $f(x)$ at x , though it implies (C, α) summability for every $\alpha > 1$.

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In this note, by applying Hahn's condition to Szász's theorem, we extend Theorem A as follows.

THEOREM. Let $\lambda \geq 2$ be any positive integer. If (i) holds, then

$$\lim_{n \rightarrow \infty} \{\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x)\} = \frac{1}{\pi} D(x) \log \lambda$$

for every $\alpha > 1$.

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The following lemmas are used.

LEMMA 1 [2]. If $\alpha > -1$ and $\bar{\tau}_n^\alpha(x)$ denotes the n -th Cesàro mean of order α of the sequence $\{n\bar{A}_n(x)\}$, then

$$\begin{aligned}
 \bar{\tau}_n^\alpha(x) &= n\{\bar{\sigma}_n^\alpha(x) - \bar{\sigma}_{n-1}^\alpha(x)\}, \\
 \bar{\tau}_n^{\alpha+1}(x) &= (\alpha+1)\{\bar{\sigma}_n^\alpha(x) - \bar{\sigma}_{n-1}^{\alpha+1}(x)\}.
 \end{aligned}$$

LEMMA 2 [2]. If $g_n^\alpha(t)$ denotes the n -th Cesàro mean of order α of the sequence $\{g_n(t)\}$, where $g_n(t) = \cos nt$ ($n \geq 1$), $g_0(t) = \frac{1}{2}$, then, for $\alpha > 0$, $0 < t < \pi$, $k = 0, 1, 2, \dots$,

$$\left| \left(\frac{d}{dt} \right)^k g_n^\alpha(t) \right| \leq \begin{cases} An^k & (k \geq 0), \\ An^{-2}t^{-k-2} & (k \leq \alpha-2), \\ An^{k-\alpha}t^{-\alpha} & (k > \alpha-2).^1 \end{cases}$$

¹ Through this paper, A denotes an absolute constant not necessarily the same at each occurrence.

This lemma can easily be proved with a similar argument used by Zygmund [4].

LEMMA 3. If

$$h_n^\alpha(t) = \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu} g_\nu^\alpha(t),$$

then, for $\alpha > 0$, $0 < t < \pi$, $k = 1, 2, \dots$,

$$\left| \left(\frac{d}{dt} \right)^k h_n^\alpha(t) \right| \leq \begin{cases} An^k & (k \geq 0), \\ An^{-2}t^{-k-2} & (k \leq \alpha-1), \\ An^{-k-\alpha-1}t^{-\alpha-1} & (k > \alpha-1). \end{cases}$$

By Lemma 1, with $g_n(t)$ in place of $\bar{S}_n(t)$, we have

$$(\alpha+1) \frac{1}{n} g_n^\alpha(t) = (\alpha+1) \frac{1}{n} g_n^{\alpha+1}(t) + g_n^{\alpha+1}(t) - g_{n-1}^{\alpha+1}(t).$$

Thus,

$$\begin{aligned} (\alpha+1) \left| \left(\frac{d}{dt} \right)^k h_n^\alpha(t) \right| &\leq (\alpha+1) \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu} \left| \left(\frac{d}{dt} \right)^k g_\nu^{\alpha+1}(t) \right| \\ &\quad + \left| \left(\frac{d}{dt} \right)^k g_{\lambda n}^{\alpha+1}(t) \right| + \left| \left(\frac{d}{dt} \right)^k g_n^{\alpha+1}(t) \right|. \end{aligned}$$

The lemma follows from Lemma 2.

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Now, we are in a position to prove the theorem. Since

$$\begin{aligned} n\bar{A}_n(x) &= \frac{n}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \sin nt dt \\ &= -\frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} \cos nt dt, \end{aligned}$$

from this, we get

$$\bar{\tau}_n^\alpha(x) = -\frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} g_n^\alpha(t) dt.$$

Now,

$$\begin{aligned} \bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) &= \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu} \bar{\tau}_\nu^\alpha(x) \\ &= -\frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} \frac{d}{dt} h_n^\alpha(t) dt. \end{aligned}$$

Denote

$$\omega_n = -\frac{1}{\pi} \int_0^\pi \frac{d}{dt} h_n^\alpha(t) dt.$$

Then

$$\begin{aligned} -\pi \{ \bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) - \omega_n D(x) \} &= \int_0^\pi \psi(t) \frac{d}{dt} h_n^\alpha(t) dt \\ &= \int_0^{l/n} + \int_{l/n}^\pi \\ &= I_1 + I_2, \end{aligned}$$

say, where $0 < l < n$. Let $\beta = \min(\alpha, 2)$, then, by Lemma 3, we have

$$\begin{aligned} |I_2| &\leq \left| \left\{ \Psi(t) \frac{d}{dt} h_n^\alpha(t) \right\}_{l/n}^\pi \right| + \left| \int_{l/n}^\pi \Psi(t) \left(\frac{d}{dt} \right)^2 h_n^\alpha(t) dt \right| \\ &\leq An^{-\beta} \left\{ |\Psi(\pi)| \pi^{-\beta-1} + \left| \Psi\left(\frac{l}{n}\right) \left(\frac{l}{n}\right)^{-\beta-1} \right| \right\} + An^{-\beta+1} \int_{l/n}^\pi |\Psi(t)| t^{-\beta-1} dt \\ &\leq An^{-\beta+1} + Al^{-\beta+1}, \end{aligned}$$

and

$$\begin{aligned} I_1 &= \left\{ \Psi(t) \frac{d}{dt} h_n^\alpha(t) \right\}_0^{l/n} - \int_0^{l/n} \Psi(t) \left(\frac{d}{dt} \right)^2 h_n^\alpha(t) dt \\ &= o\left(\frac{1}{n}\right) O(n) + o\left(n^2 \int_0^{l/n} t dt\right) \\ &= o(1) \end{aligned}$$

as $n \rightarrow \infty$. Hence,

$$\limsup_{n \rightarrow \infty} \pi |\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) - \omega_n D(x)| \leq Al^{-\beta-1}$$

for all $l > 0$. Since $\beta \geq \alpha > 1$, it follows that

$$\lim_{n \rightarrow \infty} \{ \bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x) - \omega_n D(x) \} = 0.$$

Now,

$$\begin{aligned} \omega_n &= -\frac{1}{\pi} \{ h_n^\alpha(\pi) - h_n^\alpha(0) \} \\ &= -\frac{1}{\pi} \sum_{\nu=n+1}^{\lambda n} \frac{1}{\nu A_\nu^\alpha} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{\alpha-1} \{ (-1)^\mu - 1 \}, \end{aligned}$$

where

$$A_n^\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Since the sequence $\{(-1)^n - 1\}$ is summable (C, α) to -1 for every $\alpha > 0$, we may write

$$\frac{1}{A_\nu^\alpha} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{\alpha-1} \{(-1)^\mu - 1\} = -1 + \varepsilon_\nu,$$

where $\varepsilon_\nu \rightarrow 0$ as $\nu \rightarrow \infty$. Therefore,

$$\omega_n = -\frac{1}{\pi} \sum_{\nu=n+1}^{\lambda n} \frac{-1 + \varepsilon_\nu}{\nu} = \frac{1}{\pi} \log \lambda + o(1)$$

as $n \rightarrow \infty$. I.e.,

$$\lim_{n \rightarrow \infty} \{\bar{\sigma}_{\lambda n}^\alpha(x) - \bar{\sigma}_n^\alpha(x)\} = \frac{D(x)}{\pi} \log \lambda.$$

This completes the proof of the theorem.

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