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On the Number of Representations of an Integer as a Sum of Primes belonging to given Arithmetical Progressions

by

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1. Introduction: Let $K_1, \ldots, K_s$ be $s$ given positive integers, and $K$ their least common multiple. Let further $a_1, \ldots, a_s$ be given integers, $(a_\sigma, K_\sigma) = 1$ ($\sigma = 1, \ldots, s$), and denote by $\kappa(n)$ the number of sets of residues $x_1, \ldots, x_s$ (mod $K$) which (i) are relatively prime to $K$, and (ii) satisfy the following system of congruences.

\[
\begin{align*}
x_\sigma & \equiv a_\sigma \pmod{K_\sigma} \quad (\sigma = 1, \ldots, s), \\
\sum_{\sigma=1}^{s} x_\sigma & \equiv n \pmod{K}.
\end{align*}
\]

Finally let $N(n)$ denote the number of representations of the positive integer $n$ in the form

\[
n = p_1 + p_2 + \ldots + p_s, \quad p_\sigma \equiv a_\sigma \pmod{K_\sigma} \quad (\sigma = 1, \ldots, s)
\]

where the $p_\sigma$ are odd prime numbers.

I have proved $^1$ that if $n \equiv s \pmod{2}$, and $s > 2$, then

\[
N(n) = \kappa(n) \frac{1}{\varphi(n)(s-1)!} \sum_{n} \log^* n + O\left(\frac{n \log \log n}{\log^{s+1} n}\right),
\]

where $\varphi$ denotes Euler’s function,

\[
\sum^*_n = \Omega K \prod_{p \mid \n} \left(1 - \left[\frac{-1}{p-1}\right]^s\right) \times \prod_{p \mid \n} \left(1 - \left[\frac{-1}{p-1}\right]^{s-1}\right),
\]

where $p$ runs through the odd prime numbers, and where $\Omega = 1$, or $= 2$, according as $K$ is even, or odd.

In this paper we shall prove an explicit formula for $\kappa(n)$.

2. Lemma: Let $p$ be a prime number $\geq 2$, let $s \geq 1$, $1 \leq l \leq L$, and let $\kappa_p(s, l, L; n)$ denote the number of sets of residues $x_1, \ldots, x_s$ (mod $p^L$) which satisfy simultaneously

\[^1\text{) see [1], p. 228.}\]

\[^2\text{) If } s = 2 \text{ this results holds for “almost all” } n. \text{ See [3] and [4].}\]
Then
\[ \tilde{\kappa}_p(s, l, L; n) = \begin{cases} 
   p^{t(L-1)-1} (p-1) \{(p-1)^{s-1} - (-1)^{s-1}\} & \text{if } p | n \\
   p^{t(L-1)-1} \{(p-1)^s - (-1)^s\} & \text{if } p \not| n.
\end{cases} \]

**Proof.** The lemma is evidently true for \( s = 1 \), since the system
\[ \{(x_1, p) = 1; \ x_1 \equiv n \ (\text{mod } p^t)\} \]
has \( p^{t-1} \), or no, solutions \( x_1 \ (\text{mod } p^L) \) according as \( p | n \) or \( p \not| n \).

Next, assuming the lemma true for \( s = t \), we shall prove that it is also true for \( s = t + 1 \).

The system
\[ \{(x_{t+1}, p) = 1; \ x_{t+1} \equiv n - \sum_{\sigma=1}^{t} x_\sigma \ (\text{mod } p^t)\} \]
has \( p^{t-1} \), or no, solutions \( x_{t+1} \ (\text{mod } p^L) \) according as
\[ \sum_{\sigma=1}^{t} x_\sigma \not\equiv n \ (\text{mod } p), \]
or not. Thus

\[ (3) \quad \tilde{\kappa}_p(t+1, l, L; n) = p^{t-1} \sum_{m=1 \atop m \equiv n \ (p)}^{\varphi} \tilde{\kappa}_p(t, l, L; m). \]

If \( p | n \) then \( p | m \) for every \( m \not\equiv n \ (\text{mod } p) \), and we obtain, by assumption, from (3)
\[ \tilde{\kappa}_p(t+1, l, L; n) = p^{t-1}(p-1)p^{t(L-1)-1}\{(p-1)^t - (-1)^t\} \]
\[ = p^{(t+1)(L-1)-1}(p-1)\{(p-1)^t - (-1)^t\} \]

If \( p \not| n \) then \( p \not| m \) for \( (p-2) \) residues \( m \not\equiv n \ (\text{mod } p) \), and \( p | m \) for one residue \( m \not\equiv n \ (\text{mod } p) \). In this case we obtain, therefore, from (3)
\[ \tilde{\kappa}_p(t+1, l, L; n) = p^{t-1+t(L-1)-1}\{(p-2)((p-1)^t - (-1)^t) \]
\[ + (p-1)((p-1)^{t-1} - (-1)^{t-1})\} \]
\[ = p^{(t+1)(L-1)-1}(p-1)^{t+1} - (-1)^{t+1}. \]

It follows, by induction, that our lemma is true for all \( s \geq 1 \).

3. **Theorem.** Let \( \kappa(n) \) be defined as in the introduction. Let \( k = (K_1, \ldots, K_s) \), let \( q \) run through all prime factors of \( K \), and put
\[ m_q = n - \sum_{\sigma=1 \atop q | K_\sigma}^{s} a_\sigma, \quad s_q = \sum_{\sigma=1 \atop q | K_\sigma}^{s} 1. \]
Then
\[ \kappa(n) = K^{s-1}k \times \prod_{\sigma=1}^{s} K_{\sigma}^{-1} \times \prod_{q \mid k} q^{-s_q}((q-1)^{s_q} - (-1)^{s_q}) \times \prod_{q \mid k} q^{-s_q}(q-1)((q-1)^{s_q-1} - (-1)^{s_q-1}), \]

or \( \kappa(n) = 0 \), according as

\[ n \equiv \sum_{\sigma=1}^{s} a_{\sigma} \pmod{k}, \]

or not.

**Proof.** Since the congruences

\[ \{x_{\sigma} \equiv a_{\sigma} \pmod{K_{\sigma}} \ (\sigma = 1, \ldots, s), \sum_{\sigma=1}^{s} x_{\sigma} \equiv n \pmod{K}\} \]

imply

\[ n \equiv \sum_{\sigma=1}^{s} x_{\sigma} \equiv \sum_{\sigma=1}^{s} a_{\sigma} \pmod{k}, \]

it is trivial that \( \kappa(n) = 0 \) if \( n \) does not satisfy the congruence (4).

Suppose now that \( n \) does satisfy the congruence (4). Let

\[ K = \prod_{q} q^{l_q}, \ k = \prod_{q} q^{l_q}, \ K_{\sigma} = \prod_{q} q^{l_{q_{\sigma}}}, \prod_{\sigma=1}^{s} K_{\sigma} = \prod_{q} q^{l_q}, \]

so that

\[ 0 \leq l_q \leq \lambda_{q_{\sigma}} \leq L_q \ (\sigma = 1, \ldots, s), \ L_q \geq 1, \sum_{\sigma=1}^{s} \lambda_{q_{\sigma}} = \Sigma_q. \]

If \( \kappa_q(n) \) denotes the number of sets of residues \( x_1, \ldots, x_s \) (mod \( q^{l_q} \)) which satisfy

\[ \begin{cases} x_{\sigma} \equiv a_{\sigma} \pmod{q^{l_{q_{\sigma}}}}, & (x_{\sigma}, q) = 1 \quad (\sigma = 1, \ldots, s), \\ \sum_{\sigma=1}^{s} x_{\sigma} \equiv n \pmod{q^{l_q}}, \end{cases} \]

then, obviously,

\[ \kappa(n) = \prod_{q} \kappa_q(n). \]

For finding the value of \( \kappa_q(n) \), there is no less of generality in assuming that

\[ l_q = \lambda_{q_1} \leq \lambda_{q_2} \leq \cdots \leq \lambda_{q_s} = L_q. \]
Consider first the case \( q | k \). If \( q | k \) we have \( s_q = 0 \) and

\[
1 \leq l_q = \lambda_{q1} \leq \lambda_{q\sigma} \leq L_q \quad (\sigma = 1, \ldots, s).
\]

Since \( \lambda_{q1} = l_q \), and hence

\[
\sum_{\sigma=2}^{s} (L_q - \lambda_{q\sigma}) = (s-1)L_q + l_q - \Sigma_q,
\]

the number of different sets of residues \( x_2, \ldots, x_s (\text{mod } q^L_q) \) satisfying

\[
x_{\sigma} \equiv a_{\sigma} \pmod{q^{\lambda_{q\sigma}}} \quad (\sigma = 2, \ldots, s)
\]

is evidently given by

\[
q^{(s-1)L_q + l_q - \Sigma_q}.
\]

Now the congruence

\[
x_1 \equiv n - \sum_{\sigma=2}^{s} x_{\sigma} \pmod{q^L_q}
\]

uniquely determines a residue \( x_1 \) (mod \( q^L_q \)), and if \( x_1 \) is so determined, then, by (6), (7) and (4), also

\[
x_1 \equiv n - \sum_{\sigma=2}^{s} x_{\sigma} \equiv n - \sum_{\sigma=2}^{s} a_{\sigma} \equiv a_1 \pmod{q^{\lambda_{q1}}}.
\]

Since \((a_{\sigma}, K_\sigma) = 1 \) and \( \lambda_{q\sigma} \geq 1 \), the congruences (7) and (8) imply

\((x_{\sigma}, q) = 1 \quad (\sigma = 1, \ldots, s)\),

and hence we conclude that

\[
\kappa_q(n) = q^{(s-1)L_q + l_q - \Sigma_q}
\]

if \( q | k \), and \( n \) satisfies (4).

Now consider the case \( q \nmid k \). If \( q \nmid k \) we have \( 1 \leq s_q \leq s-1 \), and

\[
0 = l_q = \lambda_{q1} = \cdots = \lambda_{q_{s_q}} < \lambda_{q(s_q+1)} \leq \cdots \leq \lambda_{qs} = L_q.
\]

Since \( 0 = l_q = \lambda_{q1} = \cdots = \lambda_{q_{s_q}} \), and hence

\[
\sum_{\sigma=s_q+1}^{s} (L_q - \lambda_{q\sigma}) = (s-s_q)L_q + l_q - \Sigma_q,
\]

the number of different sets of residues \( x_{s_q+1}, \ldots, x_s (\text{mod } q^L_q) \) which satisfy

\[
x_{\sigma} \equiv a_{\sigma} \pmod{q^{\lambda_{q\sigma}}} \quad (s_q < \sigma \leq s)
\]

is evidently given by

\[
q^{(s-s_q)L_q + l_q - \Sigma_q}.
\]
It follows that there are

\[
q^{(s-s_q)L_q+\Sigma_q-E_q\kappa_q(s_q, L_q, L_q; n-\sum_{\sigma=s+1}^s x_\sigma)}
\]
different sets of residues \(x_1, \ldots, x_s \pmod{q^{L_q}}\) which satisfy the congruences (11) and

\[
\begin{align*}
(x_\sigma, q) &= 1 \quad (\sigma = 1, \ldots, s_q), \\
\sum_{\sigma=1}^s x_\sigma &\equiv n \pmod{q^{L_q}}.
\end{align*}
\]

But \(\lambda \equiv 0\) and, consequently,

\[x_\sigma \equiv a_\sigma \pmod{q^{L_q}} \quad \text{for } \sigma = 1, \ldots, s_q.
\]

Further, the congruences (11) and the conditions \((a_\sigma, K) = 1\) imply

\[\quad (x_\sigma, q) = 1 \quad \text{for } s_q < \sigma \leq s,
\]

since then \(\lambda \equiv 1\). Hence, if \(q \nmid k\), then \(\kappa_q(n)\) is given by the expression (12). By (10) and (11), the condition

\[q | (n-\sum_{\sigma=s+1}^s x_\sigma)
\]

is equivalent to \(q | m_q\). We deduce, therefore, from (12) and the lemma that, in the case \(q \nmid k\),

\[
\kappa_q(n) = \begin{cases} q^{(s-1)L_q+\Sigma_q-E_q-s_q}(q-1)(q-1)^{s-1}(-1)^{s-1} & \text{if } q | m_q \\
q^{(s-1)L_q+\Sigma_q-E_q-s_q}(q-1)^{s-1}(-1)^{s-1} & \text{if } q \nmid m_q.
\end{cases}
\]

The truth of our theorem, when \(n\) satisfies (4), is thus established by (5), (9) and (13).

4. Conclusion. We have \(\kappa(n) > 0\) if simultaneously

\[
\begin{align*}
\text{(14a)} \quad & n \equiv s \pmod{2}, \\
\text{(14b)} \quad & n \equiv \sum_{\sigma=1}^s a_\sigma \pmod{k}, \\
\text{(14c)} \quad & n \equiv \sum_{\sigma=1, \sigma \neq s_q}^s a_\sigma \pmod{q} \text{ \{for every odd prime number } q \text{ which divides all } K_{\sigma} \text{ except one, } K_{s_q} \text{ say.}\}
\end{align*}
\]

(Condition (14c) may be stated as \(q \mid m_q\) for every odd prime number \(q \mid K\) for which \(s_q = 1\)).
It follows that all sufficiently large integers $n$ satisfying the conditions (14) can be represented as a sum of primes in the form (1), and (2) will be an asymptotic formula for the number of such representations. 1)

To prove the above statement about $\kappa(n) > 0$, we observe that, since $(a_\sigma, K_\sigma) = 1$,

$$m_2 = n - \sum_{\sigma=1}^{s} a_\sigma \equiv n - (s - s_2) \equiv s_2 \pmod{2}$$

provided that $n$ satisfies (14a). Hence $s_2$ is odd if $2 \nmid m_2$, and $(s_2 - 1)$ is odd if $2 \mid m_2$. It follows that, if (14a) and (14b) are satisfied, then $\kappa(n)$ vanishes only if there is an odd prime number $q$ for which $s_q = 1$ and $q \mid m_\sigma$.

1) The above conclusions could also be drawn from general results proved in my paper [2].

REFERENCES

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