

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 17 (1965-1966), p. 190-206

[<http://www.numdam.org/item?id=CM_1965-1966__17__190_0>](http://www.numdam.org/item?id=CM_1965-1966__17__190_0)

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Extremal points in summability theory

by

J. W. Baker and G. M. Petersen

1.

The main purpose of this paper is to study the norms of regular summability methods introduced by Brudno in [4]. In this paper, the term *matrix* will be reserved for regular summability matrices. We shall be studying the summability of bounded sequences throughout. The usual norm for the space of bounded sequences is $||\{s_n\}|| = \sup |s_n|$, and the *unit sphere* is the set of bounded sequences $s = \{s_n\}$ with $||s|| \leq 1$. If A is a matrix, then $\mathcal{A}$ denotes the set of bounded sequences which are summed by A , $\mathcal{A}$ is called the *summability field* of A . If $\{s_n\}$ is summed by A , $A\text{-lim } s_n$ denotes the value to which it is summed. Two matrices, A and B , are said to be *b-consistent* if whenever a sequence is summed by both matrices, it is summed to the same value by A as by B ; A is said to be *b-stronger* than B if we have $\mathcal{A} \supset \mathcal{B}$. The following result relates these last two definitions, see [3] and [7].

THEOREM 1. *If A and B are matrices with A b-stronger than B , then A and B are b-consistent.*

From this theorem it is clear that if $\mathcal{A} = \mathcal{B}$ then A and B sum exactly the same bounded sequences to the same values. In this case we say that A and B are *b-equivalent*. By the *summability method*, $\mathfrak{U}$, we mean the set of all matrices which have $\mathcal{A}$ as their summability field.

2.

Let A be a matrix, then we have,

$$h(A) = \sup_m \sum_{n=1}^{\infty} |a_{m,n}| < \infty;$$

$h(A)$ is called the *matrix norm* of A . We may define,

$$(1) \quad N_A = \sup |A\text{-lim } s_n| \leq h(A),$$

where the supremum is taken over all the bounded sequences s , which are in the unit sphere and summed by A . If B is b -equivalent to A , then $N_B = N_A$, so that N_A is a function of the summability field $\mathcal{A}$ rather than of the matrices. For this reason, for each matrix we define the *field norm*, $N(\mathcal{A})$ by

$$N(\mathcal{A}) = N_A.$$

Since every matrix must sum the unit sequence to unity, the following theorem is an immediate consequence of Theorem 1.

THEOREM 2. *If A and B are matrices with A b -stronger than B , we have,*

$$N(\mathcal{A}) \geq N(\mathcal{B}) \geq 1.$$

We can also consider

$$||\mathfrak{U}|| = \inf h(A),$$

where the infimum is taken over the matrices, A , which have $\mathcal{A}$ as their summability field. The following fundamental theorem is due to Brudno, [4].

THEOREM 3. *For every matrix A ,*

$$||\mathfrak{U}|| = N(\mathcal{A}).$$

If there is an A' in $\mathfrak{U}$, such that

$$h(A') = ||\mathfrak{U}|| = N(\mathcal{A}),$$

we shall say that the field norm of $\mathcal{A}$ is *attained* by the matrix A' ; otherwise, we shall say that the norm is *not attained*.

We shall subsequently require the following constructions and notation. Suppose that $A^1, A^2, \dots, A^n, \dots$ is a countable family of matrices. Firstly, we may obtain a sequence, $\{\mu(k)\}$, of natural numbers, $\mu(k) \uparrow \infty$, such that,

$$(2) \quad \sum_{p=\mu(k)}^{\infty} |a_{m,p}^n| < \frac{1}{k}, \quad (1 \leq m, n \leq k),$$

then it follows that,

$$(3) \quad \lim_{m \rightarrow \infty} \sum_{p=\mu(m)}^{\infty} |a_{m,p}^n| = 0, \quad (n = 1, 2, \dots).$$

Since,

$$\lim_{m \rightarrow \infty} |a_{m,p}^n| = 0, \quad (n, p = 1, 2, \dots),$$

it is clear that there is a sequence, $\{m(r)\}$, of natural numbers with $m(r) > m(r-1)$, ($r = 2, 3, \dots$), such that

$$\sum_{p=1}^r |a_{m,p}^n| < \frac{1}{r}, \quad (n \leq r, r = 1, 2, \dots),$$

if $m \geq m(r)$. Then a sequence $\{\lambda(m)\}$ may be selected so that $\lambda(m) = 1$, ($m < m(2)$), and $\lambda(m) = r$, ($m(r) \leq m < m(r+1)$). Then,

$$(4) \quad \lim_{m \rightarrow \infty} \sum_{p=1}^{\lambda(m)} |a_{m,p}^n| = 0, \quad (n = 1, 2, \dots),$$

and $\{\lambda(m)\}$, $\lambda(m) \nearrow \infty$ satisfies,

$$(5) \quad \lambda(m) - \lambda(m-1) \leq 1, \quad (m = 2, 3, \dots).$$

If necessary, we may also increase the terms of $\{\mu(m)\}$ so that $\lambda(m) < \mu(m)$, ($m = 1, 2, \dots$), without changing the other properties of $\{\mu(m)\}$. This construction of $\{\lambda(m)\}$ and $\{\mu(m)\}$ is of course possible in particular for a finite set of matrices, $A^1, A^2, \dots, A^N$.

Before proceeding further, we require two auxiliary theorems. We first make some definitions.

DEFINITION 1. Let E be a set of bounded sequences, the set is said to be *uniformly summable* to s by the matrix A , if there exists a sequence, $\{\varepsilon_n\}$, $\varepsilon_n \searrow 0$, such that,

$$\left| \sum_{k=1}^{\infty} a_{m,k} s_k - s \right| < \varepsilon_m, \quad (m = 1, 2, \dots),$$

for every sequence, $\{s_n\}$, in E . The set is said to be *uniformly bounded* if there exists an H such that $\|\{s_n\}\| \leq H$, for every $\{s_n\}$ in E .

DEFINITION 2. A sequence $\{s_n\}$ is *eventually bounded* by the sequence $\{\xi_n\}$, if there exists an N such that $|s_n| \leq |\xi_n|$, ($n \geq N$).

If $\{\zeta_n^k\}$, $\zeta_n^k \searrow 0$, ($k = 1, 2, \dots$), is a countable set of sequences, let $m(k)$, ($k = 1, 2, \dots$) be the index such that,

$$|\zeta_n^r| < \frac{1}{k}, \quad (n \geq m(k), 1 \leq r \leq k).$$

Let $\{\xi_n\}$ be the sequence defined by

$$\xi_n = 1, \quad (n < m(1)), \quad \xi_n = \frac{1}{k}, \quad (m(k) \leq n < m(k+1)),$$

then $\{\zeta_n^k\}$ is eventually bounded by $\{\xi_n\}$, for each k , ($k = 1, 2, \dots$), and $\xi_n \searrow 0$.

THEOREM 4. *Let $A^1, A^2, \dots, A^n, \dots$ be a countable set of matrices, and let $\{s_k^n\}$, ($n = 1, 2, \dots$) be a countable set of uniformly bounded sequences, such that $\{s_k^n\}$, ($n \geq p$) is summed to zero by A^p , ($p = 1, 2, \dots$). Then, there exists a set of bounded sequences, $\{t_k^n\}$, ($n = 1, 2, \dots$) which are uniformly summed to zero by A^p , ($p \leq n$, $p = 1, 2, \dots$), and such that,*

$$\begin{aligned} s_k^n &= t_k^n, & (k \geq \bar{N}(n), n = 1, 2, \dots), \\ |t_k^n| &\leq |s_k^n|, & (n, k = 1, 2, \dots). \end{aligned}$$

PROOF. Let $\zeta_m^{n,p} = \sup_{r \geq m} |A_r^p(s^n)|$, ($m, n = 1, 2, \dots; n \geq p$), where,

$$A_r^p(s^n) = \sum_{k=1}^{\infty} a_{r,k}^p s_k^n.$$

Since $\zeta_m^{n,p} \searrow 0$ ($n = 1, 2, \dots; p \leq n$), there exists a sequence $\{\xi_m\}$, $\xi_m \searrow 0$, such that $\{A_m^p(s^n)\}$ is eventually bounded by $\{\xi_m\}$ for each n and each p , ($n = 1, 2, \dots; p \leq n$). Therefore, there exists an $N(n)$ such that,

$$|A_m^p(s^n)| < \xi_m, \quad (m > N(n), p \leq n).$$

Let $|s_k^n| \leq H$, ($k, n = 1, 2, \dots$), $h(A^p) = M_p$, ($p = 1, 2, \dots$), and let $\{\lambda(m)\}$ and $\{\mu(m)\}$ be chosen as above. Then we have,

$$\sum_{k=1}^{\lambda(m)} |a_{m,k}^p| < \varepsilon_{m,p}, \quad \sum_{k=\mu(m)}^{\infty} |a_{m,k}^p| < \varepsilon_{m,p},$$

where $\varepsilon_{m,p} \searrow 0$, ($p = 1, 2, \dots$). Let $\{\nu(r)\}$ be the sequence of indices such that $\lambda(\nu(r)) = \mu(\nu(r)-1)$, ($r = 2, 3, \dots$), $\nu(1) = 1$. To construct $\{t_k^n\}$, select q so that,

$$\lambda \left(\nu \left[\frac{q(q-1)}{2} \right] \right) > \mu(N(n)),$$

and set,

$$\begin{aligned} t_k^n &= 0, & \left(k \leq \lambda \left(\nu \left[\frac{q(q-1)}{2} \right] \right) \right), \\ t_k^n &= \frac{r}{q} s_k^n, & \left(\lambda \left(\nu \left[\frac{q(q-1)}{2} + r - 1 \right] \right) < k \leq \lambda \left(\nu \left[\frac{q(q-1)}{2} + r \right] \right) \right), \\ t_k^n &= s_k^n, & \left(k > \lambda \left(\nu \left[\frac{q(q+1)}{2} \right] \right) \right); \end{aligned} \quad (1 \leq r \leq q),$$

and note that $|t_k^n| \leq |s_k^n|$, ($k, n = 1, 2, \dots$).

i) If $m \leq N(n)$, $p \leq n$, we have,

$$\left| \sum_{k=1}^{\infty} a_{m,k}^p t_k^n \right| \leq H \sum_{k=\mu(m)+1}^{\infty} |a_{m,k}^p| \leq H \varepsilon_{m,p}.$$

ii) If $m > \nu[q(q+1)/2]$, and $p \leq n$, we have

$$\begin{aligned} \left| \sum_{k=1}^{\infty} a_{m,k}^p t_k^n \right| &\leq \left| \sum_{k=1}^{\lambda(m)} a_{m,k}^p t_k^n \right| + \left| \sum_{k=\lambda(m)+1}^{\infty} a_{m,k}^p s_k^n \right| \\ &\leq 2H \sum_{k=1}^{\lambda(m)} |a_{m,k}^p| + \left| \sum_{k=1}^{\infty} a_{m,k}^p s_k^n \right| \\ &\leq 2H \varepsilon_{m,p} + \xi_m. \end{aligned}$$

iii) If $\nu[q(q-1)/2+r-1] < m \leq \nu[q(q-1)/2+r]$, ($1 \leq r \leq q$), $p \leq n$, and if $\lambda'(r)$ denotes $\lambda(\nu[q(q-1)/2+r])$, we have,

$$\begin{aligned} \left| \sum_{k=1}^{\infty} a_{m,k}^p t_k^n \right| &\leq \left| \sum_{k=1}^{\lambda(m)} a_{m,k}^p t_k^n \right| + \left| \sum_{k=\lambda(m)+1}^{\mu(m)} \frac{r}{q} a_{m,k}^p s_k^n \right| \\ &\quad + \left| \sum_{k=\lambda'(r)+1}^{\mu(m)} \frac{1}{q} a_{m,k}^p s_k^n \right| + \left| \sum_{k=\mu(m)+1}^{\infty} a_{m,k}^p t_k^n \right| \\ &\leq H \left(1 + \frac{r}{q} \right) \sum_{k=1}^{\lambda(m)} |a_{m,k}^p| + \frac{r}{q} \left| \sum_{k=1}^{\infty} a_{m,k}^p s_k^n \right| \\ &\quad + \frac{H}{q} \sum_{k=\lambda'(r)+1}^{\mu(m)} |a_{m,k}^p| + H \left(1 + \frac{r}{q} \right) \sum_{k=\mu(m)+1}^{\infty} |a_{m,k}^p| \\ &\leq 4H \varepsilon_{m,p} + \frac{r}{q} \xi_m + \frac{HM_p}{q} \\ &\leq 4H \varepsilon_{m,p} + \xi_m + \frac{HM_p}{q}. \end{aligned}$$

iv) If $N(n) < m \leq \nu[q(q-1)/2]$, and if $\lambda'(0)$ denotes $\lambda(\nu[q(q-1)/2])$, we have,

$$\left| \sum_{k=1}^{\infty} a_{m,k}^p t_k^n \right| \leq \frac{1}{q} \left| \sum_{k=\lambda'(0)+1}^{\mu(m)} a_{m,k}^p s_k^n \right| + \left| \sum_{k=\mu(m)+1}^{\infty} a_{m,k}^p t_k^n \right|,$$

(the first sum of the right hand side being taken as zero if $\lambda'(0) \geq \mu(m)$). This implies that,

$$\left| \sum_{k=1}^{\infty} a_{m,k}^p t_k^n \right| \leq \frac{HM_p}{q} + H \varepsilon_{m,p}.$$

It follows that for every m and n , ($m, n = 1, 2, \dots$), and for $p \leq n$,

$$|\sum_{k=1}^{\infty} a_{m,k}^p t_k^n| \leq 4H\varepsilon_{m,p} + \xi_m + \frac{HM_p}{q},$$

where q is such that $\nu[q(q-1)/2] < m \leq \nu[q(q+1)/2]$. For fixed p and n , $4H\varepsilon_{m,p} + \xi_m + HM_p/q$ converges to zero as $m \nearrow \infty$, and the proof is completed by observing that if $n \geq p$, these terms are independent of n .

DEFINITION 3. Consider the partial sums $\{u_r\}$ of the series

$$1 - 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n} + \dots \\ + \frac{1}{n} - \frac{1}{n} - \frac{1}{n} - \dots - \frac{1}{n} + \dots,$$

where $1/n$ appears in the series $2n$ times, the first n times with the plus sign, the second n times with the minus sign. We shall call the sequence $\{u_r\}$ the c -sequence.

It is evident that $0 \leq u_r \leq 1$, ($r = 1, 2, \dots$), and that $u_{q(q+1)} = 0$, $u_{q^2} = 1$, ($q = 1, 2, \dots$); if $\eta_r = |u_r - u_{r+1}|$, then $\lim_{r \rightarrow \infty} \eta_r = 0$.

Suppose that we have a countable set of matrices, $A^1, A^2, \dots, A^n, \dots$, and that $\{\lambda(m)\}$, $\{\mu(m)\}$ and $\{m(r)\}$ have been chosen as above. Let $v = \{v_r\}$ be a sequence of real numbers with $0 \leq v_r \leq 1$, ($r = 1, 2, \dots$). Suppose that the set of zeroes of this sequence is infinite so that $v_{z(k)} = 0$, ($k = 1, 2, \dots$), say, $z(k) \nearrow \infty$. For each natural number r , $r \geq m[z(1)]$, there exists $q = q(r)$, such that,

$$\lambda(m[z(q)]) \leq \lambda(r) \leq \lambda(r+1) < \lambda(m[z(q+1)]).$$

For each r , $r \geq m[z(1)]$, and each $q(r)$, let $I(v, r, q)$ denote the interval,

$$[\lambda(r), \lambda(r+1)] \subset (\lambda(m[z(q)]), \lambda(m[z(q+1)])].$$

Every natural number k , $k \geq \lambda(m[z(1)])$, belongs to a unique interval $I(v, r, q)$.

In particular, if $u = \{u_r\}$ is the c -sequence, then $I(u, r, q)$ is the interval,

$$[\lambda(r), \lambda(r+1)] \subset (\lambda(m[q(q-1)]), \lambda(m[q(q+1)]));$$

and if $u' = \{1 - u_r\}$, then $I(u', r, q)$ is the interval,

$$[\lambda(r), \lambda(r+1)] \subset (\lambda(m[q^2]), \lambda(m[(q+1)^2])).$$

We now state the following theorem, which is a generalization of Theorem 4 of [2].

THEOREM 5. Let $A = (a_{m,k})$ be a matrix. Choose $\{\lambda(m)\}$, $\{\mu(m)\}$ and $\{m(r)\}$ as in the beginning of this section. Let $v = \{v_r\}$ be a

sequence as above, with $0 \leq v_r \leq 1$, ($r = 1, 2, \dots$) and $\lim_{r \rightarrow \infty} \gamma_r = 0$, where $\gamma_r = |v_r - v_{r+1}|$, ($r = 2, 3, \dots$). Then, if $\{s_k^q\}$, ($q = 1, 2, \dots$) is a countable set of uniformly bounded sequences, uniformly summed to zero by A , the bounded sequence $\{s_k\}$ is A summable to zero, where,

$$\begin{aligned} s_k &= v_r s_k^q \quad \text{for } k \in I(v, r, q), \\ s_k &= s_k^1 \quad \text{for } k < \lambda(m[z(1)]). \end{aligned}$$

PROOF. Let $\{s_k\}$ be the sequence described in the theorem, $h(A) = M$, and $|s_k^q| \leq H$, ($k, q = 1, 2, \dots$). Let $\{\zeta_m\}$ be a sequence, $\zeta_m \searrow 0$, such that,

$$\left| \sum_{k=1}^{\infty} a_{m,k} s_k^q \right| < \zeta_m, \quad (m, q = 1, 2, \dots).$$

For m with $\lambda(m)$ in $I(v, r, q)$,

$$\begin{aligned} \left| \sum_{k=1}^{\infty} a_{m,k} s_k \right| &\leq \left| \sum_{k=1}^{\lambda(m)} a_{m,k} s_k \right| + v_r \left| \sum_{k=\lambda(m)+1}^{\mu(m)} a_{m,k} s_k^q \right| \\ &\quad + |v_r - v_{r+1}| \left| \sum_{k=\lambda(m(r+1))}^{\mu(m)} a_{m,k} s_k^q \right| + \left| \sum_{k=\mu(m)+1}^{\infty} a_{m,k} s_k \right| \end{aligned}$$

(the third sum to be taken as zero if $\lambda(m(r+1)) \geq \mu(m)$).

It follows that,

$$\left| \sum_{k=1}^{\infty} a_{m,k} s_k \right| \leq 2H \sum_{k=1}^{\lambda(m)} |a_{m,k}| + 2H \sum_{k=\mu(m)}^{\infty} |a_{m,k}| + v_r \left| \sum_{k=1}^{\infty} a_{m,k} s_k^q \right| + \gamma_r MH,$$

and the sequence $\{s_k\}$ is A summable to zero.

This theorem is true in particular if $v = \{u_r\}$ or $v = \{1 - u_r\}$. We shall describe the operations performed in Theorem 4 as *interweaving the sequences* $\{s_k^q\}$.

Now suppose that A is a matrix, and that $s^n = \{s_k^n\}$, ($n = 1, 2, \dots$), are sequences in the unit sphere with $A\text{-lim } s^n = \alpha^n$. Suppose that $\lim_{n \rightarrow \infty} \alpha^n = \alpha$, and that the sequences $\{s_k^n - \alpha^n\}$, ($n = 1, 2, \dots$) are summed uniformly to zero by A . Let $\{t_k^2\}$ be defined by

$$t_k^2 = u_r (s_k^{2^q-1} - \alpha^{2^q-1}) \quad \text{for } k \text{ in } I(u, r, q)$$

and

$$t_k^2 = 0, \quad k < \lambda(m(2));$$

and $\{t_k^1\}$ by

$$t_k^1 = (1 - u_r)(s_k^{2^q} - \alpha^{2^q}) \quad \text{for } k \text{ in } I(u', r, q),$$

and

$$t_k^1 = 0, \quad k < \lambda(m(1)),$$

where u is the c -sequence $\{u_r\}$, and u' the sequence, $\{1 - u_r\}$.

From Theorem 4 it follows that $\{t_k^1\}$ and $\{t_k^2\}$ are both summed to zero by A . Let $t_k = t_k^1 + t_k^2$, ($k = 1, 2, \dots$), then $\{t_k\}$ is also summed to zero by A , and $\{t_k + \alpha\}$ is summed to α by A . Now if $\lambda(m(q^2)) < \lambda(m(r)) \leq k \leq \lambda(m(r+1)) \leq \lambda(m[q(q+1)])$, we have k in $I(u, r, q) \cap I(u', r, q)$, and

$$t_k + \alpha = u_r(s_k^{2q-1} - \alpha^{2q-1}) + (1 - u_r)(s_k^{2q} - \alpha^{2q}) + \alpha,$$

and if $\lambda(m[q(q+1)]) < \lambda(m(r)) \leq k \leq \lambda(m(r+1)) \leq \lambda(m[(q+1)^2])$, we have k in $I(u, r, q+1) \cap I(u', r, q)$, and,

$$t_k + \alpha = u_r(s_k^{2q+1} - \alpha^{2q+1}) + (1 - u_r)(s_k^{2q} - \alpha^{2q}) + \alpha.$$

Now the sequence $\{\nu_k\}$ defined by

$$\nu_k = \alpha - (1 - u_r)\alpha^{2q} - u_r\alpha^{2q-1} \text{ for } k \text{ in } I(u, r, q) \cap I(u', r, q),$$

and

$$\nu_k = \alpha - (1 - u_r)\alpha^{2q} - u_r\alpha^{2q+1} \text{ for } k \text{ in } I(u, r, q+1) \cap I(u', r, q)$$

converges to zero. Thus the sequence $\{s_k\}$, defined by,

$$s_k = t_k + \alpha - \nu_k, \quad (k = 1, 2, \dots),$$

is A summable to α , further, $\{s_k\}$ is in the unit sphere.

Hence, we have the following:

THEOREM 6. *Let A be a matrix, and suppose that $\{s_k^n\}$, ($n = 1, 2, \dots$) are sequences in the unit sphere with $A\text{-lim } s^n = \alpha^n$. Suppose that $\lim_{n \rightarrow \infty} \alpha^n = \alpha$, and that the sequences $\{s_k^n - \alpha^n\}$, ($n = 1, 2, \dots$) are summed uniformly to zero by A . Then there is a sequence in the unit sphere, $\{s_k\}$, such that $A\text{-lim } s_k = \alpha$.*

From this theorem we can immediately deduce the following, which is originally due to Brudno, see [4].

THEOREM 7. *Let A be a matrix. Then A sums a sequence in the unit sphere, s , with*

$$A\text{-lim } s = N(\mathcal{A}).$$

PROOF. By definition of N , we have sequences $s^n = \{s_k^n\}$, ($n = 1, 2, \dots$), in the unit sphere with $A\text{-lim } s^n = \alpha^n$, and $\lim_{n \rightarrow \infty} \alpha^n = N(\mathcal{A})$. After theorem 4, we may assume that these sequences are uniformly summable by A , without altering other properties. Theorem 6 then implies that we can construct a sequence s in the unit sphere with

$$A\text{-lim } s = N(\mathcal{A}).$$

In [3], see also [7], the following has been shown.

THEOREM 8. *Let $A^1, A^2, \dots, A^n, \dots$ be a countable set of regular matrices, with $\mathcal{A}^n \supset \mathcal{A}^{n+1}$, ($n = 1, 2, \dots$), then the intersection of their bounded convergence fields is the bounded convergence field of some matrix A .*

Theorem 6 enables us to obtain more information in this direction.

THEOREM 9. *Let $A^1, A^2, \dots, A^n, \dots$, be a countable set of regular matrices with $\mathcal{A}^n \supset \mathcal{A}^{n+1}$, ($n = 1, 2, \dots$), then if $\mathcal{A} = \bigcap_{n=1}^{\infty} \mathcal{A}^n$,*

$$N(\mathcal{A}) = \lim_{n \rightarrow \infty} N(\mathcal{A}^n).$$

PROOF. From Theorem 7, we can construct sequences $s^n = \{s_k^n\}$ in the unit sphere with

$$A^r\text{-}\lim s^n = N(\mathcal{A}^n) = \alpha^n, \quad (r \leq n).$$

Now $\{\alpha^n\}$ is a decreasing sequence of positive numbers and bounded from below by Theorem 2. Hence, $\lim_{n \rightarrow \infty} \alpha^n = \alpha$, say. It is clear from Theorem 4, that we may assume that for each r , ($r = 1, 2, \dots$), the sequences $\{s_k^n\}$, ($n = r, r+1, \dots$), are uniformly summed by A^r . We may construct a sequence $\{s_k\}$ by using functions $\lambda(m)$ and $\mu(m)$ that are appropriate for all the matrices A^r , ($r = 1, 2, \dots$), (as in (3), (4) and (5)) and using the sequences $\{s_k^n\}$, ($n = 1, 2, \dots$) for an interweaving as in the proof of Theorem 6. Since only finitely many terms of $\{s_k\}$ are derived from the sequences $\{s_k^n\}$, ($n < r$), it is clear that as in Theorem 6, we have,

$$A^r\text{-}\lim s = \alpha, \quad (r = 1, 2, \dots).$$

Thus s belongs to $\bigcap_{n=1}^{\infty} \mathcal{A}^n = \mathcal{A}$, and from Theorem 1 we have $A\text{-}\lim s = \alpha$. But s is in the unit sphere, hence, $N(\mathcal{A}) \geq \alpha$. From Theorem 2 we see that $N(\mathcal{A}) \leq \lim_{n \rightarrow \infty} N(\mathcal{A}^n) = \alpha$ and this implies that $N(\mathcal{A}) = \lim_{n \rightarrow \infty} N(\mathcal{A}^n)$ as required.

3.

We are now going to consider some results which follow from Theorem 7.

DEFINITION 4. If A is a matrix, a sequence s in the unit sphere is called an *extremal point* of $\mathcal{A}$ if

$$A\text{-}\lim s = N(\mathcal{A}).$$

Hence, Theorem 7 tells us that every bounded convergence field of a matrix has at least one extremal point.

THEOREM 10. *Suppose that A is a matrix and that s is an extremal point of $\mathcal{A}$. Then $\limsup_{n \rightarrow \infty} s_n = 1$. Further, if $N(\mathcal{A}) > 1$, we also have $\liminf_{n \rightarrow \infty} s_n = -1$. Before proceeding to the proof we remark that the condition $N(\mathcal{A}) > 1$ is essential to obtain $\liminf_{n \rightarrow \infty} s_n = -1$, since any matrix with $N(\mathcal{A}) = 1$ will sum the unit sequence $\{1\}$, to the value $1 = N(\mathcal{A})$.*

PROOF. Let s be an extremal point of $\mathcal{A}$. Suppose that $\limsup_{n \rightarrow \infty} s_n = U < 1$. We can find an n_0 such that $s_n \leq U + L/2$, for $n \geq n_0$. Define $s' = \{s'_n\}$ by

$$\begin{aligned} s'_n &= 0, & (n < n_0), \\ s'_n &= s_n + \frac{1-U}{2}, & (n \geq n_0). \end{aligned}$$

Then $\{s'_n\}$ is the unit sphere. If $N(\mathcal{A}) = N$, then $A\text{-}\lim s' = N + 1 - U/2 > N$. This contradiction establishes the first part of the theorem.

Now suppose that $N > 1$, and that $\liminf_{n \rightarrow \infty} s_n = L > -1$. Let $\varepsilon = (L+1)/8 > 0$. We have $s_n \geq -1 + 4\varepsilon$, if $n \geq n_1$, say. Let

$$\begin{aligned} t_n &= 0, & (n < n_1), \\ t_n &= \frac{s_n - \varepsilon/N}{1 - \varepsilon/N} & (n \geq n_1). \end{aligned}$$

Then $t_n \leq \frac{1 - \varepsilon/N}{1 - \varepsilon/N} = 1$, also,

$$t_n \geq \frac{-1 + 4\varepsilon - \varepsilon/N}{1 - \varepsilon/N} \geq \frac{-1 + 3\varepsilon/N}{1 - \varepsilon/N} \geq -1.$$

Thus, $\{t_n\}$, is in the unit sphere. Also,

$$A\text{-}\lim t = \frac{N - \varepsilon/N}{1 - \varepsilon/N} = N + \frac{\varepsilon - \varepsilon/N}{1 - \varepsilon/N} > N, \text{ since } N > 1.$$

This contradiction establishes that $L = -1$, and the theorem is proved.

THEOREM 11. *Let $\mathcal{A}$ be a summability method such that $N(\mathcal{A}) > 1$. If the norm of $\mathcal{A}$ is attained, there is an extremal point of $\mathcal{A}$, $\{s_n\}$, such that $s_n = 1$, or $s_n = -1$, ($n = 1, 2, \dots$).*

PROOF. Without loss of generality we may assume that A is a matrix with $h(A) = N(\mathcal{A})$. Let s be an extremal point of $\mathcal{A}$. Then $\|s\| = 1$, $A\text{-}\lim s = N(\mathcal{A})$ and $\liminf_{n \rightarrow \infty} s_n = -1$, $\limsup_{n \rightarrow \infty} s_n = 1$. We define $s' = \{s'_n\}$ by

$$s'_n = 1 \text{ if } s_n \geq 0; \quad s'_n = -1 \text{ if } s_n < 0, \quad (n = 1, 2, \dots).$$

We shall show that $\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{m,n} s'_n = N(\mathcal{A}) = N$. Now, $h(A) = N$, so that if $\{t_n\}$ is in the unit sphere, then,

$$(6) \quad \left| \sum_{n=1}^{\infty} a_{m,n} t_n \right| \leq N, \quad (m = 1, 2, \dots).$$

Suppose that

$$\liminf_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{m,n} s'_n = N_1 < N,$$

then,

$$\limsup_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{m,n} (2s_n - s'_n) = 2N - N_1 > N.$$

However, $|2s_n - s'_n| \leq 1$, ($n = 1, 2, \dots$), so in view of (6) we must have,

$$\liminf_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{m,n} s'_n \geq N.$$

But (6) is also applicable to s' , so we have

$$\limsup_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{m,n} s'_n \leq N.$$

Hence, A sums s' to N , and s' is the required sequence.

4.

We now consider some examples and applications. Firstly, it is possible that a summability method does not attain its norm. Consider the matrix, A , which transforms the sequence $\{s_n\}$ into the sequence $\{t_n\}$, as follows:

$$t_{2n-1} = 6s_{4n-3} - 5s_{4n-2}, \quad t_{2n} = 2s_{4n-1} - s_{4n}, \quad (n = 1, 2, \dots).$$

If $\{s_n\}$ is the sequence with $s_{4n-3} = \frac{1}{2}$, $s_{4n-2} = 0$, $s_{4n-1} = 1$, $s_{4n} = -1$, then A sums $\{s_n\}$ to the value 3. Hence $N(\mathcal{A}) \geq 3 > 1$, but the matrix defined by the even numbered transformations is b -stronger than A and has a matrix norm 3. From Theorems 2 and 3 we see that $N(\mathcal{A}) = 3$. It is clear that no sequence $\{s'_n\}$ of 1's and -1's with $\limsup_{n \rightarrow \infty} s_n = 1$, $\liminf_{n \rightarrow \infty} s_n = -1$ can be summed by A . In view of Theorem 11, we conclude that $\mathcal{A}$ does not attain its norm. Further, if B is any matrix such that $\mathcal{B} \subset \mathcal{A}$, and $N(\mathcal{B}) > 1$, the above argument shows us that $\mathcal{B}$ cannot attain its norm. Thus A is a matrix such that no

method weaker than $\mathcal{A}$ which has field norm strictly greater than unity, can attain its norm.

A slightly different example is the matrix A defined by the transformations,

$$t_n = 2s_{2n} - s_{2n+1}, \quad (n = 1, 2, \dots).$$

Here, $N(\mathcal{A}) = 3$ and the reader can show by arguments similar to the above that no weaker matrix attains its norm; however, in this case $h(A) = N(\mathcal{A})$.

The following result which is proved in [6], Theorem 4, is relevant here. In the terminology of this paper it is as follows:

THEOREM 12. *For each summability method $\mathcal{B}$, there is a method $\mathcal{B}^A$, with $N(\mathcal{B}) = N(\mathcal{B}^A)$ such that the norm of $\mathcal{B}^A$ is not attained.*

We remark that, as is clear from the proof given in [6], and was meant to be included in the statement of the theorem, *the method $\mathcal{B}^A$ is b -stronger than $\mathcal{B}$.*

This result means that the converse of Theorem 11 is not true in all cases. In fact, let A be a matrix which sums a sequence $s = \{s_n\}$ with $s_n = 1$ or $s_n = -1$, ($n = 1, 2, \dots$), to the value $N(\mathcal{A})$. Suppose that the norm of $\mathcal{A}$ is attained. Choose a matrix A^A such that $N(\mathcal{A}) = N(\mathcal{A}^A)$, and such that the norm of $\mathcal{A}^A$ is not attained. Then A^A sums s to $N(\mathcal{A}) = N(\mathcal{A}^A)$. Thus every sequence which is an extremal point for some method is an extremal point for a method for which the norm is not attained.

We now turn to another application. We recall the following theorem, see [8], Theorem 2.

THEOREM 13. *Let A be a matrix. Then there exists a positive sequence $\{x_n\}$ such that $\sum_{n=1}^{\infty} x_n = \infty$, $x_n \searrow 0$, and such that a bounded sequence $\{s_n\}$ is summed to zero by A if and only if, every sequence of the form $\{\xi_n s_n\}$ is a member of $\mathcal{A}$, where $\{\xi_n\}$ is bounded and $\xi_n - \xi_{n-1} = 0(x_n)$.*

In fact a glance at the proof shows that the sequence $\{x_n\}$ which is constructed has the property that $\{\xi_n s_n\}$ is not only summed by A , but is summed to zero by A .

Now suppose that we have a matrix A , and a sequence $s = \{s_n\}$ summed to zero by A in $\mathcal{A}$, where $\limsup_{n \rightarrow \infty} s_n = \alpha$, $\liminf_{n \rightarrow \infty} s_n = \beta$, with $\alpha > \beta > 0$. Then we can find sequences $\{n_k\}$ and $\{p_k\}$ of natural numbers, strictly increasing for which $\lim_{k \rightarrow \infty} s_{n_k} = \alpha$, $\lim_{k \rightarrow \infty} s_{p_k} = \beta$. Suppose that $\alpha' > \beta' > 0$, define $\alpha_0 = \alpha'/\alpha$, $\beta_0 = \beta'/\beta$. We may clearly construct a sequence $\{\xi_n\}$, as in Theorem 13, for which $A\text{-}\lim \xi_n s_n = 0$,

$$\limsup_{k \rightarrow \infty} \{\xi_{n_k}\} = \alpha_0, \quad \liminf_{k \rightarrow \infty} \{\xi_{p_k}\} = \beta_0$$

and also $\alpha_0 \geq \xi_n \geq \beta_0$, ($n = 1, 2, \dots$). Then we have

$$\limsup_{n \rightarrow \infty} \xi_n s_n = \alpha', \quad \liminf_{n \rightarrow \infty} \xi_n s_n = \beta'.$$

THEOREM 14. *Suppose that A is a matrix, and that $N(\mathcal{A}) > 1$. Then for each a , $1 < a < N(\mathcal{A})$, there exists a sequence $t = \{t_n\}$ in $\mathcal{A}$ with $A\text{-}\lim t = a$, $\limsup_{n \rightarrow \infty} t_n = 1$, $\liminf_{n \rightarrow \infty} t_n = -1$ and $\|t\| = 1$.*

PROOF. Let $s = \{s_n\}$ be an extremal point of $\mathcal{A}$, so that $A\text{-}\lim s = N(\mathcal{A}) = N$, $\limsup_{n \rightarrow \infty} s_n = 1$, $\liminf_{n \rightarrow \infty} s_n = -1$. Let $s'_n = N - s_n > 0$, ($n = 1, 2, \dots$). Then $\limsup_{n \rightarrow \infty} s'_n = N+1$, $\liminf_{n \rightarrow \infty} s'_n = N-1$, $A\text{-}\lim s'_n = 0$, and

$$N+1 \geq s'_n \geq N-1, \quad (n = 1, 2, \dots).$$

As indicated above, we may find a sequence $\{\xi_n\}$ with $A\text{-}\lim \{\xi_n s'_n\} = 0$,

$$\limsup_{n \rightarrow \infty} \xi_n s'_n = N+1, \quad \liminf_{n \rightarrow \infty} \xi_n s'_n = (N+1) \frac{a-1}{a+1},$$

and $1 \geq \xi_n \geq (N+1)(a-1)(N-1)^{-1}(a+1)^{-1}$, ($n = 1, 2, \dots$).

Define,

$$\begin{aligned} t_n &= \frac{\frac{a}{a+1} (N+1) - \xi_n s'_n}{(N+1) \left(1 - \frac{a}{a+1}\right)}, \quad (n = 1, 2, \dots), \\ &= \frac{a(N+1) - (a+1)\xi_n s'_n}{(N+1)}. \end{aligned}$$

Now,

$$\limsup_{n \rightarrow \infty} t_n = \frac{a(N+1) - (a-1)(N+1)}{N+1} = 1,$$

and

$$\liminf_{n \rightarrow \infty} t_n = \frac{a(N+1) - (a+1)(N+1)}{N+1} = -1,$$

also $-1 \leq t_n \leq 1$, ($n = 1, 2, \dots$). Further, $A\text{-}\lim t_n = a$, and thus $\{t_n\}$ is the required sequence.

THEOREM 15. *If A is a matrix, and $N(\mathcal{A}) > 1$, then for each a , $1 < a < N(\mathcal{A})$, there exists a matrix B , with $N(\mathcal{A}) = a$, and such that A is b -stronger than B .*

PROOF. Let $s = \{s_n\}$ be a sequence with $\|s\| = 1$, $\liminf_{n \rightarrow \infty} s_n = -1$, $\limsup_{n \rightarrow \infty} s_n = 1$, and $A\text{-}\lim s = a$. We may construct two disjoint sequences of natural numbers, $\{n_k\}$ and $\{p_k\}$, strictly increasing, such that,

$$\lim_{k \rightarrow \infty} s_{n_k} = 1, \quad \lim_{k \rightarrow \infty} s_{p_k} = -1.$$

Define the matrix $C = (c_{m,n})$ by,

$$c_{k,n_k} = \frac{1+a}{2}, \quad c_{k,p_k} = \frac{1-a}{2}, \quad (k = 1, 2, \dots)$$

and

$$c_{k,n} = 0, \quad n \neq n_k \text{ or } p_k.$$

Then,

$$N(\mathcal{C}) \leq h(C) = a,$$

also $C\text{-}\lim s = a$. We may then define a matrix $B = (b_{m,n})$ by the equations,

$$b_{2k,n} = a_{k,n}, \quad b_{2k-1,n} = c_{k,n}, \quad (k, n = 1, 2, \dots).$$

Then we have $\mathcal{B} \subset \mathcal{A} \cap \mathcal{C}$. Thus, by Theorem 2, $N(\mathcal{B}) \leq N(\mathcal{C}) \leq a$, but on the other hand $B\text{-}\lim s = a$, so that $\|\mathcal{B}\| = N(\mathcal{B}) = a$, and B is the required matrix.

As a final application, we prove the following:

THEOREM 16. *If $\mathcal{A}$ is a summability method, and if the norm of $\mathcal{A}$ is attained, there is a matrix B belonging to the method, such that,*

- i) $h(B) = N(\mathcal{A})$,
- ii) *in every column of B , all the non-zero elements have the same sign.*

PROOF. We assume that $h(A) = N(\mathcal{A})$, (if $\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} |a_{m,n}| = N(\mathcal{A})$ but $h(A) \neq N(\mathcal{A})$, this result may be achieved by multiplying the rows of A by the appropriate factors). Now there is an extremal point, $s = \{s_n\}$, of $\mathcal{A}$ for which $s_n = 1$ or $s_n = -1$, ($n = 1, 2, \dots$). We define the matrix B , by

$$b_{m,n} = |a_{m,n}| \operatorname{sign} s_n, \quad (n, m = 1, 2, \dots).$$

This matrix clearly satisfies condition ii, also $h(B) = h(A) = N(\mathcal{A})$. Thus, we only have to verify that $\mathcal{B} = \mathcal{A}$. Let ω_m be the set of indices, n , for which $s_n a_{m,n} < 0$, ($m = 1, 2, \dots$). Then, if $t = \{t_n\}$ is any bounded sequence,

$$\left| \sum_{n=1}^{\infty} (b_{m,n} - a_{m,n}) t_n \right| = \left| \sum_{\omega_m} (b_{m,n} - a_{m,n}) t_n \right|.$$

Hence, we have,

$$(7) \quad \left| \sum_{n=1}^{\infty} (b_{m,n} - a_{m,n}) t_n \right| \leq \sum_{\omega_m} |b_{m,n} - a_{m,n}| \|t\|,$$

and,

$$(8) \quad \sum_{\omega_m} |b_{m,n} - a_{m,n}| = 2 \sum_{\omega_m} |a_{m,n}| = \left| \sum_{n=1}^{\infty} (b_{m,n} - a_{m,n}) s_n \right|.$$

Now,

$$\sum_{n=1}^{\infty} b_{m,n} s_n = \sum_{n=1}^{\infty} |b_{m,n}| = \sum_{n=1}^{\infty} |a_{m,n}| \geq \sum_{n=1}^{\infty} a_{m,n} s_n.$$

However,

$$\left| \sum_{n=1}^{\infty} b_{m,n} s_n \right| \leq h(B) = N(\mathcal{A})$$

and it follows that, since

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} a_{m,n} s_n = N(\mathcal{A}),$$

then

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} b_{m,n} s_n = N(\mathcal{A}).$$

Hence, B sums s and

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} (b_{m,n} - a_{m,n}) s_n = 0;$$

with (7) and (8), this implies that

$$\lim_{m \rightarrow \infty} \sum_{n=1}^{\infty} (b_{m,n} - a_{m,n}) t_n = 0$$

and that $\mathcal{A} = \mathcal{B}$ as required.

5.

We wish to make some brief comments on ‘*diagonalization*’. By this latter term, we mean the following: Suppose $\alpha = \{A^1, A^2, \dots, A^n, \dots\}$ is a countably infinite set of matrices. In [1] and [2], we have obtained necessary and sufficient conditions for the existence of a matrix A , which is b -stronger than, (and b -consistent with) every matrix in α . However, these conditions did not indicate a method of construction of such a matrix. When we speak of a *diagonalization* we shall mean that there are sequences $\{n(k)\}$, and $\{p(k)\}$, $n(k) \nearrow \infty$ and $p(k) \nearrow \infty$, of natural numbers so that the $n(k)$ th row of the matrix $A^{p(k)}$ is the k th row of A . It is clear

that in general, such a diagonalization does not give a matrix b -stronger than every member of α .

We recall the following definition, see [2]. If $\alpha = \{A^1, A^2, \dots, A^n, \dots\}$ and $\beta = \{B^1, B^2, \dots, B^n, \dots\}$ are countable infinite sets of matrices, we shall say that α and β are b -equivalent if A^n is b -equivalent to B^n , ($n = 1, 2, \dots$).

The existence of a matrix A , b -stronger than all members of α was shown in [2] to depend on the properties of the sets of matrices which are b -equivalent to α .

THEOREM 17. *Suppose that α is a countably infinite set of matrices, and that there exists a matrix A which is b -stronger than each member of α . Then there exists a set of matrices, β , b -equivalent to α , such that every diagonalization from β gives a matrix b -stronger than each member of α . In fact every such diagonalization will be b -stronger than A .*

PROOF. Choose a sequence $\{\varepsilon_n\}$, $\varepsilon_n > 0$, ($n = 1, 2, 3, \dots$) such that

$$\lim_{n \rightarrow \infty} \varepsilon_n h(A^n) = 0.$$

We define the matrix B^n , ($n = 1, 2, \dots$), by the equations,

$$b_{2m,k}^n = a_{m,k}, \quad b_{2m-1,k}^n = (1 - \varepsilon_n) a_{m,k} + \varepsilon_n a_{m,k}^n, \quad (m, k = 1, 2, \dots).$$

It is clear that if A is b -stronger than A^n , B^n is b -equivalent to A^n . Let B be a diagonalization from β , it is clear that B is b -stronger than A and hence b -stronger than every member of α . This completes the proof of the theorem.

This problem has also been discussed by Brudno, [5].

6.

We close by stating two important questions which arise in this paper.

1. We have seen that if the norm of a method is attained, there is necessarily a sequence of 1's and -1 's summed by the method to $N(\mathcal{A})$. If the norm is not attained, is there necessarily a sequence in the unit sphere, $\{s_n\}$ which is summed by the method to $N(\mathcal{A})$ and such that

$$\liminf_{n \rightarrow \infty} |s_n| = L < 1?$$

We have seen of course, that if this is the case, the norm is not attained.

2. A question proposed by Brudno also remains. For each method, is there necessarily a stronger method of the same norm which attains the norm?

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