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Fourier series for Meijer's G -functions

by

Roop Narain Kesarwani

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The object of this paper is to establish the following two Fourier series expansions for the Meijer's G -functions.

$$(1.1) \quad \sum_{r=0}^{\infty} G_{p+2, q+2}^{m+1, n+1} (z |_{\frac{1}{2}, b_1, \dots, b_q}^{1-r, a_1, \dots, a_p, 2+r}) \sin (2r+1) \theta \\ = \sqrt{\pi}/2 \sin \theta G_{p, q}^{m, n} (z / \sin^2 \theta |_{b_1, \dots, b_q}^{a_1, \dots, a_p}),$$

where $0 \leq \theta \leq \pi$, $|\arg z| < (m+n-p/2-q/2)\pi$.

$$(1.2) \quad G_{p+1, q+1}^{m+1, n} (z |_{\frac{1}{2}, b_1, \dots, b_q}^{a_1, \dots, a_p, 1}) + 2 \sum_{r=0}^{\infty} G_{p+2, q+2}^{m+1, n+1} (z |_{\frac{1}{2}, b_1, \dots, b_q}^{1-r, a_1, \dots, a_p, 1+r}) \cos r \theta \\ = \sqrt{\pi} G_{p, q}^{m, n} (z / \sin^2 (\theta/2) |_{b_1, \dots, b_q}^{a_1, \dots, a_p}),$$

where $0 < \theta \leq \pi$, $|\arg z| < (m+n-p/2-q/2)\pi$.

The Meijer's G -function is sum of hypergeometric functions each of which is usually an entire function. It is defined [1, p. 207] by a Mellin-Barnes type integral

$$(1.3) \quad G_{p, q}^{m, n} (z |_{b_1, \dots, b_q}^{a_1, \dots, a_p}) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds,$$

where m, n, p, q are integers with $q \geq 1, 0 \leq n \leq p, 0 \leq m \leq q$. The parameters a_j and b_j are such that no pole of $\Gamma(b_j - s)$, $j = 1, \dots, m$ coincides with any pole of $\Gamma(1 - a_j + s)$, $j = 1, \dots, n$. The poles of the integrand must be simple and those of $\Gamma(b_j - s)$, $j = 1, \dots, m$ lie on one side of the contour L and those of $\Gamma(1 - a_j + s)$, $j = 1, \dots, n$ must lie on the other side.

To prove (1.1) and (1.2) whose conditions of validity are given in section 2, we require the following Fourier series established by McRobert [2, p. 79 and 3, p. 143].

$$(1.4) \quad \frac{\sqrt{\pi} \Gamma(2-s)}{2 \Gamma(\frac{3}{2}-s)} (\sin \theta)^{1-2s} = \sum_{r=0}^{\infty} \frac{(s; r)}{(2-s; r)} \sin (2r+1) \theta,$$

where $0 \leq \theta \leq \pi$, $R(s) \leq \frac{1}{2}$. Here $(s; 0) = 1$, $(s; r) = s(s+1) \dots (s+r-1)$, $r = 1, 2, 3, \dots$

$$(1.5) \quad \frac{\sqrt{\pi}\Gamma(1-s)}{\Gamma(\frac{1}{2}-s)} (\sin(\theta/2))^{-2s} = 1 + 2 \sum_{r=0}^{\infty} \frac{(s; r)}{(1-s; r)} \cos r\theta$$

where $0 < \theta \leq \pi$.

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From (1.3), the expression on the left of (1.1) can be written as

$$\sum_{r=0}^{\infty} \frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{3}{2}-s) \prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s) \Gamma(r+s)}{\Gamma(s) \prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=n+1}^p \Gamma(a_j-s) \Gamma(2+r-s)} z^s ds \sin(2r+1)\theta.$$

Here the path L of integration runs from $c-i\infty$ to $c+i\infty$. The conditions

$$0 < c < \frac{3}{2}$$

$$Rl(b_j) > c, \quad j = 1, \dots, m$$

$$Rl(a_j) < c+1, \quad j = 1, \dots, n$$

ensure that all the poles of $\Gamma(\frac{3}{2}-s)$ and $\Gamma(b_j-s)$, $j = 1, \dots, m$ lie to the right of L and those of $\Gamma(r+s)$ and $\Gamma(1-a_j+s)$, $j = 1, \dots, n$ lie to the left of L , as required for the definition of the G -function on the left side of (1.1). The integral converges if $p+q < 2(m+n)$ and $|\arg z| < (m+n-p/2-q/2)\pi$. On changing the order of integration and summation, which is easily seen to be justified, the above expression becomes

$$\frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{3}{2}-s) \prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s)}{\Gamma(2-s) \prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=n+1}^p \Gamma(a_j-s)} \cdot \left\{ \sum_{r=0}^{\infty} \frac{(s; r)}{(2-s; r)} \sin(2r+1)\theta \right\} z^s ds$$

and on using (1.4), it takes the form

$$\sqrt{\pi/2} \sin \theta \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j-s) \prod_{j=1}^n \Gamma(1-a_j+s)}{\prod_{j=m+1}^q \Gamma(1-b_j+s) \prod_{j=n+1}^p \Gamma(a_j-s)} \left(\frac{z}{\sin^2 \theta} \right)^s ds$$

which is just the expression on the right of (1.1).

(1.1) is the Fourier sine series for the G -functions. The Fourier cosine series (1.2) is proved in an analogous manner by using (1.3) and (1.5). The conditions of validity of (1.2) are

$$\begin{aligned} 0 < c < \frac{1}{2} \\ Rl(b_j) > c, \quad j = 1, \dots, m \\ Rl(a_j) < c+1, \quad j = 1, \dots, n. \end{aligned}$$

If we make use of the relation [1, p. 215]

$$G_{p+1, q}^{q, 1}(z|_{b_1, \dots, b_q}^{1, a_1, \dots, a_p}) = E(b_1, \dots, b_q; z),$$

where $E(\cdot)$ denotes McRobert's E -function [1, p. 203], the formulae (1.1) and (1.2) reduce to the Fourier series for E -functions obtained by McRobert [2, p. 79, eqns (1) and (2)].

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From (1.1) and (1.2), we easily deduce the integrals

$$\begin{aligned} \int_0^\pi \sin(2r+1)\theta \sin \theta G_{p, q}^{m, n}(z/\sin^2 \theta|_{b_1, \dots, b_q}^{a_1, \dots, a_p}) d\theta \\ = \sqrt{\pi} G_{p+2, q+2}^{m+1, n+1}(z|_{\frac{1}{2}, b_1, \dots, b_q}^{1-r, a_1, \dots, a_p, 1^{2+r}}), \end{aligned}$$

and

$$\begin{aligned} \int_0^\pi \cos r\theta G_{p, q}^{m, n}(z/\sin^2(\theta/2)|_{b_1, \dots, b_q}^{a_1, \dots, a_p}) d\theta \\ = \sqrt{\pi} G_{p+2, q+2}^{m+1, n+1}(z|_{\frac{1}{2}, b_1, \dots, b_q}^{1-r, a_1, \dots, a_p, 1^{1+r}}), \end{aligned}$$

where $p+q < 2(m+n)$, $|\arg z| < (m+n-p/2-q/2)\pi$ and $r = 0, 1, 2, \dots$

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