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Steinitz' exchange theorem for infinite bases. II

by

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We first prove an improved form of Steinitz' Exchange Theorem for a Dependence Space, (Hughes, 1), using the same notation as in the previous paper.

Later, we define a generalized dependence space, by allowing the directly dependent sets to become infinite, and see that the Exchange Theorem becomes invalid, though, if they are of cardinal aleph 0 at most, an invariant rank still exists.

1. Steinitz' exchange theorem

THEOREM 1. If A is a basis and B an independent subset, both well ordered, of the dependence space S , there exists an explicitly defined, one-one mapping φ of B onto A' , a subset of A , such that φ is the identity map on $B \cap A$ and $B + (A - A')$ is a basis of S .

Let $C = A \cap B$, $X = A - C = (x_i)_{i \in I}$, $Y = B - C = (y_j)_{j \in J}$. It is sufficient that I and J are well ordered (by $<$).

For any $y_j \in Y$, there exists at least one relation of the form

$$(1) \quad y_j \sim (x_i) + \sum_{r < j} y_r + \sum_{s < i} x_s + \sum C.$$

We define

$$\varphi(y_j) = x_i,$$

where $i \in I$ is the least in the well ordering such that (1) is satisfied, and put $\varphi(Y) = X'$.

LEMMA 1. φ is a one-one mapping of Y onto X' .

Suppose that $j, k \in J$, $j < k$ and $\varphi(y_k) = \varphi(y_j) = x_i$. Then

$$(2) \quad y_k \sim (x_i) + \sum_{r < k} y_r + \sum_{s < i} x_s + \sum C.$$

From (1) and (2) we have

$$y_k \sim \sum_{r < k} y_r + \sum_{s < i} x_s + \sum C,$$

which is impossible if $\varphi(y_k) = x_i$.

LEMMA 2. $C + Y + (X - X')$ is independent.

If either $Y \cap (X - X')$ is not empty or the set above is dependent, for some $x_i \in X - X'$, the relation (1) is satisfied and we may suppose that $j \in J$ is the least possible in the well ordering. By the definition of φ , $\varphi(y_j) = x_k$, where $k < i$, so that

$$(3) \quad y_j \sim (x_k) + \sum_{r < j} y_r + \sum_{s < k} x_s + \sum C.$$

By (1) and (3),

$$x_i \sim \sum_{r < j} y_r + \sum_{s < i} x_s + \sum C,$$

which contradicts the minimality of j in J .

If, for all $x \in C$, $x \sim \sum A$, we say that A generates or is a set of generators of S .

LEMMA 3. $C + Y + (X - X')$ generates S .

From (1), for all $x_i \in X'$,

$$x_i \sim \sum Y + \sum C + \sum_{s < i} x_s,$$

and hence, by transfinite induction, for all $x_i \in X$,

$$x_i \sim \sum Y + \sum C + \sum (X - X').$$

Since $C + X$ generates S , the Lemma follows.

Theorem 1 now follows from the lemmas, defining φ on C to be the identity mapping.

2. Generalized dependence space

We shall call the set S a generalized dependence space (with respect to Δ), if it satisfies the conditions for a dependence space (Hughes, 1) except that the members of Δ , the directly dependent sets, may be infinite subsets of S .

We carry over the notation for a dependence space. Thus, for $x \in S$, $A \subset S$, x is dependent on A , ($x \sim \sum A$), if and only if, either $x \in A$ or there exists D , such that

$$D \in \Delta, \quad x \in D, \quad D \subset A + (x).$$

We see, by induction on the cardinal of Y , that, provided $Y = B - A$ is finite, Theorem 1 remains valid.

Hence, if S has a finite basis (or, equivalently, a finite set of generators), then any subset of S of greater cardinal is dependent and, if $x \sim \sum A$, then $x \sim \sum A'$, where A' is a finite subset of A . In fact, A' may be any maximal independent subset of A . Thus if

Δ' denotes the set of those elements of Δ , which are finite subsets of S , then S is a dependence space with respect to Δ' , having exactly the same dependence relations $x \sim \sum A$.

THEOREM 2. If S is a generalized dependence space with respect to Δ and every member of Δ has cardinal aleph 0 at most, then every set of generators of S contains a set of generators of minimum cardinal (the rank of S). Any two bases of S have the same cardinal.

Let A be a set of generators of S , of minimum cardinal, which we may assume to be infinite, and B be any set of generators of S .

For every $a \in A$, there exists a set B_a , such that,

$$B_a \subset B, \quad a \sim \sum B_a, \quad \text{card}(B_a) \leq \text{aleph } 0.$$

If $B' = \bigcup_{a \in A} B_a$, then $B' \subset B$ and, for any $a \in A$, $a \sim \sum B'$, so that B' generates S . Also

$$\text{card}(B') \leq \text{card}(A) \times \text{aleph } 0 = \text{card}(A),$$

so that $\text{card}(B') = \text{card}(A)$.

Since a basis is a minimal set of generators, the last part follows.

3. Examples of generalized dependence space

I. If S is an infinite set and Δ consists of all subsets of S of cardinal aleph 0, then S is a generalized dependence space with no basis, for a subset of S generates S if and only if it is infinite and is independent if and only if it is finite.

II. We call the subsets A and B of the infinite set S almost equal, ($A \equiv B$), if there exists a one-one mapping φ of A onto B , such that the set of those $a \in A$, such that $\varphi(a) \neq a$, is finite.

Let S contain a system of disjoint, infinite sets P_i , ($i \in I$), and Γ be the set of all sets C , such that

$$C \equiv P_i, \text{ for some } i \in I.$$

Now let Δ denote the set of all subsets D of S , such that

$$(1) \quad D = C + (x), \quad C \in \Gamma, \quad x \notin C.$$

We see that, if $D \in \Delta$, D has the form (1) for every $x \in D$.

S is a generalized dependence space and X is a basis of S if and only if $X \in \Gamma$.

Let $I = (1, 2)$, then P_1 and P_2 are bases but may have different cardinals.

Now let I be infinite and, for all $i \in I$,

$$\text{card}(P_i) = \aleph_0.$$

Then every $D \in \mathcal{A}$ has cardinal $\aleph_0$. However, if, for every $i \in I$, $a_i \in P_i$, then $A = (a_i)_{i \in I}$ is independent, but is not contained in any basis. We may also have

$$\text{card}(A) = \text{card}(I) > \aleph_0.$$

Thus Theorem 1 is invalid in a generalized dependence space as is Theorem 2 without the condition that the members of $\mathcal{A}$ have cardinal at most $\aleph_0$.

REFERENCE

HUGHES

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