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by

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The intention of this article is to present a survey of typical and basic ideas and results concerning abstract theory of uniform distribution. As in other cases, the idea underlying any abstract formulation of uniform distribution is to find a concept of uniform distribution as general as possible satisfying the following two requirements: 1) it contains uniform distribution of a sequence of reals mod 1 as a special case; 2) certain theorems on uniformly distributed sequences mod 1 appear as special cases of theorems on the abstract concept of uniform distribution.

In order to be able to trace these theorems through various generalizations of uniform distribution mod 1 as well as to recognize the limitations of these generalizations it may be allowed to collect in a very rough and informal way the concepts and theorems concerning uniform distribution mod 1 under the following headings (for details the reader is referred to [12] and [4]):

a) *Defining properties.* (What is a uniformly distributed sequence: definition by measures (relative frequency in intervals) and functionals (averaging of Riemann-integrable functions), behaviour in extended classes of sets and functions etc.)

b) *Criteria for uniform distribution.* (How to recognize uniformly distributed sequences: Weyl's criterion, fundamental theorem of van der Corput, Fejér's theorem etc.)

c) *Constructions of uniformly distributed sequences.* (How to get uniformly distributed sequences: sequences obtained from special functions, rearrangement of dense sequences etc.)

d) *Comparison of uniformly distributed sequences.* (How good is the uniform distribution of a sequence: discrepancy, estimates of trigonometric sums, well distributed sequences, completely uniformly distributed sequences etc.)

e) *Metric theorems on uniform distribution.* (How many sequences are uniformly distributed: almost all sequences are uniformly

* Nijenrode lecture.

distributed mod 1, special sequences depending on a parameter are uniformly distributed for almost all values of this parameter etc.)

f) *Distribution functions of sequences.* (What about sequences that are not uniformly distributed: distribution functions as studied by van der Corput, Koksma, Schoenberg etc.)

g) *Applications of uniformly distributed sequences.* (How to use uniformly distributed sequences.)

The abstract concept of uniform distribution which one is going to use will clearly depend on the choice and formulation of the definition and of the theorems concerning uniform distribution which one is going to start with. Thus two generalizations introduced by Bundgaard [1] and by Eckmann [5] aim in quite different directions.

For systematic reasons let us first consider uniform distribution in compact groups as studied by Eckmann [5], Hlawka [9], and others. Here we start out with the following observation on Weyl's theorem on uniform distribution mod 1 of the multiples of an irrational number: in the proof by means of exponential functions enter only concepts also quite familiar in the theory of topological groups. In fact, if we replace the interval [0, 1] (endpoints identified) by any compact (Hausdorff) topological group X having a countable base, if we replace Lebesgue measure on [0, 1] by normed Haar measure μ on X, and if we denote by $\mathfrak{C}(X)$ the set of complex-valued continuous functions on X, then a sequence (x_n) in X may be called *uniformly distributed in X* if

$$(1) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_X f(x) d\mu(x) \text{ for all } f \in \mathfrak{C}(X)$$

or, equivalently, if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_M(x_n) = \mu(M)$$

for all closed subsets M of X whose boundary is a μ-null-set (here χ_M denotes the characteristic function of the set M). If, furthermore, the set of exponential functions $\{\exp(2\pi ikx) : k = 0, \pm 1, \pm 2, \ldots\}$ is replaced by a complete system of inequivalent irreducible unitary representations of X $\{D^{(\kappa)} : \kappa = 0, 1, 2, \ldots\}$, then Weyl's criterion for uniform distribution obtains the following form: the sequence (x_n) is uniformly distributed in X if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} D^{(\kappa)}(x_n) = 0 \text{ for all } \kappa \neq 0$$

(0 being the null matrix and $D^{(0)}$ denoting the trivial representation $D^{(0)}(x) = 1$ for all $x \in X$). Using this criterion it is easy to establish the following generalization of Weyl's theorem: the sequence (a^n) $(a \in X)$ is uniformly distributed in X if $det|D^{(\kappa)}(a) - E^{(\kappa)}| \neq 0$ for all $\kappa \neq 0$ ($E^{(\kappa)}$ being the identity matrix of same degree as $D^{(\kappa)}$) or, equivalently, if (a^n) is dense in X. Also van der Corput's fundamental theorem carries over, even with Korobov-Postnikov's sharper conclusion: if $(x_{n+h}x_n^{-1})$ is uniformly distributed for all $h \geqq 1$, then (x_{kn+l}) is uniformly distributed for all $k \geqq 1$ and $l \geqq 0$. This allows applications as in the mod 1 case: if (x^n) is uniformly distributed in X, if X is connected and if $p(n)$ is any non-constant integral polynomial of degree $\geqq 1$ in n, then also $(x^{p(n)})$ is uniformly distributed in X. In this context it may be mentioned that even if the second axiom of contability for X (accounting for the countability of the system of representations $D^{(\kappa)}$) is omitted, there may exist uniformly distributed sequences in the sense of (1) (cf. [7]). Also, with some modification of this definition, one may study sequences, uniformly distributed in locally compact groups (cf. S. Hartman "*Remarks on equidistribution on non-compact groups*", p. 66, and J. H. B. Kempermann "*on the distribution of a sequence in a compact group*", p. 138).

While important concepts and facts mentioned under the headings *a*)—*c*) may be generalized successfully to the compact group case, it seems hard, in the absence of any euclidean structure in the underlying space X, to formulate a useful concept of a discrepancy whose vanishing is necessary and sufficient for a sequence (x_n) to be uniformly distributed in X (cf. E. Hlawka "*Discrepancy and uniform distribution of sequences*", p. 83). There are estimates for the analoga of trigonometric sums; for instance

$$\left\|\frac{1}{N}\sum_{n=1}^{N} D^{(\kappa)}(x_n)\right\| = 0(N^{-\frac{1}{2}}\log^{\frac{1}{2}+\varepsilon} N) \text{ for all } \kappa \neq 0$$

holds for almost all sequences (x_n) in X (here $||A|| = (\sum_{i,k}|a_{ik}|^2)^{\frac{1}{2}}$ for $A = (a_{ik})$ and "almost all" refers to the infinite product space $(X_\infty, \mu_\infty) = \prod_1^\infty (X, \mu)$). However, there is one more useful concept related with those falling under heading *d*): a family $\mathfrak{S} = \{(x_{\sigma,n}) : \sigma \in \Sigma\}$ of sequences in X is called a *family of equi-uniformly distributed sequences* if, for every $\varepsilon > 0$ and for every $f \in \mathfrak{C}(X)$, there exists an integer $N(f, \varepsilon)$ depending on f and ε only, such that

$$\left| \int_X f(x) d\mu(x) - \frac{1}{N} \sum_{n=1}^{N} f(x_{\sigma,n}) \right| < \varepsilon \text{ for all } \sigma \in \Sigma \text{ and for all } N \geqq N(f, \varepsilon).$$

This definition, as it stands, does not make essential use of any group theoretic concepts. These turn up in the statement that, if (x_n) is uniformly distributed in X, the sequences of the family $\{(x_n s) : s \in X\}$ are equi-uniformly distributed. Applying this to a uniformly distributed sequence of the form (a^n) $(a \in X)$ we see that the sequences of the family $\{(a^{n+h}) : h \geqq 0\}$ (i.e. the sequences obtained by cutting off the first h terms) are equi-uniformly distributed. In general, any sequence (x_n) in X having the property that $\{(x_{n+h}) : h \geqq 0\}$ is a family of equi-uniformly distributed sequences is called *well distributed.* An application of well distributed sequences has been given by Hlawka in generalization of a theorem by Fatou: let (x_n) be well distributed in X, let σ be any positive real number, and let (ρ_n) be a sequence of reals having the property that $0 < \rho_{n+1} \leqq \sigma\rho_n$ for all n. Let, furthermore, f be a complex-valued function, continuous almost everywhere on X and not almost everywhere equal to zero. If $\sum_{n=1}^{\infty} |\rho_n f(x_n)| < \infty$, then $\sum_{n=1}^{\infty} \rho_n < \infty$.

A result of metric nature, implying that almost every sequence is uniformly distributed in X, has been given above. In connection with uniform distribution in compact groups the following metric results are of special interest: Schreier [15] has shown that a connected compact metric group X is generated by almost every pair of elements of X (in the product space $(X, \mu) \times (X, \mu)$). Strangely enough, the abelian case seems to have been treated explicitly only later when Halmos and Samelson [6] showed that a connected compact abelian group X satisfying the second axiom of countability is generated by almost every element a ("generated" means that finite products of arbitrary finite powers are dense in X). Thus, in view of the theorem mentioned above, in the abelian case (a^n) is uniformly distributed for almost all a in X, whereas in the non-abelian case obviously (a^n) is not uniformly distributed for any a. Incidentally, even in this case it may still be possible to use sequences of this type in order to construct explicitly sequences that are uniformly distributed in X (cf. G. Helmberg "*A class of criteria concerning uniform distribution in compact groups*", p. 196).

However, the questions falling under the headings *e*) and *f*) may be treated successfully in an even more general setting, as has been done by Hlawka [10], based on the work of van der

Corput, Hill, and others. Let X be any compact (Hausdorff) topological space satisfying the second axiom of countability. Now there is no reason to single out any measure on X as preferred to the others (all measures mentioned in the sequel will be assumed to be regular nonnegative normed Borel-measures). Moreover, we may regard the formation of $\lim_{N\to\infty} 1/N \sum_{n=1}^{N} f(x_n)$ as just one possible summation method for sequences of the form $(f(x_n))$ $(f \in \mathfrak{C}(X),\ (x_n) \subset X)$ and replace it by any other regular Toeplitz method given by an infinite matrix $A = (a_{nk})$ whose coefficients satisfy the following conditions:

(2a) $$a_{nk} \geqq 0 \text{ for all } n \geqq 1 \text{ and for all } k \geqq 1$$

(2b) $$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$$

(2c) $$\lim_{n\to\infty} a_{nk} = 0 \text{ for all } k \geqq 1$$

(cf. J. Cigler "*Methods of summability and uniform distribution*". p. 44). We identify finite measures and bounded linear functionals on $\mathfrak{C}(X)$ as elements of $\mathfrak{C}^*(X)$ by means of the equation $\nu(f) = \int_X f(x)d\nu(x)$ for all $f \in \mathfrak{C}(X)$. If A is any non-negative regular Toeplitz matrix as above and if ν is any measure on X we say that the sequence $(x_n) \subset X$ is *(A, ν)-uniformly distributed in X* if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} f(x_k) = \nu(f) \text{ for all } f \in \mathfrak{C}(X).$$

There arises the question as to the existence of (A, ν)-uniformly distributed sequences. Let us assume $\sum_{k=1}^{\infty} a_{nk} \leqq M < \infty$ for all $n \geqq 1$. Every row $(a_{nk})_{k=1}^{\infty}$ of A defines a linear functional α_n on $\mathfrak{C}(X)$ given by $\alpha_n(f) = \sum_{k=1}^{\infty} a_{nk} f(x_k)$ and bounded by M. Since the sphere of radius M in $\mathfrak{C}^*(X)$ ist weakly compact, the sequence (α_n) has cluster points in $\mathfrak{C}^*(X)$ which by (2a) and (2b) must necessarily be non-negative normed measures. Any such measure we shall call an *A-distribution measure* of the sequence (x_n). Thus, for any given sequence (x_n) in X and for any non-negative regular Toeplitz matrix A there exists a corresponding non-empty set $\mathfrak{W} = \mathfrak{W}(A, (x_n)) \subset \mathfrak{C}^*(X)$ of A-distribution measures of (x_n). A sequence (x_n) is (A, ν)-uniformly distributed in X if $\mathfrak{W}(A, (x_n))$ consists of the measure ν only. It turns out that the set $\mathfrak{W}(A, (x_n))$ is always weakly closed in $\mathfrak{C}^*(X)$ and that under suitable restrictions, imposed on (x_n), A and $\mathfrak{W}$, given any two of these three objects there is a third one such that $\mathfrak{W} = \mathfrak{W}(A, (x_n))$. In particular, let $\mathfrak{W}$ be a closed and convex set of measures and let A satisfy the following condition (called *Hill condition*)

$$\sum_{n=1}^{\infty} e^{-(\delta^2)/a_n} < \infty \text{ for all } \delta \neq 0$$

where $a_n = \sum_{k=1}^{\infty} a_{nk}^2$ (this condition is satisfied in particular for the matrix of usual arithmetic means). Then any sequence (x_n) dense in X may be rearranged to a sequence (x'_n) such that $\mathfrak{W} = \mathfrak{W}(A, (x'_n))$.

The Hill condition plays an important role not only in several theorems of this type, answering essentially questions falling under the heading *f*), but also in the following important metric result. Let $(X_\infty, \nu_\infty) = \prod_1^\infty (X, \nu)$. If A satisfies the Hill-condition, then, for every measure ν, ν_∞-almost all sequences (x_n) are (A, ν)-uniformly distributed (this fact is also expressed by saying that A has the *Borel property*). Still, the sequences (x_n) being A-uniformly distributed with respect to any measure ν at all form a set of first category in the infinite product space X_∞.

In a similar way Hlawka [11] has introduced a very general concept of normal or completely uniform distribution of a sequence in a compact space X. Essentially the same results as mentioned above hold for sequences of this type, too.

It may be noticed that even in the absence of any algebraic structure *a priori* in the space X, it is possible to make use of the algebraic structure of the set of Borel-measurable transformations in X as has been done by Cigler [2] (cf. also J. Cigler "*Applications of the individual ergodic theorem to problems in number theory*". p. 000). Let, for a moment, X be a compact group again and let (a^n) be a uniformly distributed sequence in X. Then left multiplication by a may be regarded as a Borel-measurable transformation T in X that is measure-preserving and ergodic with respect to Haar measure μ. The sequence $(a^n x)$ is uniformly distributed in X for every $x \in X$ and may be written as $(T^n x)$. If X is now any compact space satisfying the second axiom of countability, then there exists a countable subset $\mathfrak{H}$ of $\mathfrak{C}(X)$ having the property that every function $f \in \mathfrak{C}(X)$ may be uniformly approximated by finite linear combinations of elements of $\mathfrak{H}$. Let T be any Borel-measurable transformation in X, ergodic and measure-preserving with respect to a measure ν on X. Then, by the individual ergodic theorem, applied to the elements of $\mathfrak{H}$, for ν-almost all x in X, the sequence $(T^n x)$ is (A, ν)-uniformly distributed (where A is again the matrix of arithmetic means). The following generalization of a theorem of Postnikov and Pjatezkij gives information on the behaviour of individual elements $x \in X$: if there exists a real number $c \geqq 1$ such that

$$\overline{\lim_{N \to \infty}} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \leqq c\nu(f) \text{ for all } f \in \mathfrak{H},$$

then $(T^n x)$ is (A, ν)-uniformly distributed in X.

In our considerations of abstract concepts of uniform distribution we started out with a topological group and then forgot about its algebraic structure. Now let us see whether it makes sense to omit the topological structure instead. Since the concepts of continuous functions and measures will disappear then, too, there arises the question whether we can generalize the concept of uniform distribution so as to be able to speak about uniform distribution even in the absence of any topology in the underlying set X, for instance in the purely group theoretic case. In order to find an answer to this question we look back to the definition of a sequence (x_n), being uniformly distributed in a compact group X. We may regard this sequence as the range of a mapping $\varphi : n \to x_n$ of the set P of positive integers into X. This mapping φ induces a mapping $\varphi' : f \to f \circ \varphi$ that maps the Banach space $\mathfrak{C}(X)$ (with respect to uniform norm) into a set of functions on P. The functions $f \circ \varphi$ belong to the Banach space $\mathfrak{M}'$ (with respect to uniform norm) of bounded complex-valued functions f' on P for which $\lim_{N \to \infty} 1/N \sum_{n=1}^{N} f'(n)$ exists. This limit constitutes a bounded linear funtional ϕ' on $\mathfrak{M}'$, and the bounded linear functional ϕ on $\mathfrak{C}(X)$ defined by $\phi(f) = \phi'(f \circ \varphi)$ coincides with Haar measure on X. Thus we are led to the following definition: let X_1, X_2 be arbitrary given sets and let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be Banach spaces of bounded complex-valued functions on X_1 and X_2 respectively. Let, furthermore, ϕ_1 and ϕ_2 be bounded linear functionals on $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. A mapping φ of X_2 into X_1 is called a *uniform distribution of* X_2 *into* X_1 (depending, of course, on $\mathfrak{M}_1$, $\mathfrak{M}_2$, ϕ_1, ϕ_2), if the following two conditions are satisfied:

(3a) $\qquad f \circ \varphi \in \mathfrak{M}_2 \qquad$ for all $f \in \mathfrak{M}_1$,

(3b) $\qquad \phi_2(f \circ \varphi) = \phi_1(f)$ for all $f \in \mathfrak{M}_1$.

This definition may still be extended (cf. [8]), but even in this setting a generalization of Weyl's criterion applies: let $\mathfrak{H}$ be a subset of $\mathfrak{M}_1$ with the property that finite linear combinations of elements of $\mathfrak{H}$ are dense in $\mathfrak{M}_1$ (for instance, the functions exp $(2\pi ikx)$ in the mod 1 case, or the coefficients of irreducible unitary representations in the compact group case). Then the above mentioned mapping φ is a uniform distribution of X_2 into X_1 if conditions (3a) and (3b) are satisfied for all $f \in \mathfrak{H}$.

Coming back to the original question what to do in the (non-topological) group case, we may choose $\mathfrak{M}_1$ to be the Banach algebra (or more general any full module) of almost periodic functions on X (in the sense of von Neumann and Maak) and ϕ_1 to be the corresponding mean value. Since (in the presence of sufficiently many almost periodic functions) the group X admits an almost periodic compactification X' characterized by the fact that X is a dense subgroup of X' and the functions in $\mathfrak{M}_1$ are precisely the restrictions of the functions in $\mathfrak{C}(X')$ to X (their mean value being essentially Haar measure on X'), in some sense we are back to the compact group case again. However, there are (at least) two reasons for stressing the purely group theoretic aspect of this situation. In the first place, there is the difference in formulation. This is supported by the fact that in many cases (as for instance in the case of rationals or reals) it seems easier to picture a set of almost periodic functions on the given group X than to picture the almost periodic compactification of X which might be a rather complicated object. Secondly we still have free choice of X_2, $\mathfrak{M}_2$, ϕ_2, and φ. The main point of interest here is not necessarily to study sequences in X by which to compute the mean value of almost periodic functions but rather to investigate the relationship between the group structure of X and certain methods of decomposition of this mean value. The questions falling under the headings $d)$—$f)$ therefore seem to lose interest in this particular situation.

The following theorem, due to Maak [13], is a generalization of the two-dimensional version of Weyl's theorem on equidistribution of linear functions mod 1. It extends a result of Bundgaard [1] referred to in the beginning. Let $X_2 = X$ be a group and let D, D' be two independent unitary irreducible representations of X (i.e. the smallest full modules of almost periodic functions, containing the coefficients of D and D' respectively, have an intersection consisting of the constant functions only; a full module of almost periodic functions is a Banach algebra of almost periodic functions with respect to uniform norm, closed under formation of complex conjugates and left-and right-translation by elements of X). Let $X_1 = X \times X$ and let $\mathfrak{M}_1$ be the smallest full module of almost periodic functions $f(x, y)$ on X_1 containing all coefficients $d_{ik}(x)$ and $d'_{jl}(y)$ of the representations D and D' respectively. Let, furthermore, $\mathfrak{M}_2$ be the set of all almost periodic functions on X and let $M_{x,y}$, M_x be the mean values on $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. Then the mapping $\varphi : x \to (x, x)$ is a uniform distribution of X into $X \times X$ in the above sense: we have $f(x, x) \in \mathfrak{M}_2$ and $M_x f(x, x) = M_{x,y} f(x, y)$ for all $f(x, y) \in \mathfrak{M}_1$. As an application of this theorem we get the following generalization of Kronecker's approximation theorem: for any elements a, a' in X and any $\varepsilon > 0$ there is an element x in X such that $||D(a)-D(x)|| < \varepsilon$ and $||D'(a')-D'(x)|| < \varepsilon$. As another consequence of this theorem the coefficients of independent representations may in some sense be regarded as independent random variables.

Having considered a mapping of X into $X \times X$ from the point of view of uniform distribution, let us now consider a mapping going the opposite way (cf. [8]). If Y and Z are subgroups of X (not necessarily abelian), there is a natural mapping φ of $X_2 = Y \times Z$ into $X_1 = X$ defined by $\varphi(y, z) = yz$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the sets of all almost periodic functions on X_1 and X_2 respectively and let $\phi_1 = M_x$ and $\phi_2 = M_{y,z}$ be the corresponding mean values. If, for instance, Y and Z are normal subgroups of X such that $X = YZ$ (however, X need not be the direct product of Y and Z), then φ is a uniform distribution of $Y \times Z$ into X, i.e. we have $f(yz) \in \mathfrak{M}_2$ and $M_{y,z} f(yz) = M_x f(x)$ for all $f \in \mathfrak{M}_1$. If Y and Z are closed subgroups of a compact group X and if $\mathfrak{M}_1 = \mathfrak{C}(X)$ and $\mathfrak{M}_2 = \mathfrak{C}(Y \times Z)$, then this fact may also be described by saying that the Haar measure on X is the convolution of the Haar measures on Y and Z, extended in an obvious way to measures on the whole group X. Theorems of this type may also be formulated for infinite sets of subgroups, leading for instance to the formula

$$M_x f(x) = \lim_{m\to\infty} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f\left(\frac{n}{m!}\right)$$

for the mean value of any almost periodic function f on the rationals.

If X_1 is the group of integers and if X_2 is the set of positive integers, we are led to study sequences (x_n) of integers as has been done by Niven [14] (cf. also I. Niven "*Uniform distribution of sequences of integers*". p. 158). The characterization of these sequences as having relative frequency $1/m$ in every residue class mod m for all m may be reformulated in the following way: for every almost periodic function f on X_1 whose Fourier exponents are rational multiples of $2\pi i$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = M_x f(x).$$

Recently Cigler [3] has studied another interesting generalisation of uniformly distributed sequences mod 1. Let us come back once more to the case of a compact group satisfying the second axiom of countability and having Haar measure μ. Looking at a sequence (x_n), uniformly distributed in X, we may identify every point $x \in X$ with the measure ε_x concentrated in x. Thus we arrive at a sequence of measures (ε_{x_n}) having the property that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varepsilon_{x_n}(f) = \mu(f)$$

for all $f \in \mathfrak{C}(X)$. In general, let us call a sequence (ν_n) of measures *uniformly distributed with respect to a measure* ν if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \nu_n(f) = \nu(f) \text{ for all } f \in \mathfrak{C}(X).$$

This definition admits an elegant treatment of questions concerning convolutions of measures within the theory of uniform distribution. Let us close by mentioning two significant results. We define convolution of measures as usual by $\nu_1\nu_2(f) = \int_X \int_X f(xy) d\nu_1(x) d\nu_2(y)$ and "inversion" by $\nu^*(f) = \int_X f(x^{-1}) d\nu(x)$. Just as the sequence of positive powers of any element $a \in X$ is uniformly distributed with respect to Haar measure on the smallest closed subgroup of X containing a, the sequence of positive powers of any measure ν is uniformly distributed with respect to Haar measure on the smallest closed subgroup of X containing the support of ν. While this may be regarded as another generalization of Weyl's theorem on uniform distribution mod 1 of the multiples of an irrational number, also the following generalization of van der Corput's fundamental theorem holds: if (ξ_n) is a sequence of measures having the property that, for every $h \geqq 1$, $(\xi_{n+h}\xi_n^*)$ is uniformly distributed with respect to a measure ν_h, and if the sequence (ν_n) is uniformly distributed with respect to μ, then also the sequence (ξ_n) is uniformly distributed with respect to μ.

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