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On a function, which is a special case of Meijer's *G*-function

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On a function, which is a special case of Meijer's G-function[1])

by

J. Boersma

1. Introduction.

In the present paper a function, called $K(z)$ is studied, which is defined by means of a power series in the following way:

$$K(z) = \sum_{h=0}^{\infty} \frac{\Gamma(a_1+\mu_1 h)\ldots\Gamma(a_r+\mu_r h)}{\Gamma(b_1+\nu_1 h)\ldots\Gamma(b_t+\nu_t h)} \frac{z^h}{h!} =$$

$$= \sum_{h=0}^{\infty} \frac{\prod_{j=1}^{r} \Gamma(a_j+\mu_j h)}{\prod_{k=1}^{t} \Gamma(b_k+\nu_k h)} \frac{z^h}{h!} \quad \ldots\ldots\ldots \quad (1)$$

μ_j and ν_k are supposed to be real and positive; a_j and b_k may be complex. We assume that a_j and μ_j have such values that $\Gamma(a_j+\mu_j h)$ is finite for $h = 0, 1, 2, \ldots$; so

$$a_j+\mu_j h \neq 0, -1, -2, \ldots \text{ for } j = 1, \ldots, r \text{ and } h = 0, 1, 2, \ldots . \quad (2)$$

It is clear that for $\mu_1 = \ldots = \mu_r = \nu_1 = \ldots = \nu_t = 1$ we have

$$K(z) = \frac{\prod_{j=1}^{r} \Gamma(a_j)}{\prod_{k=1}^{t} \Gamma(b_k)} {}_rF_t(a_1, \ldots, a_r; b_1, \ldots, b_t; z) \quad \ldots \quad (3)$$

in which ${}_rF_t(a_1, \ldots, a_r; b_1, \ldots, b_t; z)$ represents the generalized hypergeometric function.

In section 2 the convergence of the power series (1) is examined. In section 3 an integral representation for the function $K(z)$ is derived, by means of which in the case of positive rational parameters μ_j and ν_k the function $K(z)$ may be written as a G-function. By means of this connection with the G-function the differential equation, the analytic continuation and the asymptotic

[1]) The present article is the elaboration of an answer to a prize-competition of the University of Groningen. This answer was awarded in September 1958.

expansion of the function $K(z)$ are discussed in resp. sections 4, 5 and 6, again for the case of rational μ_j and ν_k. In section 7 the case of a divergent power series (1) is studied. Then a G-function may be constructed, which has $K\left(\frac{1}{z}\right)$ as its asymptotic expansion. In section 8 the results of the foregoing sections are applied to the case of the generalized hypergeometric function i.e. the case with $\mu_1 = \ldots = \mu_r = \nu_1 = \ldots = \nu_t = 1$ (see (8)). In section 9 a comparison is made between the results of this paper and the researches of E. M. Wright [2]) about the asymptotic behaviour of this same function, but for real positive values of the parameters μ_j and ν_k. The methods used by Wright are however much more complicated than the method followed in this paper.

Following Wright we shall introduce the abbreviations

$$\left.\begin{aligned} \rho &= \frac{\prod_{j=1}^{r} \mu_j^{\mu_j}}{\prod_{k=1}^{t} \nu_k^{\nu_k}}, \\ \kappa &= \sum_{k=1}^{t} \nu_k - \sum_{j=1}^{r} \mu_j + 1, \\ \vartheta &= \sum_{j=1}^{r} a_j - \sum_{k=1}^{t} b_k + \tfrac{1}{2}(t-r). \end{aligned}\right\} \quad \ldots\ldots \quad (4)$$

Instead of ρ Wright uses the symbol h.

2. Convergence of the Series.

We examine the convergence of the power series (1). Use is made of the following well-known formula [3]) for the gamma function

$$\frac{\Gamma(\zeta+\alpha)}{\Gamma(\zeta+\beta)} = \zeta^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2\zeta} + O\left(\frac{1}{\zeta^2}\right)\right] \quad . \; . \quad (5)$$

valid for large values of $|\zeta|$ with $|\arg \zeta| < \pi$.

We write the power series in the right-hand side of (1) shortly as

$$\sum_{h=0}^{\infty} c_h z^h \text{ with } c_h = \frac{\prod_{j=1}^{r} \Gamma(a_j+\mu_j h)}{\prod_{k=1}^{t} \Gamma(b_k+\nu_k h)} \frac{1}{h!}.$$

[2]) E. M. Wright, Journ. Lond. Math. Soc. 10, pp. 287—293, (1935) and Proc. Lond. Math. Soc. (2), 46, pp. 389—408, (1940).

[3]) Cf. A. Erdélyi, Higher Transcendental Functions, Bateman Manuscript Project, Vol. I, p. 47, (1953).

We consider $\frac{c_{h+1}}{c_h}$ and make use of (5) and (4).

$$\frac{c_{h+1}}{c_h} = \frac{\prod_{j=1}^{r}\Gamma(a_j+\mu_j h+\mu_j)}{\prod_{j=1}^{r}\Gamma(a_j+\mu_j h)} \frac{\prod_{k=1}^{t}\Gamma(b_k+\nu_k h)}{\prod_{k=1}^{t}\Gamma(b_k+\nu_k h+\nu_k)} \frac{1}{h+1} =$$

$$= \frac{\prod_{j=1}^{r}(\mu_j h)^{\mu_j}}{\prod_{k=1}^{t}(\nu_k h)^{\nu_k}} \frac{1}{h}\left\{1+O\left(\frac{1}{h}\right)\right\} = \rho h^{-\kappa}\left\{1+O\left(\frac{1}{h}\right)\right\}.$$

We now distinguish three cases:

1°. $\kappa < 0$; then we have $\lim_{h\to\infty}\left|\frac{c_{h+1}}{c_h}\right| = \infty.$

The radius of convergence of the power series (1) is now zero, so that the series (1) is divergent for all values of z except $z = 0$.

2°. $\kappa = 0$; then we have $\lim_{h\to\infty}\left|\frac{c_{h+1}}{c_h}\right| = \rho.$

The radius of convergence of the power series (1) is now equal to $\frac{1}{\rho}$. For $|z| < \frac{1}{\rho}$ the series (1) is absolutely convergent. For $|z| > \frac{1}{\rho}$ the series (1) is divergent. The behaviour of the series on the circle of convergence will be discussed after 3°.

3°. $\kappa > 0$; then we have $\lim_{h\to\infty}\left|\frac{c_{h+1}}{c_h}\right| = 0.$

The radius of convergence of the power series (1) is infinite, that means, the series (1) is absolutely convergent for all finite values of z.

Finally we examine the convergence of the series (1) in the case 2° ($\kappa = 0$) for $|z| = \frac{1}{\rho}$.

We put $z = \frac{w}{\rho}$, then the series (1) becomes

$$\sum_{h=0}^{\infty} \frac{\prod_{j=1}^{r}\Gamma(a_j+\mu_j h)}{\prod_{k=1}^{t}\Gamma(b_k+\nu_k h)} \frac{1}{\rho^h}\frac{w^h}{h!} \quad \ldots\ldots\ldots (6)$$

The power series (6) in w has a radius of convergence equal to 1.

Now we make use of a convergence test of Weierstrass [4]). We write the power series (6) shortly in the form

$$\sum_{h=0}^{\infty} d_h w^h \text{ with } d_h = \frac{\prod_{j=1}^{r} \Gamma(a_j+\mu_j h)}{\prod_{k=1}^{t} \Gamma(b_k+\nu_k h)} \frac{1}{\rho^h} \frac{1}{h!}.$$

We now consider $\dfrac{d_{h+1}}{d_h}$; by means of (4) and (5) we find

$$\frac{d_{h+1}}{d_h} = \frac{\prod_{j=1}^{r} \Gamma(a_j+\mu_j h+\mu_j)}{\prod_{j=1}^{r} \Gamma(a_j+\mu_j h)} \frac{\prod_{k=1}^{t} \Gamma(b_k+\nu_k h)}{\prod_{k=1}^{t} \Gamma(b_k+\nu_k h+\nu_k)} \frac{1}{\rho} \frac{1}{h+1} =$$

$$= \frac{\prod_{j=1}^{r} (\mu_j h)^{\mu_j} \left\{1+\dfrac{\mu_j(2a_j+\mu_j-1)}{2\mu_j h}+O\left(\dfrac{1}{h^2}\right)\right\}}{\prod_{k=1}^{t} (\nu_k h)^{\nu_k} \left\{1+\dfrac{\nu_k(2b_k+\nu_k-1)}{2\nu_k h}+O\left(\dfrac{1}{h^2}\right)\right\}} \frac{1}{\rho h}\left\{1-\frac{1}{h}+O\left(\frac{1}{h^2}\right)\right\} =$$

$$= \frac{\rho h^{-\kappa}}{\rho}\left(1-\frac{1}{h}\right)\prod_{j=1}^{r}\left(1+\frac{a_j+\frac{1}{2}\mu_j-\frac{1}{2}}{h}\right)\times$$

$$\times \prod_{k=1}^{t}\left(1-\frac{b_k+\frac{1}{2}\nu_k-\frac{1}{2}}{h}\right)\left\{1+O\left(\frac{1}{h^2}\right)\right\}=$$

$$= 1+\frac{\sum_{j=1}^{r} a_j-\sum_{k=1}^{t} b_k+\frac{1}{2}\sum_{j=1}^{r}\mu_j-\frac{1}{2}\sum_{k=1}^{t}\nu_k-\frac{1}{2}r+\frac{1}{2}t-1}{h}\left\{1+O\left(\frac{1}{h^2}\right)\right\}=$$

$$= 1+\frac{\vartheta-\frac{1}{2}(\kappa+1)}{h}+O\left(\frac{1}{h^2}\right) = 1+\frac{\vartheta-\frac{1}{2}}{h}+O\left(\frac{1}{h^2}\right).$$

Application of Weierstrass' test gives three subcases:

1°. $\mathrm{Re}\,\vartheta-\frac{1}{2} < -1$.

The power series (6) is absolutely convergent for $|w| = 1$, so the original power series (1) is absolutely convergent for $|z| = \dfrac{1}{\rho}$.

2°. $-1 \leqq \mathrm{Re}\,\vartheta-\frac{1}{2} < 0$.

The power series (6) is convergent for $|w| = 1$, except for $w = 1$, so the original power series (1) is convergent for $|z| = \dfrac{1}{\rho}$, except for $z = \dfrac{1}{\rho}$.

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[4]) Cf. K. Knopp, Infinite Sequences and Series, Dover Publications, p. 139, (1956).

3°. $\operatorname{Re}\vartheta-\frac{1}{2}\geqq 0$.

The power series (6) is divergent for $|w| = 1$, so the original power series (1) is divergent for $|z| = \dfrac{1}{\rho}$.

3. Integral representation and connection with Meijer's G-function.

We consider the following integral

$$\int_C \frac{\Gamma(-s)\prod_{j=1}^{r}\Gamma(a_j+\mu_j s)}{\prod_{k=1}^{t}\Gamma(b_k+\nu_k s)}(-z)^s\,ds \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where C is a contour, which runs from $\infty - i\tau$ to $\infty + i\tau$ (τ is a positive number) enclosing all the poles

$$0,\ 1,\ 2,\ 3, \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

but none of the poles

$$-\frac{a_j}{\mu_j},\ -\frac{a_j+1}{\mu_j},\ -\frac{a_j+2}{\mu_j},\ \ldots\ (j = 1, \ldots, r) \quad . \quad . \quad (9)$$

of the integrand.[5])

We assume $\sum_{j=1}^{r}\mu_j \leqq \sum_{k=1}^{t}\nu_k+1$ or $\kappa \geqq 0$.

We suppose that z satisfies the inequalities

$$z \neq 0 \text{ and } |z| < \frac{1}{\rho} \text{ if } \kappa = 0$$
$$z \neq 0 \text{ if } \kappa > 0.$$

Now the only singularities of the integrand within the contour are simple poles at the points (8). The residue of the integrand at the point $s = h$ is equal to

$$-\frac{\prod_{j=1}^{r}\Gamma(a_j+\mu_j h)}{\prod_{k=1}^{t}\Gamma(b_k+\nu_k h)}\,\frac{z^h}{h!}.$$

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[5]) Such a contour C exists according to (2):

$$a_j+\mu_j h \neq -m \text{ with } h,\ m = 0, 1, 2, \ldots \text{ and } j = 1, \ldots, r.$$

So we have $h \neq -\dfrac{a_j+m}{\mu_j}$, that means none of the poles (8) coincides with one of the poles (9).

It may be proved by means of the asymptotic expansion of the Γ-function that the integral (7) is convergent and that we have from the theorem of residues

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(-s)\prod_{j=1}^{r}\Gamma(a_j+\mu_j s)}{\prod_{k=1}^{t}\Gamma(b_k+\nu_k s)}(-z)^s ds = \sum_{h=0}^{\infty}\frac{\prod_{j=1}^{r}\Gamma(a_j+\mu_j h)}{\prod_{k=1}^{t}\Gamma(b_k+\nu_k h)}\frac{z^h}{h!} = K(z). \quad (10)$$

We now restrict ourselves to positive rational values of μ_j and ν_k. Then each of the parameters μ_j and ν_k can be written as a fraction with positive integral numerator and denominator, which are mutually indivisible. The least common multiple of the denominators of these fractions is called q. Further we put

$$\left.\begin{aligned} &q\cdot\mu_j = \gamma_j \ (j = 1, \ldots, r),\\ &q\cdot\nu_k = \delta_k \ (k = 1, \ldots, t),\\ &\sum_{j=1}^{r}\gamma_j = \gamma,\\ &\sum_{k=1}^{t}\delta_k = \delta, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (11)$$

(γ_j and δ_k are integers).
From (4) it follows that

$$q+\delta-\gamma = \kappa q \quad \ldots\ldots\ldots\ldots \quad (12)$$

We make use of the multiplication formula [6]) of Gauss and Legendre for the Γ-function

$$\prod_{r=0}^{m-1}\Gamma\left(\zeta+\frac{r}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)}m^{\frac{1}{2}-m\zeta}\Gamma(m\zeta), \ (m = 1, 2, 3, 4, \ldots) \ . \quad (13)$$

We transform the integral representation (10) into the integral definition of the G-function as presented in Meijer's "On the G-function" [7]), p. 229.

We put $s = qv$, then C is transformed into a contour C^* enclosing all poles $0, \frac{1}{q}, \frac{2}{q}, \ldots$ but none of the poles $-\frac{a_j}{\gamma_j}, -\frac{a_j+1}{\gamma_j}, -\frac{a_j+2}{\gamma_j}, \ldots (j = 1, \ldots, r)$ of the transformed integrand,

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[6]) Cf. A. Erdélyi, loc. cit. [3]), p. 4.

[7]) Meijer's "On the G-function" refers to a collection of eight papers written by C. S. Meijer under the same title, which appeared in Proc. Ned. Akad. Wetensch. 49, pp. 227—237, 344—356, 457—469, 632—641, 765—772, 936—943, 1063—1072, 1165—1175, (1946).

$$K(z)=\frac{q}{2\pi i}\int_{C^*}\frac{\Gamma(-qv)\prod_{j=1}^{r}\Gamma\left\{\gamma_j\left(\frac{a_j}{\gamma_j}+v\right)\right\}}{\prod_{k=1}^{t}\Gamma\left\{\delta_k\left(\frac{b_k}{\delta_k}+v\right)\right\}}(-z)^{qv}\,dv=$$

$$=\frac{1}{2\pi i}\,\frac{\prod_{k=1}^{t}(2\pi)^{\frac{1}{2}(\delta_k-1)}\,\delta_k^{\frac{1}{2}-b_k}}{(2\pi)^{\frac{1}{2}(q-1)}\,q^{-\frac{1}{2}}\prod_{j=1}^{r}(2\pi)^{\frac{1}{2}(\gamma_j-1)}\,\gamma_j^{\frac{1}{2}-a_j}}\times$$

$$\times\int_{C^*}\frac{\prod_{m=0}^{q-1}\Gamma\left(\frac{m}{q}-v\right)\prod_{j=1}^{r}\prod_{m_j=0}^{\gamma_j-1}\Gamma\left(\frac{a_j+m_j}{\gamma_j}+v\right)}{\prod_{k=1}^{t}\prod_{l_k=0}^{\delta_k-1}\Gamma\left(\frac{b_k+l_k}{\delta_k}+v\right)}\left\{\frac{(-z)^q\prod_{j=1}^{r}\gamma_j^{\gamma_j}}{q^q\prod_{k=1}^{t}\delta_k^{\delta_k}}\right\}^{v}dv=$$

$$=MG^{q,\gamma}_{\gamma,q+\delta}\left(\frac{(-z)^q\prod_{j=1}^{r}\gamma_j^{\gamma_j}}{q^q\prod_{k=1}^{t}\delta_k^{\delta_k}}\,\middle|\,\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_\gamma\\ \beta_1,\beta_2,\ldots,\beta_{q+\delta}\end{matrix}\right)\quad.\;.\;.\;.\;.\quad(14)$$

in which

$$M=\frac{\prod_{k=1}^{t}(2\pi)^{\frac{1}{2}(\delta_k-1)}\,\delta_k^{\frac{1}{2}-b_k}}{(2\pi)^{\frac{1}{2}(q-1)}\,q^{-\frac{1}{2}}\prod_{j=1}^{r}(2\pi)^{\frac{1}{2}(\gamma_j-1)}\,\gamma_j^{\frac{1}{2}-a_j}}=$$

$$=\frac{(2\pi)^{\frac{1}{2}(\delta-t)}\,q^{\frac{1}{2}t-\sum_{k=1}^{t}b_k}\prod_{k=1}^{t}\nu_k^{\frac{1}{2}-b_k}}{(2\pi)^{\frac{1}{2}(q-1)}\,q^{-\frac{1}{2}}\,(2\pi)^{\frac{1}{2}(\gamma-r)}\,q^{\frac{1}{2}r-\sum_{j=1}^{r}a_j}\prod_{j=1}^{r}\mu_j^{\frac{1}{2}-a_j}}$$

$$=(2\pi)^{\frac{1}{2}q(\kappa-2)+\frac{1}{2}r-\frac{1}{2}t+\frac{1}{2}}\,q^{\vartheta+\frac{1}{2}}\prod_{j=1}^{r}\mu_j^{a_j-\frac{1}{2}}\prod_{k=1}^{t}\nu_k^{\frac{1}{2}-b_k}.\quad.\;.\;.\;.\quad(15)$$

and

$$\left.\begin{array}{lll}\alpha_1=1-\dfrac{a_1}{\gamma_1}, & \alpha_{\gamma_1+1}=1-\dfrac{a_2}{\gamma_2}, & \alpha_{\gamma_1+\gamma_2+\cdots+\gamma_{r-1}+1}=1-\dfrac{a_r}{\gamma_r},\\[2ex] \alpha_2=1-\dfrac{a_1+1}{\gamma_1} & \alpha_{\gamma_1+2}=1-\dfrac{a_2+1}{\gamma_2}, & \alpha_{\gamma_1+\gamma_2+\cdots+\gamma_{r-1}+2}=1-\dfrac{a_r+1}{\gamma_r},\\ \vdots & \vdots\qquad\text{etc.} & \vdots\\ \alpha_{\gamma_1}=1-\dfrac{a_1+\gamma_1-1}{\gamma_1}, & \alpha_{\gamma_1+\gamma_2}=1-\dfrac{a_2+\gamma_2-1}{\gamma_2}, & \alpha_\gamma=1-\dfrac{a_r+\gamma_r-1}{\gamma_r},\end{array}\right\}(16)$$

$$\left.\begin{aligned}
&\beta_1 = 0, \quad \beta_{q+1} = 1-\frac{b_1}{\delta_1}, \quad \beta_{q+\delta_1+1} = 1-\frac{b_2}{\delta_2},\\
&\beta_2 = \frac{1}{q}, \quad \beta_{q+2} = 1-\frac{b_1+1}{\delta_1}, \quad \beta_{q+\delta_1+2} = 1-\frac{b_2+1}{\delta_2},\\
&\quad\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \text{etc.}\\
&\beta_q = \frac{q-1}{q}, \quad \beta_{q+\delta_1} = 1-\frac{b_1+\delta_1-1}{\delta_1}, \quad \beta_{q+\delta_1+\delta_2} = 1-\frac{b_2+\delta_2-1}{\delta_2},\\
&\qquad \beta_{q+\delta_1+\delta_2+\cdots+\delta_{t-1}+1} = 1-\frac{b_t}{\delta_t},\\
&\qquad \beta_{q+\delta_1+\delta_2+\cdots+\delta_{t-1}+2} = 1-\frac{b_t+1}{\delta_t},\\
&\qquad\qquad \vdots\\
&\qquad \beta_{q+\delta} = 1-\frac{b_t+\delta_t-1}{\delta_t}.
\end{aligned}\right\} \tag{17}$$

The argument of the G-function in (14) may be written by means of (4), (11) and (12) in the following way

$$\frac{(-z)^q \prod_{j=1}^{r} \gamma_j^{\gamma_j}}{q^q \prod_{k=1}^{t} \delta_k^{\delta_k}} = \frac{(-z)^q q^\gamma \prod_{j=1}^{r} \mu_j^{q\mu_j}}{q^q q^\delta \prod_{k=1}^{t} \nu_k^{q\nu_k}} =$$

$$= (-z)^q q^{\gamma-q-\delta} \rho^q = \{-z\, q^{-\kappa} \rho\}^q \quad . \; . \; . \; . \; . \; . \; . \; . \tag{18}$$

We remark that the condition, which Meijer assumes in defining the G-function, viz. $a_j - b_h \neq 1, 2, 3, \ldots (j = 1, \ldots, n; h = 1, \ldots, m)$ becomes in this case

$$\alpha_j - \beta_h \neq 1, 2, 3, \ldots \; (j = 1, \ldots, \gamma;\; h = 1, \ldots, q)$$

$$1 - \frac{a_j+m_j}{\gamma_j} - \frac{k}{q} \neq 1, 2, 3, \ldots \; (j = 1, \ldots, r;\; m_j = 0, \ldots, \gamma_j - 1 \text{ and } k = 0, \ldots, q-1).$$

Multiplication with γ_j yields

$$\gamma_j - a_j - m_j - \mu_j k \neq \gamma_j,\; 2\gamma_j,\; 3\gamma_j, \ldots$$
$$a_j + \mu_j k + m_j \neq 0,\; -\gamma_j,\; -2\gamma_j, \ldots$$

Because $m_j = 0, \ldots, \gamma_j - 1$ we may write

$a_j + \mu_j k \neq 0, -1, -2, -3, \ldots$ for $k = 0, \ldots, q-1$ and $j = 1, \ldots, r$.

This condition is certainly fulfilled according to (2).

4. Differential equation.

The function

$$y = G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right)$$

satisfies the following differential equation

$$\{(-1)^{p-m-n}\, z \prod_{j=1}^{p}(\Theta-a_j+1)-\prod_{j=1}^{q}(\Theta-b_j)\}\, y = 0, \quad . \quad . \; (19)$$

where Θ denotes the operator $z\,\dfrac{d}{dz}$ (Meijer's "On the G-function", p. 344).

In the G-function (14) the variable z is replaced by

$$\frac{(-z)^{q}\prod_{j=1}^{r}\gamma_j^{\gamma_j}}{q^{q}\prod_{k=1}^{t}\delta_k^{\delta_k}}$$

from which it follows that the operator $z\,\dfrac{d}{dz}$ must be replaced by $\dfrac{z}{q}\,\dfrac{d}{dz}$.

After substitution of the values (16) and (17) for α_j and β_k in (19) and multiplication with

$$q^{q}\prod_{k=1}^{t}\delta_k^{\delta_k}$$

we obtain the following homogeneous linear differential equation of order $q+\delta$ for $y = K(z)$:

$$\left\{z^{q}\prod_{j=1}^{r}\prod_{m_j=0}^{\gamma_j-1}(\mu_j\Theta+a_j+m_j)-\prod_{m=0}^{q-1}(\Theta-m)\prod_{k=1}^{t}\prod_{l_k=0}^{\delta_k-1}(\nu_k\Theta+b_k-l_k-1)\right\}y=0 \quad (20)$$

where Θ denotes again the operator $z\,\dfrac{d}{dz}$.

5. Analytic continuation.

In the case 2° of section 2 i.e. $\kappa = 0$ or

$$\sum_{j=1}^{r}\mu_j = \sum_{k=1}^{t}\nu_k+1$$

the power series (1) is divergent for $|z| > \dfrac{1}{\rho}$, so the function $K(z)$

is not defined for that region. Inside the circle of convergence $K(z)$ is equal to a G-function of the type $G^{m,n}_{p,p}(z)$, for the condition $\kappa = 0$ is on account of (12) equivalent to $\gamma = q+\delta$.

We apply now theorem D (p. 235) of Meijer's "On the G-function". The condition $m+n \geqq p+1$ becomes here $q+\gamma \geqq \gamma+1$ and is certainly fulfilled. In the sector

$$|\arg \{-z\, q^{-\kappa} \rho\}^q| < q\pi \text{ or } |\arg(-z)| < \pi$$

$K(z)$ is according to theorem D an analytic function of z and may be represented by the integral

$$K(z) = \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\prod_{j=1}^{r}\Gamma(a_j+\mu_j s)}{\prod_{k=1}^{t}\Gamma(b_k+\nu_k s)}(-z)^s ds \quad . \quad . \quad (21)$$

where L is a contour, which runs from $-\infty i+\sigma$ to $+\infty i+\sigma$ (σ is an arbitrary real number) and is curved, if necessary, so that the points (8) lie on the right and the points (9) lie on the left of the contour.

For $|z| < \frac{1}{\rho}$ we may bend round the contour L to the contour C and so we obtain the integral representation (10). In this way the function $K(z)$ is defined and analytic in the whole complex z-plane with exception of a cut, which has to be made along the positive real axis from $\frac{1}{\rho}$ to ∞.

According to theorem E (p. 236) of Meijer's "On the G-function", the analytic continuation of $K(z)$ outside the circle $|z| = \frac{1}{\rho}$ in the sector $|\arg(-z)| < \pi$ may also be written in the form

$$M \cdot G^{\gamma,q}_{q+\delta,\gamma}\left(\frac{1}{(-z)^q q^{-\kappa q}\rho^q}\,\middle|\,\begin{matrix}1-\beta_1,\ldots,1-\beta_{q+\delta}\\ 1-\alpha_1,\ldots,1-\alpha_\gamma\end{matrix}\right). \quad . \quad . \quad (22)$$

If the condition

$$\alpha_j - \alpha_h \neq 0, \ \pm 1, \ \pm 2, \ldots \text{ for } j \neq h;\ j,\ h = 1,\ldots,\gamma$$

or (see (16))

$$\frac{a_j+m_j}{\gamma_j} - \frac{a_h+m_h}{\gamma_h} \neq 0,\ \pm 1,\ \pm 2,\ldots \text{ for } \left.\begin{matrix} j=1,\ldots,r;\ m_j=0,\ldots,\gamma_j-1 \\ h=1,\ldots,r;\ m_h=0,\ldots,\gamma_h-1 \\ j\neq h \text{ or } j=h,\ m_j\neq m_h \end{matrix}\right\} (23)$$

is fulfilled, then we may apply to (22) the "sum" definition of the G-function (Meijer's "On the G-function", p. 230). The values of the α's and β's are substituted from (16) and (17) and the value of M from (15). By means of formula (13) and some other elementary formulae for the Γ-function we can write the analytic continuation of $K(z)$ in the following form

$$P\left(-\frac{1}{z}\right)=\sum_{h=1}^{r}\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{a_h+n}{\mu_h}\right)\prod\limits_{\substack{j=1\\ j\neq h}}^{r}\Gamma\left\{a_j-\frac{\mu_j}{\mu_h}(a_h+n)\right\}}{\mu_h\prod\limits_{k=1}^{t}\Gamma\left\{b_k-\frac{\nu_k}{\mu_h}(a_h+n)\right\}}\frac{(-1)^n}{(-z)^{\frac{a_h+n}{\mu_h}}n!} \tag{24}$$

defined for $|z|>\frac{1}{\rho}$ and $|\arg(-z)|<\pi$.

The conditions (2) and (23) must be fulfilled in order that the series in the right-hand side of (24) will be defined.

From (2) we have

$$a_j+\mu_j h\neq -n \quad \text{with } j=1,\ldots,r;\ n,\ h=0,1,2,\ldots,$$

$$a_j+n\neq -\mu_j h,$$

$$\frac{a_j+n}{\mu_j}\neq -h,$$

so $\Gamma\left(\frac{a_j+n}{\mu_j}\right)$ is defined for $j=1,\ldots,r$ and $n=0,1,2,\ldots$.

In the case $j\neq h$ condition (23) gives

$$\frac{a_j+m_j}{\gamma_j}-\frac{a_h+m_h}{\gamma_h}\neq l \text{ where } l \text{ is an arbitrary integer,}$$

or

$$a_j+m_j-\frac{\mu_j}{\mu_h}(a_h+m_h)\neq l\gamma_j.$$

Because $m_j=0,\ldots,\gamma_j-1$ we may write

$$a_j-\frac{\mu_j}{\mu_h}(a_h+m_h)\neq l \text{ where } l \text{ is again an arbitrary integer.}$$

We subtract from both sides $p\gamma_j$ (p is also an arbitrary integer). According to (11) we then have

$$a_j-\frac{\mu_j}{\mu_h}(a_h+m_h+p\gamma_h)\neq l-p\gamma_j=l'.$$

Because $m_h = 0, \ldots, \gamma_h - 1$ we may write

$$a_j - \frac{\mu_j}{\mu_h}(a_h + n) \neq l'$$

with $j,\ h = 1, \ldots, r;\ j \neq h;\ l'$ and n arbitrary integers.

So $\Gamma\left\{a_j - \frac{\mu_j}{\mu_h}(a_h+n)\right\}$ is defined for $j,\ h = 1, \ldots, r;\ j \neq h;$ $n = 0, 1, 2, \ldots$.

It is clear that the condition (23) is equivalent to the condition

$$\left.\begin{array}{l} \dfrac{a_j+l}{\mu_j} \neq \dfrac{a_h+n}{\mu_h} \text{ for } j \neq h;\ j,\ h = 1, \ldots, r \\ \qquad\qquad\qquad \text{and } l,\ n = 0,\ \pm 1,\ \pm 2, \ldots. \end{array}\right\} \quad (25)$$

This condition denotes that all points (9) are different, hence, all these points are simple poles of the integrand in (7) and (21). It is obvious that, under this condition, the series (24) also could have been derived directly from the integral (21) by means of the theorem of residues.

It may be remarked that the above results (21) and (24) (the latter under the condition (25)) are also valid in the case of real, not necessary rational, positive values μ_j and ν_k with

$$\sum_{j=1}^{r} \mu_j = \sum_{k=1}^{t} \nu_k + 1,$$

as can easily be proved.

6. Asymptotic expansion.

In the case $\kappa > 0$ or

$$\sum_{j=1}^{r} \mu_j < \sum_{k=1}^{t} \nu_k + 1$$

or $\gamma < q + \delta$ the power series (1) is convergent for all finite values of z. The function $K(z)$ is then an analytic function of z in the whole complex plane. We will examine the asymptotic behaviour of the function $K(z)$ by means of the relation (14) and the known asymptotic behaviour of the G-function. The argument of the G-function will be given by (18). The applied theorems are to be found in Meijer's "On the G-function", viz. the theorems B, 19, 17, 18. We distinguish two cases:

I. $\kappa < 2$ or $\sum_{j=1}^{r} \mu_j + 1 > \sum_{k=1}^{t} \nu_k$ (see (4)) or $q + \gamma > \delta$ (see (11)).

There are now three subcases:

a. We apply theorem B (p. 233).

Assumptions: $1 \leqq \gamma \leqq \gamma < q+\delta$, $1 \leqq q \leqq q+\delta$ and
$q+\gamma > \frac{1}{2}\gamma+\frac{1}{2}q+\frac{1}{2}\delta$ or $q+\gamma > \delta$
$\alpha_j-\alpha_h \neq 0, \pm 1, \pm 2, \ldots$ for $j \neq h$;
$j, h = 1, \ldots, \gamma$ i.e. condition (23).

Assertions: The function $K(z)$ possesses for large values of $|z|$ with

$$|\arg \{-z\, q^{-\kappa}\, \rho\}^q| < (\tfrac{1}{2}q+\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta)\pi$$

or according to (4) and (11)

$$|\arg(-z)| < (2-\kappa)\frac{\pi}{2}$$

the asymptotic expansion

$$K(z) \sim M \sum_{j=1}^{\gamma} e^{(\gamma-\delta-1)\pi i \alpha_j} \Delta_{q+\delta}^{q,\gamma}(j) E_{\gamma,q+\delta}\left((-z)^q q^{-\kappa q} \rho^q e^{(\delta-\gamma+1)\pi i} \| \alpha_j\right). \quad (26)$$

Herein is

$$\Delta_{q+\delta}^{q,\gamma}(j) = (-1)^{\gamma-\delta-1} \frac{\prod\limits_{\substack{h=1\\h\neq j}}^{\gamma} \{\Gamma(\alpha_j-\alpha_h)\Gamma(1+\alpha_h-\alpha_j)\}}{\prod\limits_{h=q+1}^{q+\delta} \{\Gamma(\alpha_j-\beta_h)\Gamma(1+\beta_h-\alpha_j)\}}$$

and

$$E_{\gamma,q+\delta}(x \| \alpha_j) = \frac{x^{-1+\alpha_j} \prod\limits_{h=1}^{q+\delta} \Gamma(1+\beta_h-\alpha_j)}{\prod\limits_{\substack{h=1\\h\neq j}}^{\gamma} \Gamma(1+\alpha_h-\alpha_j)} \times$$

$$\times\, {}_{q+\delta}F_{\gamma-1}\left(\begin{matrix}1+\beta_1-\alpha_j, \ldots, 1+\beta_{q+\delta}-\alpha_j; \\ 1+\alpha_1-\alpha_j, \ldots * \ldots, 1+\alpha_\gamma-\alpha_j;\end{matrix} \; -\frac{1}{x}\right)$$

the asterisk denoting that the number $1+\alpha_j-\alpha_j$ is to be omitted in the sequence $1+\alpha_1-\alpha_j, \ldots, 1+\alpha_\gamma-\alpha_j$.

We substitute the values of the α's and β's from (16) and (17) and the value of M from (15). In the same way as has been done at the derivation of (24), the asymptotic expansion (26) may be written in the following way:

For large values of $|z|$ with

$$|\arg(-z)| < (2-\kappa)\frac{\pi}{2}$$

$K(z)$ possesses the asymptotic expansion

$$K(z) \sim P\left(-\frac{1}{z}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where $P\left(-\frac{1}{z}\right)$ stands for the, in this case, formal expansion in the right-hand side of (24).

The asymptotic expansion (27) is valid for large values of $|z|$ with $|\arg(-z)| < (2-\kappa)\frac{\pi}{2}$. This sector does not cover the whole complex plane; therefore another theorem is needed in order to derive an asymptotic expansion for the remaining part of the complex plane. This is done in subcase *b*.

b. We apply theorem 19 (p. 1066).

Assumptions: $0 \leqq \gamma \leqq \gamma < q+\delta$, $1 \leqq q \leqq q+\delta$ and $q+\gamma > \frac{1}{2}\gamma+\frac{1}{2}q+\frac{1}{2}\delta$ or $q+\gamma > \delta$.

In the last two assertions, viz. the formulae (30) and (31) we assume besides that the condition (23) is fulfilled.

Assertions: The function $K(z)$ possesses for large values of $|z|$ with

$$(q+\gamma-\delta)\frac{\pi}{2} < \arg\{-z\, q^{-\kappa}\rho\}^{q} < (q+\lambda)\pi$$

or according to (4) and (11)

$$(2-\kappa)\frac{\pi}{2} < \arg(-z) < \pi + \frac{\lambda}{q}\pi$$

in which

$\lambda = \frac{1}{2}$ if $q+\delta = \gamma+1$ or (see (12)) $q\kappa = 1$ or $\kappa = \frac{1}{q}$,

$\lambda = 1$ if $q+\delta \geqq \gamma+2$ or $\kappa \geqq \frac{2}{q}$

(Meijer's "On the G-function", p. 234), the asymptotic expansion

$$K(z) \sim M \cdot A_{q+\delta}^{q,\gamma} H_{\gamma,q+\delta}\left((-z)^{q} q^{-\kappa q} \rho^{q} e^{(\delta-\gamma)\pi i}\right); \quad . \quad . \quad (28)$$

for large values of $|z|$ with

$$-\pi - \frac{\lambda}{q}\pi < \arg(-z) < -(2-\kappa)\frac{\pi}{2}$$

the asymptotic expansion

$$K(z) \sim M \cdot \bar{A}^{q,\gamma}_{q+\delta} H_{\gamma,q+\delta}((-z)^q q^{-\kappa q} \rho^q e^{(\gamma-\delta)\pi i}); \quad . \quad . \quad (29)$$

for large values of $|z|$ with

$$\arg(-z) = (2-\kappa)\frac{\pi}{2}$$

the asymptotic expansion

$$K(z) \sim M \cdot A^{q,\gamma}_{q+\delta} H_{\gamma,q+\delta}((-z)^q q^{-\kappa q} \rho^q e^{(\delta-\gamma)\pi i}) + P\left(-\frac{1}{z}\right); \quad (30)$$

for large values of $|z|$ with

$$\arg(-z) = -(2-\kappa)\frac{\pi}{2}$$

the asymptotic expansion

$$K(z) \sim M \cdot \bar{A}^{q,\gamma}_{q+\delta} H_{\gamma,q+\delta}((-z)^q q^{-\kappa q} \rho^q e^{(\gamma-\delta)\pi i}) + P\left(-\frac{1}{z}\right). \quad (31)$$

$P\left(-\frac{1}{z}\right)$ stands again for the, also in this case, formal expansion in the right-hand side of (24). $A^{q,\gamma}_{q+\delta}$ and $\bar{A}^{q,\gamma}_{q+\delta}$ become (see Meijer's "On the G-function", p. 349) in our case

$$\left.\begin{aligned} A^{q,\gamma}_{q+\delta} &= (2\pi)^{\gamma-\delta} e^{-\vartheta\pi i} \\ \bar{A}^{q,\gamma}_{q+\delta} &= (2\pi)^{\gamma-\delta} e^{\vartheta\pi i} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

From Meijer's "On the G-function", p. 234 we have

$$H_{\gamma,q+\delta}(x) = \left[\exp\left((\gamma-q-\delta)x^{\frac{1}{q+\delta-\gamma}}\right)\right] \times$$

$$\times\, x^{\Omega} \cdot \left\{\frac{(2\pi)^{\frac{q+\delta-\gamma-1}{2}}}{\sqrt{q+\delta-\gamma}} + \frac{M_1}{x^{\frac{1}{q+\delta-\gamma}}} + \frac{M_2}{x^{\frac{2}{q+\delta-\gamma}}} + \cdots\right\} =$$

$$= \left[\exp\left((-\kappa q)x^{\frac{1}{\kappa q}}\right)\right] x^{\Omega} \left\{\frac{(2\pi)^{\frac{1}{2}\kappa q-\frac{1}{2}}}{\sqrt{\kappa q}} + \frac{M_1}{x^{\frac{1}{\kappa q}}} + \frac{M_2}{x^{\frac{2}{\kappa q}}} + \cdots\right\} \quad (33)$$

where Ω becomes in our case

$$\Omega = \frac{\vartheta}{\kappa q} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

c. In the subcase *a* the condition $\gamma \geqq 1$ has been assumed; hence, we have still to consider the asymptotic behaviour of $K(z)$ for large values of $|z|$ with $|\arg(-z)| < (2-\kappa)\frac{\pi}{2}$ under the condition

$\gamma = 0$. $\gamma = 0$ means $r = 0$ so that the product

$$\prod_{j=1}^{r} \Gamma(a_j + \mu_j h)$$

is an empty product.

First we mention the (trivial) case: $\delta = 0$. Then we have $t = 0$, hence

$$K(z) = \sum_{h=0}^{\infty} \frac{z^h}{h!} = e^z.$$

From now on we shall assume $\delta > 0$, so because δ is an integer, $\delta \geqq 1$. The power series (1) now takes the form

$$K(z) = \sum_{h=0}^{\infty} \frac{1}{\prod_{k=1}^{t} \Gamma(b_k + \nu_k h)} \frac{z^h}{h!}$$

and the relation (14) becomes

$$K(z) = M' \cdot G_{0,q+\delta}^{q,0}\left((-z)^q q^{-\kappa q} \rho^q \,\middle|\, \beta_1, \beta_2, \ldots, \beta_{q+\delta}\right) \quad . \quad (35)$$

in which is

$$M' = (2\pi)^{\frac{1}{2}q(\kappa-2)-\frac{1}{2}t+\frac{1}{2}} q^{\vartheta+\frac{1}{2}} \prod_{k=1}^{t} \nu_k^{\frac{1}{2}-b_k} \quad . \quad . \quad . \quad . \quad (36)$$

and with in (35) and (36)

$$\left.\begin{aligned} \rho &= \frac{1}{\prod_{k=1}^{t} \nu_k^{\nu_k}} \\ \kappa &= \sum_{k=1}^{t} \nu_k + 1 \\ \vartheta &= -\sum_{k=1}^{t} b_k + \tfrac{1}{2}t. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

We apply theorem 17 (p. 1063).

Assumptions: $0 \leqq \gamma \leqq q + \delta - 2$ and $\gamma + 1 \leqq q \leqq q + \delta - 1$, which are fulfilled if $\gamma = 0$, for q and δ are at least equal to one.

Besides we have still the condition of the case I $q + \gamma > \delta$, which becomes here $q > \delta$. It is clear that this condition is certainly compatible with the assumptions of theorem 17. Strictly speaking, by means of theorem 17 the case $\gamma = 0$ can be treated for the subcase I*b* and for the case II. Further it is assumed that

$$-(q+1)\pi < \arg \{-z\, q^{-\kappa} \rho\}^q < (q+1)\pi$$

or

$$-\pi - \frac{\pi}{q} < \arg(-z) < \pi + \frac{\pi}{q}.$$

Assertions: The function $K(z)$ possesses for large values of $|z|$ with

$$0 < \arg(-z) < \pi + \frac{\pi}{q}$$

the asymptotic expansion

$$K(z) \sim M' \cdot A^{q,0}_{q+\delta}\, H_{0,\,q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{\delta\pi i}\right); \quad . \; . \; (38)$$

for large values of $|z|$ with

$$-\pi - \frac{\pi}{q} < \arg(-z) < 0$$

the asymptotic expansion

$$K(z) \sim M' \cdot \bar{A}^{q,0}_{q+\delta}\, H_{0,\,q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{-\delta\pi i}\right); \quad . \; . \; (39)$$

for large values of $|z|$ with

$$\arg(-z) = 0$$

the asymptotic expansion

$$K(z) \sim M' \cdot A^{q,0}_{q+\delta}\, H_{0,\,q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{\delta\pi i}\right) +$$
$$+ M' \cdot \bar{A}^{q,0}_{q+\delta}\, H_{0,\,q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{-\delta\pi i}\right). \; . \; . \; (40)$$

In (38), (39) and (40) M' is given by (36); $A^{q,0}_{q+\delta}$, $\bar{A}^{q,0}_{q+\delta}$ and $H_{0,\,q+\delta}$ are given by the right-hand sides of (32) and (33) with $\gamma = 0$ and ρ, κ and ϑ in accordance with formula (37).

It is clear that in the sector $(2-\kappa)\frac{\pi}{2} < \arg(-z) < \pi + \frac{\lambda}{q}\pi$ the asymptotic expansion (38) is the same as the asymptotic expansion (28) in subcase I*b* with $\gamma = 0$. A similar result holds for the asymptotic expansions (39) and (29) in the sector $-\pi - \frac{\lambda}{q}\pi < \arg(-z) < -(2-\kappa)\frac{\pi}{2}$. If $\gamma = 0$ the formal expansion $P\left(-\frac{1}{z}\right)$ in the right-hand side of (24) is an empty sum ($\gamma = 0$ implies $r = 0$, see (11)) and may be put equal to zero. Then if $\arg(-z) = \pm(2-\kappa)\frac{\pi}{2}$ the asymptotic expansions (30) and (31) also pass into the asymptotic expansions (38) resp. (39).

II. $\kappa \geqq 2$ or $\sum_{j=1}^{r} \mu_j + 1 \leqq \sum_{k=1}^{t} \nu_k$ (see (4)) or $q+\gamma \leqq \delta$ (see (11)).

We apply theorem 18 (p. 1064).

Assumptions: $0 \leqq \gamma \leqq \gamma \leqq q+\delta-2$ and

$$\gamma+1 \leqq \gamma+q \leqq \tfrac{1}{2}\gamma+\tfrac{1}{2}q+\tfrac{1}{2}\delta.$$

The first assumption is true; for from $q+\gamma \leqq \delta$ it follows that $\gamma \leqq q+\delta-2q$; owing to $q \geqq 1$, $-2q \leqq -2$ we have therefore $\gamma \leqq q+\delta-2$.

In the last assertion (formula (42)) we assume besides that the condition (23) is fulfilled. Further it is assumed that

$$-(q+1)\pi < \arg \{-z\, q^{-\kappa}\rho\}^q < (q+1)\pi$$

or

$$-\pi - \frac{\pi}{q} < \arg(-z) < \pi + \frac{\pi}{q}.$$

Assertions: The function $K(z)$ possesses for large values of $|z|$ with

$$0 < \arg(-z) < \pi + \frac{\pi}{q}$$

the asymptotic expansion (28);
for large values of $|z|$ with

$$-\pi - \frac{\pi}{q} < \arg(-z) < 0$$

the asymptotic expansion (29).

If $\gamma+1 \leqq q+\gamma < \tfrac{1}{2}\gamma+\tfrac{1}{2}q+\tfrac{1}{2}\delta$, so if $q+\gamma < \delta$ or

$$\sum_{j=1}^{r} \mu_j + 1 < \sum_{k=1}^{t} \nu_k$$

or $\kappa > 2$, the function $K(z)$ possesses for large values of $|z|$ with $\arg(-z) = 0$ the asymptotic expansion

$$K(z) \sim M \cdot A^{q,\gamma}_{q+\delta}\, H_{\gamma,\, q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{(\delta-\gamma)\pi i}\right) + \\ + M \cdot \bar{A}^{q,\gamma}_{q+\delta}\, H_{\gamma,\, q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{(\gamma-\delta)\pi i}\right). \quad (41)$$

If $q+\gamma = \tfrac{1}{2}\gamma+\tfrac{1}{2}q+\tfrac{1}{2}\delta$, so if $\kappa = 2$, the function $K(z)$ possesses for large values of $|z|$ with $\arg(-z) = 0$ the asymptotic expansion

$$K(z) \sim M \cdot A^{q,\gamma}_{q+\delta}\, H_{\gamma,\, q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{(\delta-\gamma)\pi i}\right) + \\ + M \cdot \bar{A}^{q,\gamma}_{q+\delta}\, H_{\gamma,\, q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{(\gamma-\delta)\pi i}\right) + P\left(-\frac{1}{z}\right) \quad (42)$$

where $P\left(-\frac{1}{z}\right)$ denotes again the formal expansion in the right-hand side of (24).

In the case $\gamma = 0$ the first two asymptotic expansions of case II pass into the asymptotic expansions (38) and (39). The formal expansion $P\left(-\frac{1}{z}\right)$ is then equal to an empty sum (see (24)) and may be put equal to zero, so both asymptotic expansions (41) and (42) pass into the asymptotic expansion (40).

7. The case $\kappa < 0$ or $\sum_{j=1}^{r} \mu_j > \sum_{k=1}^{t} \nu_k + 1$ or $\gamma > q + \delta$.

In this case the power series (1) is divergent for every value of $z \neq 0$. Now it may be that the power series corresponding to $K\left(\frac{1}{z}\right)$ is the asymptotic expansion of some other function. In this case the relation (14) is not true but we may formally write (see Meijer's "On the G-function", p. 230):

$$K\left(\frac{1}{z}\right) = M \cdot G^{q,\gamma}_{\gamma,q+\delta}\left(\frac{q^{-\kappa q}\,\rho^q}{(-z)^q}\,\middle|\, \begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_\gamma \\ \beta_1, \beta_2, \ldots, \beta_{q+\delta}\end{matrix}\right) =$$

$$= M \cdot \sum_{h=1}^{q} \frac{\prod\limits_{\substack{j=1\\ j\neq h}}^{q} \Gamma(\beta_j-\beta_h) \prod\limits_{j=1}^{\gamma} \Gamma(1+\beta_h-\alpha_j)}{\prod\limits_{j=q+1}^{q+\delta} \Gamma(1+\beta_h-\beta_j)} \left\{\frac{q^{-\kappa q}\,\rho^q}{(-z)^q}\right\}^{\beta_h} \times$$

$$\times\ {}_\gamma F_{q+\delta-1}\left(\begin{matrix}1+\beta_h-\alpha_1, \ldots, 1+\beta_h-\alpha_\gamma; \\ 1+\beta_h-\beta_1, \ldots * \ldots, 1+\beta_h-\beta_{q+\delta};\end{matrix}\ \frac{q^{-\kappa q}\,\rho^q}{z^q}\right). \quad . \quad . \quad (43)$$

This expansion might be an asymptotic expansion of the type as discussed in theorem B (p. 233) of Meijer's "On the G-function".

We consider the G-function:

$$M \cdot G^{\gamma,q}_{q+\delta,\gamma}\left(\frac{(-z)^q}{q^{-\kappa q}\,\rho^q}\,\middle|\, \begin{matrix}1-\beta_1, 1-\beta_2, \ldots, 1-\beta_{q+\delta} \\ 1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_\gamma\end{matrix}\right). \quad . \quad (44)$$

In the following this function is shortly written as $M\cdot G^{\gamma,q}_{q+\delta,\gamma}\left(\frac{(-z)^q}{q^{-\kappa q}\,\rho^q}\right)$; it has a meaning for every value of $z \neq 0$ because $\gamma > q+\delta$.

We now apply theorem B (p. 233) of Meijer's "On the G-function" to the function (44); then we must assume that

$$1 \leqq q \leqq q+\delta < \gamma,\ 1 \leqq \gamma \leqq \gamma$$

and $q+\gamma > \frac{1}{2}q+\frac{1}{2}\delta+\frac{1}{2}\gamma$ or $q+\gamma > \delta$ which is certainly fulfilled. Further there must be

$$(1-\beta_j)-(1-\beta_h) \neq 0,\ \pm 1,\ \pm 2, \ldots$$

or $\beta_j-\beta_h \neq 0,\ \pm 1,\ \pm 2, \ldots$ for $j \neq h$; $j, h = 1, \ldots, q$. This condition is also fulfilled according to (17).

Then the function (44) possesses for large values of $|z|$ with

$$\left| \arg \left\{ \frac{(-z)}{q^{-\kappa}\rho} \right\}^q \right| < (q+\gamma-\delta)\frac{\pi}{2}$$

or $|\arg(-z)| < (2-\kappa)\frac{\pi}{2}$ (this sector covers the whole complex plane because $\kappa < 0$) the asymptotic expansion

$$M \cdot G^{\gamma,q}_{q+\delta,\gamma}\left(\frac{(-z)^q}{q^{-\kappa q}\,\rho^q}\right) \sim M \cdot \sum_{h=1}^{q} e^{(q-1)\pi i(1-\beta_h)}\, \Delta^{\gamma,q}_{\gamma}(h) \times$$

$$\times\, E_{q+\delta,\gamma}\left(\frac{(-z)^q}{q^{-\kappa q}\,\rho^q}\, e^{-(q-1)\pi i}\,||\,1-\beta_h\right) \quad . \quad (45)$$

in which

$$\Delta^{\gamma,q}_{\gamma}(h) = (-1)^{q-1} \prod_{\substack{j=1\\ j\neq h}}^{q} \{\Gamma(\beta_j-\beta_h)\,\Gamma(1+\beta_h-\beta_j)\}$$

and

$$E_{q+\delta,\gamma}(x\,||1-\beta_h) = \frac{x^{-\beta_h}\prod\limits_{j=1}^{\gamma}\Gamma(1+\beta_h-\alpha_j)}{\prod\limits_{\substack{j=1\\ j\neq h}}^{q+\delta}\Gamma(1+\beta_h-\beta_j)} \times$$

$$\times\, {}_{\gamma}F_{q+\delta-1}\left(\begin{matrix}1+\beta_h-\alpha_1, , \ldots, 1+\beta_h-\alpha_\gamma; \\ 1+\beta_h-\beta_1, \ldots * \ldots, 1+\beta_h-\beta_{q+\delta};\end{matrix}\ -\frac{1}{x}\right).$$

After substitution of these expressions we see that the right-hand side of (45) is equal to the formal expression on the right of (43); we find therefore

$$M \cdot G^{\gamma,q}_{q+\delta,\gamma}\left(\frac{(-z)^q}{q^{-\kappa q}\,\rho^q}\right) \sim K\left(\frac{1}{z}\right) \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The function (44) is the same as the function considered in (22) of section 5 [8]) after replacement of z by $\frac{1}{z}$. Hence, if the

[8]) In section 5 the case with $\gamma = q+\delta$ is treated; here we consider the function (22) with $\gamma > q+\delta$.

condition (23) is fulfilled we may write (46) in the following way:
For large values of $|z|$ with

$$|\arg(-z)| < (2-\kappa)\frac{\pi}{2}$$

the function $P(-z)$ defined by (24) possesses the asymptotic expansion

$$P(-z) \sim K\left(\frac{1}{z}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

8. The special case $\mu_1 = \ldots = \mu_r = \nu_1 = \ldots = \nu_t = 1$.

As has been stated in (3) the function $K(z)$ reduces to the generalized hypergeometric function. Section 2 passes into the well-known convergence behaviour of this function. Section 3 gives us the generalized hypergeometric function expressed in the G-function. The number q is equal to 1. The relations (11) become:

$$\begin{aligned} &\gamma_j = 1 \ (j = 1, \ldots, r), \qquad && \delta_k = 1 (k = 1, \ldots, t), \\ &\gamma = r, && \delta = t. \end{aligned}$$

From (4) we have

$$\begin{aligned} \kappa &= \sum_{k=1}^{t} \nu_k - \sum_{j=1}^{r} \mu_j + 1 = t - r + 1 \\ \rho &= 1. \end{aligned}$$

Formula (15) becomes $M = 1$.
In view of (3), (16) and (17) formula (14) becomes

$$\begin{aligned} K(z) &= \frac{\prod_{j=1}^{r} \Gamma(a_j)}{\prod_{k=1}^{t} \Gamma(b_k)} \, {}_rF_t(a_1, \ldots, a_r;\ b_1, \ldots, b_t;\ z) = \\ &= G^{1,r}_{r,t+1}\left(-z \,\middle|\, \begin{matrix} 1-a_1, \ldots, 1-a_r \\ 0,\ 1-b_1, \ldots, 1-b_t \end{matrix}\right) \quad . \quad . \quad (48) \end{aligned}$$

According to section 4 the differential equation for $y = {}_rF_t(z)$ becomes

$$\left\{ z \prod_{j=1}^{r} (\Theta + a_j) - \Theta \prod_{k=1}^{t} (\Theta + b_k - 1) \right\} y = 0. \quad . \quad . \quad . \quad (49)$$

where Θ denotes the operator $z \dfrac{d}{dz}$.

Section 5 gives the analytic continuation of ${}_rF_t(z)$ in the case $r = t+1$.

Section 6 gives the asymptotic expansion of ${}_rF_t(z)$ in the case $r < t+1$. The results agree with the results obtained in Meijer's "On the G-function", § 20 (pp. 1170—1173).

Finally, if $r > t+1$ the result of section 7 becomes:
For large values of $|z|$ with

$$|\arg(-z)| < (r-t+1)\frac{\pi}{2}$$

and under the conditions (2) and (23), which become in this case

$$a_j \neq 0,\ -1,\ -2, \ldots \text{ for } j = 1, \ldots, r$$

and $a_j - a_h \neq 0,\ \pm 1,\ \pm 2, \ldots$ for $j \neq h$; $j,\ h = 1, \ldots, r$, the following asymptotic expansion holds (see (44) and (47))

$$G^{r,1}_{t+1,r}\left(-z\,\middle|\,\begin{matrix}1,\ b_1,\ b_2, \ldots, b_t\\ a_1,\ a_2, \ldots, a_r\end{matrix}\right) =$$

$$= \sum_{h=1}^{r}\sum_{n=0}^{\infty}\frac{\Gamma(a_h+n)\prod\limits_{\substack{j=1\\ j\neq h}}^{r}\Gamma(a_j-a_h-n)}{\prod\limits_{k=1}^{t}\Gamma(b_k-a_h-n)}\frac{(-1)^n(-z)^{a_h+n}}{n!} =$$

$$= \sum_{h=1}^{r}\frac{\Gamma(a_h)\prod\limits_{\substack{j=1\\ j\neq h}}^{r}\Gamma(a_j-a_h)}{\prod\limits_{k=1}^{t}\Gamma(b_k-a_h)}(-z)^{a_h}\times$$

$$\times\ {}_{t+1}F_{r-1}\left(\begin{matrix}a_h,\ 1+a_h-b_1, \ldots, 1+a_h-b_t;\\ 1+a_h-a_1, \ldots * \ldots, 1+a_h-a_r;\ (-1)^{t-r+1}z\end{matrix}\right) \sim$$

$$\sim \frac{\prod\limits_{j=1}^{r}\Gamma(a_j)}{\prod\limits_{k=1}^{t}\Gamma(b_k)}\,{}_rF_t\left(a_1, \ldots, a_r;\ b_1, \ldots, b_t;\ \frac{1}{z}\right) \quad \ldots\ldots \quad (50)$$

It may still be remarked that the function in the left-hand side of (50) is equal to Mac Robert's E-function [9])

$$E(r;\ a_h : t;\ b_k : -z).$$

9. Comparison with Wright's results.

In a number of papers E. M. Wright has discussed the asymptotic behaviour of integral functions defined by Taylor series

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[9]) Cf. A. Erdélyi, loc. cit. [3]), p. 215.

with an application to the case of the function of this paper. The results of this application are to be found in two papers which are cited in footnote [2]) of this paper.

We recapitulate Wright's results in our notation:

THEOREM 1. If $\kappa > 0$ and $|\zeta| \leqq \frac{1}{2}\pi \min(\kappa, 2) - \varepsilon$ then

$$K(z) = I(Z).$$

THEOREM 2. If $\kappa > 2$ and $|\eta| \leqq \pi$ then

$$K(z) = I(Z_1) + I(Z_2).$$

THEOREM 3. If $\kappa = 2$ and $|\eta| \leqq \pi$ then

$$K(z) = I(Z_1) + I(Z_2) + J(-z).$$

THEOREM 4. If $0 < \kappa < 2$ and $|\eta| \leqq (2-\kappa)\dfrac{\pi}{2} - \varepsilon$, then

$$K(z) = J(-z).$$

THEOREM 5. If $0 < \kappa < 2$, $|\zeta| \leqq \min\left(\pi, \dfrac{3}{2}\pi\kappa - \varepsilon\right)$ and $|\eta| \leqq \pi$,

then

$$K(z) = I(Z) + J(-z).$$

THEOREM 6. If $f(s)$ has only a finite number of poles or none, then $\kappa \geqq 1$ and the asymptotic expansion of $K(z)$ is given by

$$K(z) = I(Z_1) + I(Z_2) + H(-z) \qquad (1 < \kappa < 2)$$
$$K(z) = I(Z) + H(-z) \qquad (\kappa = 1).$$

If $f(s)$ has no poles, then $H(-z) = 0$.

All these expansions are valid uniformly in arg z in the respective sectors. Herein is

$$f(s) = \frac{\prod_{j=1}^{r} \Gamma(a_j + \mu_j s)}{\prod_{k=1}^{t} \Gamma(b_k + \nu_k s)} \quad \ldots \ldots \ldots \quad (51)$$

The "exponential" asymptotic expression I(X) is [10])

[10]) It may be remarked that in the definition of $I(X)$ on p. 390 of Wright's second paper the coefficient A_m must be replaced by $\dfrac{A_m}{\kappa}$. The theorems 1—5 are namely proved by means of some more general theorems which are derived in E. M. Wright, Phil. Trans. Roy. Soc. (A), 238, pp. 423—451, (1940), where this factor κ indeed occurs.

$$I(X) = X^{\vartheta} e^{X} \left\{ \sum_{m=0}^{M-1} \frac{A_m}{\kappa} X^{-m} + O(X^{-M}) \right\} \quad . \quad . \quad . \quad (52)$$

where the coefficients A_0, A_1, A_2, ... have been calculated such that

$$\left| \frac{f(s)}{(\rho\kappa^{\kappa})^{s}\, \Gamma(s+1)} - \sum_{m=0}^{M-1} \frac{A_m}{\Gamma(\kappa s-\vartheta+m+1)} \right| < \left| \frac{K}{\Gamma(\kappa s-\vartheta+M+1)} \right|$$

assuming that $\arg s$, $\arg(a_j+\mu_j s)$ and $\arg(b_k+\nu_k s)$ all lie between $-\pi+\varepsilon$ and $\pi-\varepsilon$. In particular

$$A_0 = (2\pi)^{\frac{1}{2}(r-t)}\, \kappa^{\frac{1}{2}-\vartheta} \prod_{j=1}^{r} \mu_j^{a_j-\frac{1}{2}} \prod_{k=1}^{t} \nu_k^{\frac{1}{2}-b_k}. \quad . \quad . \quad . \quad (53)$$

The "algebraic" asymptotic expression $J(y)$ is

$$J(y) = \sum_{h=1}^{r} \sum_{n=0}^{N_h} P_{h,n}\, y^{-\frac{a_h+n}{\mu_h}} + O(y^{-N+\tau}) \quad . \quad . \quad . \quad . \quad (54)$$

with [11]) $N_h+\mathrm{Re}\, a_h < N\mu_h \leqq N_h+\mathrm{Re}\, a_h+1$

and

$$p P_{h,n}\, y^{-\frac{a_h+n}{\mu_h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

denoting the residue of the function $\Gamma(-s)\, f(s)\, y^s$ in the point

$$s = -\frac{a_h+n}{\mu_h} \; (h = 1, \ldots, r;\; n = 0, 1, 2, \ldots) \quad . \quad . \quad (56)$$

when this point is a pole of order p for the function $f(s)$. If the point is not a pole of $f(s)$ Wright defines $P_{h,n} = 0$.

If $f(s)$ has only a finite number of poles, then $P_{h,n} = 0$ when n is greater than some fixed n_h and Wright defines

$$H(y) = \sum_{h=1}^{r} \sum_{n=0}^{n_h} P_{h,n}\, y^{-\frac{a_h+n}{\mu_h}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

ε and τ denote arbitrary positive numbers; M and N are arbitrary

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[11]) If N_h is negative for some N and h, then $\sum_{n=0}^{N_h} P_{h,n}\, y^{-\frac{a_h+n}{\mu_h}}$ will be put equal to zero for that value of h. However, for sufficiently large N, N_h $(h = 1, \ldots, r)$ will certainly be positive.

positive integers; K is a positive number depending at most on M, a_j, μ_j, b_k, ν_k, but not on s.

Further the following abbreviations are used:

$$\left.\begin{array}{l} \zeta = \arg z,\ \eta = \arg(-z) \\ Z = \kappa(\rho|z|)^{\frac{1}{\kappa}}\, e^{\frac{i\zeta}{\kappa}} \\ Z_1 = |Z|\, e^{\frac{i(\eta+\pi)}{\kappa}},\ Z_2 = |Z|\, e^{\frac{i(\eta-\pi)}{\kappa}} \end{array}\right\} \quad \ldots\ldots (58)$$

In our paper we also obtained an algebraic type and an exponential type of asymptotic expansion. The algebraic type of asymptotic expansion is given by $P\left(-\frac{1}{z}\right)$ (see (24)). This expansion always occurs accompanied by the condition (23), which was equivalent with condition (25). The latter condition means that all poles (56) of $f(s)$ are different and therefore simple. From (55) we have in this case

$$P_{h,n}\, y^{-\frac{a_h+n}{\mu_h}} = \frac{\Gamma\left(\frac{a_h+n}{\mu_h}\right)\prod\limits_{\substack{j=1\\ j\neq h}}^{r}\Gamma\left\{a_j-\frac{\mu_j}{\mu_h}(a_h+n)\right\}}{\mu_h\prod\limits_{k=1}^{t}\Gamma\left\{b_k-\frac{\nu_k}{\mu_h}(a_h+n)\right\}}\ \frac{(-1)^n}{y^{\frac{a_h+n}{\mu_h}}\, n!}.$$

Substitution in (54) yields

$$J(y) = \sum_{h=1}^{r}\sum_{n=0}^{N_h}\frac{\Gamma\left(\frac{a_h+n}{\mu_h}\right)\prod\limits_{\substack{j=1\\ j\neq h}}^{r}\Gamma\left\{a_j-\frac{\mu_j}{\mu_h}(a_h+n)\right\}}{\mu_h\prod\limits_{k=1}^{t}\Gamma\left\{b_k-\frac{\nu_k}{\mu_h}(a_h+n)\right\}}\ \frac{(-1)^n}{y^{\frac{a_h+n}{\mu_h}}\, n!} + O(y^{-N+\tau})$$

Comparing this formula with (24) gives

$$J(y) = P\left(\frac{1}{y}\right) \quad \ldots\ldots\ldots (59)$$

The exponential type of asymptotic expansion is given by:

1°. $M \cdot A_{q+\delta}^{q,\gamma}\, H_{\gamma,q+\delta}\left((-z)^q\, q^{-\kappa q}\, \rho^q\, e^{(\delta-\gamma)\pi i}\right)$ (see (28)).

Substitution of (15), (32), (33) and (34) gives the expansion

$$\left\{(\rho|z|)^{\frac{1}{\kappa}} e^{\frac{i(\eta-\pi)}{\kappa}}\right\}^{\vartheta} \exp\left[\kappa(\rho|z|)^{\frac{1}{\kappa}} e^{\frac{i(\eta-\pi)}{\kappa}}\right] \times$$

$$\times\left[\frac{(2\pi)^{\frac{1}{2}(r-t)} \prod_{j=1}^{r} \mu_j^{a_j-\frac{1}{2}} \prod_{k=1}^{t} \nu_k^{\frac{1}{2}-b_k}}{\kappa^{\frac{1}{2}}} + \frac{M'_1}{(\rho|z|)^{\frac{1}{\kappa}} e^{\frac{i(\eta-\pi)}{\kappa}}} + \right.$$

$$\left. + \frac{M'_2}{\left\{(\rho|z|)^{\frac{1}{\kappa}} e^{\frac{i(\eta-\pi)}{\kappa}}\right\}^2} + \dots\right] =$$

$$= Z_2^{\vartheta} e^{Z_2}\left[(2\pi)^{\frac{1}{2}(r-t)} \kappa^{-\frac{1}{2}-\vartheta} \prod_{j=1}^{r} \mu_j^{a_j-\frac{1}{2}} \prod_{k=1}^{t} \nu_k^{\frac{1}{2}-b_k} + \frac{M_1^*}{Z_2} + \frac{M_2^*}{Z_2^2} + \dots\right].$$

We see that the first term of this asymptotic expansion agrees with the first term of the asymptotic expansion $I(Z_2)$. The agreement between the further, higher order terms is not proved. Because the coefficients M, which are originally introduced by Barnes [12]), and the coefficients A of Wright are defined in a much different way, the agreement is not trivial.

So the following result is obtained

$$M \cdot A_{q+\delta}^{q,\gamma} H_{\gamma, q+\delta}\left((-z)^q q^{-\kappa q} \rho^q e^{(\delta-\gamma)\pi i}\right) = I(Z_2) \quad . \quad . \ (60)$$

2°. Similarly

$$M \cdot \bar{A}_{q+\delta}^{q,\gamma} H_{\gamma, q+\delta}\left((-z)^q q^{-\kappa q} \rho^q e^{(\gamma-\delta)\pi i}\right) = I(Z_1) \quad . \quad . \ (61)$$

We now translate the results of section 6 by means of (59), (60) and (61). We assumed $\kappa > 0$ and distinguished the following cases:

I. $\kappa < 2$

a. For large values of $|z|$ with $|\eta| < (2-\kappa)\frac{\pi}{2}$

$$K(z) = J(-z) \text{ in agreement with theorem 4.}$$

b. For large values of $|z|$ with $(2-\kappa)\frac{\pi}{2} < \eta < \pi + \frac{\lambda}{q}\pi$

$$K(z) = I(Z_2);$$

for large values of $|z|$ with $-\pi - \frac{\lambda}{q}\pi < \eta < -(2-\kappa)\frac{\pi}{2}$

$$K(z) = I(Z_1);$$

[12]) Cf. E. W. Barnes, Proc. Lond. Math. Soc., (2), 5, pp. 59—116, (1907).

for large values of $|z|$ with $\eta = (2-\kappa)\frac{\pi}{2}$

$$K(z) = I(Z_2)+J(-z);$$

for large values of $|z|$ with $\eta = -(2-\kappa)\frac{\pi}{2}$

$$K(z) = I(Z_1)+J(-z).$$

If $(2-\kappa)\frac{\pi}{2} < \eta < \pi + \frac{\lambda}{q}\pi$ we may write

$$-\kappa\frac{\pi}{2} < \zeta < \frac{\lambda}{q}\pi \text{ and } Z_2 = Z.$$

If $-\pi-\frac{\lambda}{q}\pi < \eta < -(2-\kappa)\frac{\pi}{2}$ we may write

$$-\frac{\lambda}{q}\pi < \zeta < \kappa\frac{\pi}{2} \text{ and } Z_1 = Z.$$

So the first two assertions may be written in the following way:

For large values of $|z|$ with $|\zeta| < \kappa\frac{\pi}{2}$ [13])

$$K(z) = I(Z),$$

which is in agreement with theorem 1 and theorem 5, for in the sector $|\zeta| < \kappa\frac{\pi}{2}$ $J(-z)$ is negligible compared with the error term of $I(Z)$, as Wright proves.

The last two assertions may be written in the following way:

For large values of $|z|$ with $|\zeta| = \kappa\frac{\pi}{2}$

$$K(z) = I(Z)+J(-z),$$

which is in agreement with theorem 5.

c. In this subcase was supposed $\gamma = 0$. But then $f(s)$ (see (51)) has no poles. The asymptotic expansions (38) and (39) are the same as the asymptotic expansions $I(Z_2)$ resp. $I(Z_1)$.

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[13]) From the definition of λ it follows easily that we have in case I : $\kappa\frac{\pi}{2} \geqq \frac{\lambda}{q}\pi$.

Hence we obtain the following results:

For large values of $|z|$ with $0 < \eta < \pi + \frac{\pi}{q}$

$$K(z) = I(Z_2);$$

for large values of $|z|$ with $-\pi - \frac{\pi}{q} < \eta < 0$

$$K(z) = I(Z_1);$$

for large values of $|z|$ with $\eta = 0$

$$K(z) = I(Z_1)+I(Z_2).$$

We assumed $\gamma = 0$ or $\sum_{j=1}^{r} \mu_j = 0$. According to (4) we then have $\kappa > 1$.[14]) From the definition of Z_1 and Z_2 it follows that

$$\operatorname{Re} Z_2 - \operatorname{Re} Z_1 = 2\,|Z| \sin \frac{\eta}{\kappa} \sin \frac{\pi}{\kappa}.$$

If $\kappa > 1$ and $0 < \eta < \pi + \frac{\pi}{q}$ we have $\operatorname{Re} Z_1 < \operatorname{Re} Z_2$, for then

$$0 < \frac{\eta}{\kappa} < \frac{\pi(q+1)}{\kappa q} = \frac{\pi(q+1)}{q+\delta} \leqq \pi \text{ (see (12))}$$

and so $\sin \frac{\eta}{\kappa} > 0$.

In this case $I(Z_1)$ is negligible compared with the error term of $I(Z_2)$. Meijer says then that $I(Z_2)$ is dominant compared with $I(Z_1)$ (Meijer's "On the G-function", p. 941).

Similarly if $\kappa > 1$ and $-\pi - \frac{\pi}{q} < \eta < 0$ we have $\operatorname{Re} Z_2 < \operatorname{Re} Z_1$ and $I(Z_1)$ is dominant compared with $I(Z_2)$.

When these results are substituted in Wright's theorem 6 we obtain our results in the cases $0 < \eta < \pi + \frac{\pi}{q}$ and $-\pi - \frac{\pi}{q} < \eta < 0$.

In the case $\eta = 0$ we have $\operatorname{Re} Z_1 = \operatorname{Re} Z_2$ so both expansions $I(Z_1)$ and $I(Z_2)$ are of equal importance. Our result for $\eta = 0$ agrees with Wright's theorem 6.

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[14]) We might also have $\kappa = 1$, but this leads (see (4)) to the trivial case $\delta = 0$, which has been considered in section 6.

II. $\kappa \geqq 2$.

For large values of $|z|$ with $0 < \eta < \pi + \frac{\pi}{q}$

$$K(z) = I(Z_2);$$

for large values of $|z|$ with $-\pi - \frac{\pi}{q} < \eta < 0$

$$K(z) = I(Z_1);$$

for large values of $|z|$ with $\eta = 0$ and if $\kappa > 2$

$$K(z) = I(Z_1)+I(Z_2);$$

for large values of $|z|$ with $\eta = 0$ and if $\kappa = 2$

$$K(z) = I(Z_1)+I(Z_2)+J(-z).$$

If $\kappa > 2$ the first three assertions agree with Wright's theorem 2, for in the case $0 < \eta < \pi + \frac{\pi}{q}$ $I(Z_2)$ is dominant compared with $I(Z_1)$ and in the case $-\pi - \frac{\pi}{q} < \eta < 0$ $I(Z_1)$ is dominant compared with $I(Z_2)$. This may be derived in a similar way as in case I, *c*.

If $\kappa = 2$ the first two and the fourth assertion agree with Wright's theorem 3. In the case $0 < \eta < \pi + \frac{\pi}{q}$ $I(Z_2)$ is the dominant term, in the case $-\pi - \frac{\pi}{q} < \eta < 0$ $I(Z_1)$ is the dominant term. The first two assertions may also be written in such a way that they agree with Wright's theorem 1. For if $0 < \eta < \pi + \frac{\pi}{q}$ we have $-\pi < \zeta < \frac{\pi}{q}$ and $Z_2 = Z$; similarly if $-\pi - \frac{\pi}{q} < \eta < 0$ we have $-\frac{\pi}{q} < \zeta < \pi$ and $Z_1 = Z$. So for large values of $|z|$ with $|\zeta| < \pi$ then

$$K(z) = I(Z)$$

which is in agreement with theorem 1 for $\kappa \geqq 2$.

So all our results on the asymptotic behaviour of $K(z)$ agree with Wright's theorems. As has been stated, Wright's theorems occupy with asymptotic expansions valid uniformly in $\arg z$ in closed sectors, whereas our results give asymptotic expansions valid in open sectors but not uniformly in arg z. However, it may be proved that the used asymptotic expansions of Meijer's G-function, which are stated in open sectors, are also valid uniformly in arg z in closed subsectors of these open sectors.

Finally, we may conclude that the results of this paper, derived for rational positive values of the parameters, are in agreement with Wright's results for real positive values of the parameters, whereas the methods followed by Wright are much more complicated than the methods followed in this paper.

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