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by

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1.

A series $\sum a_n$ is said to be absolutely summable (A) or summable $|A|$ if the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is of bounded variation in the interval (0,1) Let σ_n^α denote the n-th Cesàro mean of order α of the series $\sum a_n$, i.e.,

$$\sigma_n^\alpha = \frac{1}{(\alpha)_n} \sum_{k=0}^{n} (\alpha)_k a_{n-k}, \quad (\alpha)_k = \Gamma(k+\alpha+1)/\Gamma(k+1)\Gamma(\alpha+1).$$

If the series

$$\sum |\sigma_n^\alpha - \sigma_{n-1}^\alpha|$$

converges, then, we say that the series $\sum a_n$ is absolutely summable (C, α) or summable $|C, \alpha|$. It is known that [2] if a series is summable $|C|$, then it is also summable $|A|$.

2.

Suppose now that $f(x)$ is a function integrable in the sense of Lebesgue and periodic with period 2π. Let

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum A_n(x).$$

Whittaker [5] proved that the series

$$\sum_{n=1}^{\infty} A_n(x)/n^\alpha \qquad (\alpha > 0)$$

is summable $|A|$ almost everywhere. Prasad [5] improved this result by showing that the series $\sum A_n(x)$ when multiplied by one of the factors:

$$1/(\log n)^{1+\varepsilon},\ 1/\log n(\log^2 n)^{1+\varepsilon}, \ldots,$$
$$1/(\log n)\cdot(\log^2 n)\ldots(\log^{k-1} n)(\log^k n)^{1+\varepsilon},$$

where ε is any positive quantity and $\log^1 n = \log n$, $\log^k n = \log(\log^{k-1} n)$, is summable $|A|$ at the point x.

Let (λ_n) be a convex and bounded sequence [3], Chow [1] has demonstrated that the series

$$\sum A_n(x)\lambda_n$$

is summable $|C, 1|$ almost everwhere, if the series $\sum n^{-1}\lambda_n$ converges.

In the present note, we are interested particularly in the case of $|C, 1|$ summability. For a fixed point of x, we write

$$\varphi(t) = \varphi_x(t) = f(x+t)+f(x-t)-2f(x).$$

We are going to establish at first the following

THEOREM 1. *If*

$$\Phi(t) = \int_0^t |\varphi(u)|\,du = O\left(\frac{t}{\log 1/t}\right)$$

as $t \to +0$, *then the series*

$$\sum_{n=2}^{\infty} \frac{A_n(x)}{(\log n)^{1+\varepsilon}}$$

is summable $|C, 1|$ *for every* $\varepsilon > 0$.

3.

In the proof of the theorem, we require the following lemmas.

LEMMA 1 [4]. *Let* $\alpha > -1$ *and let* τ_n^α *be the n-th Cesàro mean of order* α *of the sequence* (na_n), *then*

$$\tau_n^\alpha = n(\sigma_n^\alpha - \sigma_{n-1}^\alpha),$$

where σ_n^α *is the n-th Cesàro mean of order* α *of the series* $\sum a_n$.

LEMMA 2. *Write*

$$S_n(t) = \sum_{k=0}^{n} (n+2-k)\cos(n+2-k)t,$$

then

$$S_n(t) = O\begin{cases} nt^{-1} & (nt \geqq 1), \\ n^2 & (\textit{for all } t). \end{cases}$$

In fact, we have

$$
\begin{aligned}
S_n(t) &= \mathscr{I}\left\{\frac{d}{dt}\left(\overline{e^{i(n+2)t}\sum_{k=0}^{n} e^{-ikt}}\right)\right\} \\
&= \mathscr{I}\left\{\frac{d}{dt}\left(\frac{e^{i(n+2(t}}{1-e^{-it}} - \frac{e^{it}}{1-e^{-it}}\right)\right\} \\
&= O(nt^{-1})+O(t^{-2}) \\
&= O(nt^{-1}),
\end{aligned}
$$

if $nt \geqq 1$. This proves the first part of the lemma. And the second part of the lemma is evident.

4.

Now, we are in a position to prove the theorem. We use

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \varphi(t) \cos ntdt.$$

Let $\tau_n(x)$ be the n-th Cesàro mean of first order of the sequence $\{nA_n(x)(\log n)^{-(1+\varepsilon)}\}$, then

$$\frac{\pi}{2}\tau_n(x) = \int_0^{\pi} \varphi(t)\frac{1}{n+1}\sum_{\nu=0}^{n}\frac{(\nu+2)\cos(\nu+2)t}{(\log(\nu+2))^{1+\varepsilon}}dt.$$

Abel's transformation gives

$$
\begin{aligned}
\frac{\pi}{2}\tau_n(x) &= \int_0^{\pi}\varphi(t)\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} S_\nu(t)\Delta\frac{1}{(\log(\nu+2))^{1+\varepsilon}}\right\}dt \\
&\quad + \int_0^{\pi}\varphi(t)\frac{1}{n+1}\cdot\frac{S_n(t)}{(\log(n+3))^{1+\varepsilon}}dt \\
&= I_1+I_2,
\end{aligned}
$$

say. By Lemma 2, we have

$$\frac{1}{n+1}\left\{\sum_{\nu=0}^{n} S_\nu(t)\Delta\frac{1}{(\log(\nu+2))^{1+\varepsilon}}\right\} = O\begin{cases}\dfrac{1}{t(\log n)^{2+\varepsilon}} & (nt \geqq 1), \\[2ex] \dfrac{n}{(\log n)^{2+\varepsilon}} & (\text{for all } t).\end{cases}$$

Thus, on writing

$$I_1 = \int_0^{1/n} + \int_{1/n}^{\pi} = I_3+I_4,$$

say, we see that

$$I_3 = O\left\{\frac{n}{(\log n)^{2+\varepsilon}}\int_0^{1/n} |\varphi| dt\right\} = O\left\{\frac{1}{(\log n)^{2+\varepsilon}}\right\},$$

$$I_4 = O\left\{\frac{1}{(\log n)^{2+\varepsilon}}\int_{1/n}^{\pi} \frac{|\varphi|}{t}\, dt\right\}.$$

Now,

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t}\, dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2}\, dt$$
$$= O(\log n).$$

It follows that $I_4 = O\{(\log n)^{-(1+\varepsilon)}\}$. As before, we write

$$I_2 = \int_0^{1/n} + \int_{1/n}^{\pi} = I_5 + I_6,$$

say. Then,

$$I_5 = O\left\{\frac{n}{(\log n)^{1+\varepsilon}}\int_0^{1/n} |\varphi| dt\right\}$$
$$= O\left\{\frac{1}{(\log n)^{1+\varepsilon}}\right\}.$$

And

$$I_6 = O\left\{\frac{1}{(\log n)^{1+\varepsilon}}\int_{1/n}^{\pi} \frac{|\varphi|}{t}\, dt\right\}.$$

Now,

$$\int_{1/n}^{\pi} \frac{|\varphi|}{t}\, dt = \left(\frac{\Phi}{t}\right)_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi}{t^2}\, dt$$
$$= O(1) + O\left(\int_{1/n}^{\pi} \frac{dt}{1/n + \log 1/t}\right)$$
$$= O(\log^2 n).$$

It follows that

$$I_6 = O\left(\frac{\log^2 n}{(\log n)^{1+\varepsilon}}\right).$$

By Lemma 1, we need only to prove that $\sum |\tau_n(x)|/n$ converges. And from the above analysis, it concludes that

$$\sum |\tau_n(x)|/n = O\left\{\sum_{n=2}^{\infty} \frac{\log^2 n}{n(\log n)^{1+\varepsilon}}\right\}$$
$$= O(1).$$

This completes the proof of Theorem 1.

5.

For the conjugate series

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum B_n(x),$$

we can derive an analogous theorem. Write

$$\Psi(t) = \int_0^t |\psi(u)| du \equiv \int_0^t |f(x+u) - f(x-u)| du.$$

Then, we get the following

THEOREM 2. *If*

$$\Psi(t) = O\left(\frac{t}{\log 1/t}\right)$$

as $t \to +0$, *then the series*

$$\sum \frac{B_n(x)}{(\log n)^{1+\varepsilon}}$$

is summable $|C, 1|$ *at* x.

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