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A theorem on the zeros of an entire function (II)

by

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Let $f(z)$ be an entire function of order ρ and genus p . Suppose further z_1, z_2, \dots, z_n are the zeros of $f(z)$; then its Hadamard representation is:

$$f(z) = z^m e^{Q(z)} P(z),$$

where $Q(z)$ is a polynomial of degree $q \leq \rho$ and $P(z)$ is the canonical product of genus p formed with the zeros (other than $z = 0$) of $f(z)$. In a recent note the author has proved the following theorem [2]:

THEOREM A: If $P(z)$ be a canonical product of genus p and order ρ ($\rho > p$), defined by

$$P(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n} \right) \exp \left\{ z/z_n + \frac{1}{2} (z/z_n)^2 + \dots + \frac{1}{p} (z/z_n)^p \right\},$$

where z_1, z_2, \dots etc. are the zeros of $P(z)$ whose moduli r_1, r_2, \dots form a non-decreasing sequence such that $r_n > 1$ for all n and where $r_n \rightarrow \infty$, as $n \rightarrow \infty$ then for z in a domain exterior to the circles r_n^{-h} ($h > \rho$) described about the zeros z_n as centres, we have:

$$\left| \frac{P'(z)}{P(z)} \right| < K \int_0^{\infty} \frac{n(x)r^p}{x^p(x+r)^2} dx,$$

where K is a constant dependent of p and $P'(z)$ is the first derivative of $P(z)$ and $n(x)$ denotes the number of zeros within and on the circle $|z| = x$.

Again, suppose, as we may without loss of generality, that $n(r) = 0$ for $r \leq 1$. In the present note the aim of the author is to give a few important uses of the above theorem. Let

$$M(r) = \max_{|z|=r} |f(z)|; \quad M'(r) = \max_{|z|=r} |f'(z)|.$$

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We show:

THEOREM 1: If $f(z)$ is an entire function of non-integral order ρ and genus p , then for arbitrarily large r ,

$$n(r) \neq 0 \left\{ \frac{rM'(r)}{M(r)} \right\}.$$

PROOF: We shall consider two cases: according as $f(z)$ is of convergence class or divergence class¹. To whatever class $f(z)$ belongs, we always have $p < \rho < p+1$, and $\int^\infty n(t)t^{-m}dt$ diverges for $m < \rho+1$ and converges for $m > \rho+1$. Now as we are considering entire functions of non-integral order, it is sufficient to prove the theorem for the canonical product $P(z)$, of $f(z)$. Then we have from the above theorem:

$$\frac{M'(r)}{M(r)} < K \int_1^\infty \frac{n(x)r^p}{x^p(x+r)^2} dx$$

or,

$$\begin{aligned} \frac{rM'(r)}{M(r)} &< K \left\{ r^{p-1} \int_1^r \frac{n(x)}{x^p} dx + r^{p+1} \int_r^\infty \frac{n(x)}{x^{p+2}} dx \right\} \\ &\equiv K\varphi(r), \text{ (say).} \end{aligned}$$

Suppose now our result is false. Then for arbitrarily small positive ε and for almost all increasing $r > r_0$; $r_0 \in E$ ²

$$(1) \quad n(r) \leq \varepsilon\varphi(r),$$

where we have omitted K .

Take m so that $\rho+1 \leq m < p+2$, and so $\int^\infty t^{-m}n(t)dt$ converges (it converges for $m > \rho+1$ in both cases). Multiply (1) by r^{-m} , and integrate it over (R, ∞) , $R > r_0$ and belongs to E , and then change the order of integration in the resulting iterated integrals (this can easily be effected), we obtain:

$$\begin{aligned} \int_R^\infty t^{-m}n(t)dt &\leq \varepsilon \int_1^R \frac{n(u)}{u^p} du \int_R^\infty t^{p-1-m} dt + \varepsilon \int_R^\infty \frac{n(u)}{u^p} du \int_u^\infty t^{p-1-m} dt \\ &\quad + \varepsilon \int_R^\infty \frac{n(u)}{u^{p+2}} du \int_R^u t^{p-m+1} dt \\ &\leq \frac{\varepsilon R^{p-m}}{m-p} \int_1^R \frac{n(u)}{u^p} du + \frac{\varepsilon}{m-p} \int_R^\infty \frac{n(u)}{u^m} du + \frac{\varepsilon}{p-m+2} \int_R^\infty \frac{n(u)}{u^m} du \\ &= \frac{\varepsilon R^{p-m}}{m-p} \int_1^R \frac{n(u)}{u^p} du + \frac{2\varepsilon}{(m-p)(p-m+2)} \int_R^\infty \frac{n(u)}{u^m} du. \end{aligned}$$

Let

$$\varepsilon < \frac{(p-m+2)(\rho-p)}{4}; \quad \rho < m.$$

Then

$$\frac{1}{2} \int_R^\infty \frac{n(t)}{t^m} dt \leq \frac{\varepsilon R^{p-m}}{m-p} \int_1^R \frac{n(u)}{u^p} du.$$

Case (i) when $f(z)$ is of divergence class: Then letting $m \rightarrow \rho+1$, the left-hand side (2) becomes infinite whilst the right-hand side tends to a finite quantity and hence (1) gives a contradiction.

Case (ii) when $f(z)$ is of convergence class: Then from (2), taking $m = \rho+1$ to begin with, we have, since $n(r)$ increases,

$$\frac{1}{2} n(R) \rho^{-1} R^{-\rho} \leq \frac{\varepsilon R^{p-\rho-1}}{\rho+1-p} \int_1^R \frac{n(u)}{u^p} du,$$

and since this is true for almost all large R and $\varepsilon > 0$, we have

$$n(r) = 0 \left\{ r^{p-1} \int_1^r \frac{n(u)}{u^p} du \right\}.$$

Now $\int^\infty t^{-p-\alpha} n(t) dt$ diverges if $1 < \alpha < \rho+1-p$, and so for such α , as $R \rightarrow \infty$

$$\begin{aligned} \int_1^R \frac{n(r)}{r^{p+\alpha}} dr &= 0 \left\{ \int_1^R r^{-\alpha-1} dr \int_u^r \frac{n(u)}{u^p} du \right\} \\ &= 0 \left\{ \int_1^R \frac{n(u)}{u^p} dx \int_u^R r^{-\alpha-1} dr \right\} \\ &= 0 \left\{ \int_1^R \frac{n(u)}{u^{p+\alpha}} du \right\}, \end{aligned}$$

and this again shows a contradiction. Therefore (1) fails to hold good in both the cases. This proves the theorem.

THEOREM 2: If $f(z)$ is of order ρ and divergence class, the integral

$$(I_1) \quad \int^\infty r^{-1-\rho+\varepsilon} \frac{rM'(r)}{M(r)} dr = \int^\infty r^{-\rho+\varepsilon} \frac{M'(r)}{M(r)} dr$$

diverges; and if

$$(I_2) \quad \int^\infty r^{-1-\rho} \frac{rM'(r)}{M(r)} dr = \int^\infty r^{-\rho} \frac{M'(r)}{M(r)} dr.$$

¹ A function $f(z)$ is said to be of convergence or divergence class according as $\int^\infty n(x)x^{-\rho-1} dx$ converges or diverges respectively.

² E is the set of points of r for which the inequality in Theorem A holds.

diverges then $f(z)$ is of divergence class provided the order ρ of $f(z)$ is not an integer.

PROOF: The first of the theorem is obvious and so omitted (the first follows with the help of Vijayaraghvan's inequality [3]):

$$M'(r) > \frac{M(r)}{r} \frac{\log M(r)}{\log r}, \quad r > r_0(f) = r_0,$$

and the result of Boas ([1], p. 32; first part of (2.11.1)).

To prove the second part of the theorem, suppose ρ is not an integer, then $f(z)$ is dominated by its canonical product $P(z)$. Now we have from Theorem A

$$\begin{aligned} \frac{rM'(r)}{M(r)} &< K \left\{ r^{p-1} \int_1^r \frac{n(t)}{t^p} dt + r^{p+1} \int_r^\infty \frac{n(t)}{t^{p+2}} dt \right\} \\ &= K(J_1(r) + J_2(r)), \quad (\text{say}), \end{aligned}$$

where $p < \rho < p+1$. If we can prove that the convergence of $\int^\infty (n(t)/t^{\rho+1}) dt$ implies the convergence of $\int^\infty (J_j(r)/r^{\rho+1}) dr$ ($j=1, 2$), our second part of the theorem will be established. So

$$\begin{aligned} \int_R^\infty \frac{J_2(r)}{r^{\rho+1}} dr &= \int_R^\infty r^{p-\rho} dr \int_r^\infty \frac{n(t)}{t^{p+2}} dt \\ &= \int_R^\infty \frac{n(t)}{t^{p+2}} dt \int_R^t r^{p-\rho} dr \\ &\leq (p-\rho+1)^{-1} \int_R^\infty \frac{n(t)}{t^{\rho+1}} dt < \infty \end{aligned}$$

and

$$\begin{aligned} \int_1^R \frac{J_1(r)}{r^{\rho+1}} dr &= \int_1^R r^{p-\rho-2} dr \int_1^r \frac{n(t)}{t^\rho} dt \\ &\leq \int_1^R r^{p-\rho-1} dr \int_1^r \frac{n(t)}{t^{p+1}} dt \\ &\leq (\rho-p)^{-1} \int_1^R \frac{n(t)}{t^{\rho+1}} dt \end{aligned}$$

and so

$$\int^\infty r^{-\rho-1} \frac{rM'(r)}{M(r)} dr < \infty;$$

and the second part follows.

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