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On the Convergence of a Lacunary Trigonometrical Series

by

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1. A lacunary trigonometrical series is a series for which the terms different from zero are very sparse. This kind of trigonometrical series may be put in the form:

$$\sum_{\nu=1}^{\infty} (a_\nu \cos n_\nu x + b_\nu \sin n_\nu x).$$

We here assume, for simplicity, that the constant term of the series also vanishes. A series $\Sigma\, c_i$ is said to possess a gap (u, v) if $c_i = 0$ for $u < i < v$. It is known that [3, p. 251, § 10.31] *if a series $\Sigma\, C_i$ possesses infinitely many gaps (m_k, m'_k) such that $m'_k/m_k > \mu > 1$ for some μ and is summable $(C, 1)$ to sum S, then S_{m_k} and also $S_{m'_k}$ converges to S.*

2. Suppose that the above trigonometrical series is the Fourier series of an integrable function $f(x)$. This series possesses infinitely many gaps $(n_\nu, n_{\nu+1})$ such that $n_{\nu+1}/n_\nu > \lambda > 1$ for some λ. Write, for a given point x_0,

$$\varphi(t) = \varphi_{x_0}(t) = \tfrac{1}{2}\{f(x_0+t) + f(x_0-t) - 2f(x_0)\}.$$

Since

$$\varphi^*(t) = \frac{1}{t}\int_0^t |\varphi|\, du = 0(1) \quad (t \to +0)$$

holds for almost all x, the Fourier series of an integrable function is therefore almost everywhere summable $(C, 1)$ by Fejér's theorem. From this fact, we draw immediately the following well-known

Kolmogoroff's Theorem [1, 2]. *If the Fourier series of an integrable function $f(x)$ possesses infinitely many gaps $(n_\nu, n_{\nu+1})$ such that $n_{\nu+1}/n_\nu > \mu > 1$, then the partial sum S_{n_ν} converges almost everywhere to $f(x)$.*

We can also conclude that, at a given point x_0 at which $\varphi^*(t) = 0(1)$ $(t \to +0)$, if $n_{\nu+1}/n_\nu > \lambda > 1$, then $S_{n_\nu}(x_0) \to f(x_0)$ as $\nu \to \infty$.

3. We write

$$\varphi_1(t) = \frac{1}{t}\int_0^t \varphi\, du,$$

$$\varphi_2(t) = \frac{1}{t}\int_0^t \varphi_1\, du.$$

In this note, we replace the condition $\varphi^*(t) = 0(1)$ by the weaker condition $\varphi_1(t) = 0(1)$ $(t \to +0)$. We develop Kolmogoroff's theorem into the following manner.

THEOREM. *If the lacunary Fourier series of an integrable function $f(x)$ possesses infinitely many gaps $(n_\nu, n_{\nu+1})$ such that $n_{\nu+1}/n_\nu > \lambda > 1$, and if, at a given point x_0,* (i) $\varphi_1(t) = 0(1)$ $(t \to +0)$ *and*

(ii) $$\int_0^t |d\varphi_2| = O(1)$$

when $0 < t \leqq \eta$ for some η, then $S_{n_\nu}(x_0) \to f(x_0)$ as $\nu \to \infty$.

4. Now, we are in a position to prove the theorem. Take, for instance, $n_\nu = n$, $n_{\nu+1} = m$ and denote respectively by $D_n(t)$ and $K_n(t)$ Dirichlet's and Fejér's kernels. Then

$$D_n(t) = \tfrac{1}{2} + \sum_{\nu=1}^{n} \cos \nu t,$$

$$(n+1)K_n(t) = \sum_{\nu=0}^{n} D_\nu(t) = \frac{\sin^2 (n+1)t/2}{2\sin^2 t/2}.$$

Then, from the identity

$$\begin{aligned} mK_{m-1}(t) - nK_{n-1}(t) &= \sum_{\nu=n}^{m-1} D_\nu(t) \\ &= (m-n)D_n(t) + \sum_{\nu=1}^{m-n-1} (m-n-\nu)\cos(n+\nu)t \end{aligned}$$

and in virtue of the special property of the lacunary Fourier series, we obtain

$$\begin{aligned} S_n(x_0) - f(x_0) &= \frac{1}{\pi}\int_0^\pi \varphi(t) D_n(t)\, dt \\ &= \frac{1}{\pi(m-n)}\int_0^\pi \varphi(t)\,(mK_{m-1}(t) - nK_{n-1}(t))\, dt. \end{aligned}$$

Write

$$\Phi(t) = \int_0^t \varphi\, du.$$

Integration by parts gives

$$S_n(x_0)-f(x_0)=\frac{1}{\pi(m-n)}[\Phi(t)\,(mK_{m-1}(t)-nK_{n-1}(t))]_0^\pi$$

$$-\frac{1}{\pi(m-n)}\int_0^\pi \Phi(t)\frac{d}{dt}(mK_{m-1}(t)-nK_{n-1}(t))\,dt$$

$$=0(1)-\frac{1}{\pi(m-n)}\int_0^\pi \Phi(t)\,(mK'_{m-1}(t)-nK'_{n-1}(t))\,dt$$

$$=0(1)-\frac{1}{2\pi(m-n)}\left(\tfrac{1}{2}\int_0^\pi \Phi(t)\frac{m\sin mt-n\sin nt}{\sin^2 t/2}\,dt\right.$$

$$\left.-\int_0^\pi \Phi(t)\frac{\sin^2 mt/2-\sin^2 nt/2}{\sin^3 t/2}\cos t/2\,dt\right)$$

$$=0(1)-\frac{1}{2\pi(m-n)}(I_{1/2}-I_2),$$

say. We are going to estimate the orders of the integrals I_1 and I_2 respectively. We write

$$I_1=m\int_0^\pi \Phi\frac{\sin mt}{\sin^2 t/2}\,dt-n\int_0^\pi \Phi\frac{\sin nt}{\sin^2 t/2}\,dt$$

$$=mI_3-nI_4,$$

say. Since $(2\sin t/2)^{-2}-t^{-2}$ is bounded, we obtain, by Riemann-Lebesgue's theorem,

$$I_3=4\int_0^\pi \Phi\frac{\sin mt}{t^2}\,dt+0(1)$$

$$=4\int_0^\pi \varphi_1\frac{\sin mt}{t}\,dt+0(1)$$

$$=0(1)$$

as $m\to\infty$ by De la Vallée Poussin's test [3, p. 38, § 2.8] for the convergence of Fourier series at a given point by the condition (ii) and $\varphi_2(t)=0(1)$ as $t\to+0$. Similarly,

$$I_4=4\int_0^\pi \varphi_1\frac{\sin nt}{t}\,dt+0(1)$$

$$=0(1)$$

as $n \to \infty$ by the same test. It follows that

$$\begin{aligned}\frac{1}{m-n}\cdot I_1 &= 0\left(\frac{m}{m-n}\right)+0\left(\frac{n}{m-n}\right)\\ &= 0\left(\frac{1}{1-\lambda^{-1}}\right)+0\left(\frac{1}{\lambda-1}\right)\\ &= 0\,(1)\end{aligned}$$

as $n \to \infty$. In estimating the order of I_2, let us write

$$\begin{aligned}I_2 &= \int_0^{\pi} \varphi_1 \frac{\sin^2 mt/2-\sin^2 nt/2}{\sin^2 t/2}\cdot\frac{t}{\tan t/2}\,dt\\ &= \left(\int_0^{\pi/(m-n)}+\int_{\pi/(m-n)}^{\delta}+\int_{\delta}^{\pi}\right)\\ &= I_5+I_6+I_7,\end{aligned}$$

say. We have

$$\begin{aligned}|I_5| &\leqq 4\max_{0\leqq t\leqq\delta}|\varphi_1(t)|\int_0^{\pi/2(m-n)}\frac{|\sin^2 mt-\sin^2 nt|}{\sin^2 t}\left|\frac{t}{\tan t}\right|dt\\ &= 4\max_{0\leqq t\leqq\delta}|\varphi_1(t)|\,I_5',\end{aligned}$$

say. Now,

$$I_5' \leqq \int_0^{\pi/2(m-n)}\frac{|\sin(m+n)t\sin(m-n)t|}{\sin^2 t}\,dt.$$

Considering that

$$\left|\frac{\sin \alpha t}{t}\right| \leqq \alpha$$

for $0 \leqq t \leqq \pi/2$ and $\alpha \geqq 0$, we get

$$\begin{aligned}I_5' &\leqq (m^2-n^2)\int_0^{\pi/2(m-n)}dt\\ &= \frac{\pi}{2}(m+n).\end{aligned}$$

Therefore,

$$|I_5| \leqq 2\pi \max_{0\leqq t\leqq\delta}|\varphi_1(t)|\,(m+n).$$

Moreover, we have

$$
\begin{aligned}
|I_6| &\leqq 4 \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)| \int_{\pi/2(m-n)}^{\delta/2} csc^2\, t\, dt \\
&< 8 \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)| \cdot \frac{\pi}{2} \cdot \frac{2(m-n)}{\pi} \\
&= 8 \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)|\,(m-n).
\end{aligned}
$$

Last, by the second mean value theorem, we obtain

$$
|I_7| \leqq \frac{A}{\delta^3},
$$

where A is an absolute constant. From the above analysis, it follows that

$$
\begin{aligned}
|S_n(x_0)-f(x_0)| < 0(1) + \frac{1}{2\pi(m-n)}(m\,|I_3| &+ n\,|I_4| \\
&+ 2\pi \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)|\,(m+n) \\
&+ 8 \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)|\,(m-n) \\
&+ A/\delta^3) \\
&= 0(1) + \frac{1}{2\pi}\Big(\frac{m}{m-n}|I_3| + \frac{n}{m-n}|I_4| \\
&+ 2\pi \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)| \frac{m+n}{m-n} \\
&+ 8 \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3}\Big) \\
&< 0(1) + \frac{1}{2\pi}\Big(\frac{1}{1-\lambda^{-1}}|I_3| + \frac{1}{\lambda-1}|I_4| \\
&+ 2\pi \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)| \frac{1+\lambda^{-1}}{1-\lambda^{-1}} \\
&+ 8 \operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)| + \frac{A}{(m-n)\delta^3}\Big).
\end{aligned}
$$

For a given $\varepsilon > 0$, we can choose a δ so small that

$$
\operatorname*{Max}_{0\leqq t\leqq \delta} |\varphi_1(t)| < \varepsilon
$$

by the condition (i). After fixing δ, we take a sufficiently large integer n which makes $|I_3|$, $|I_4|$ and $(m-n)^{-1}\delta^{-3}$ all less than ε. Thus, we obtain finally

$$|S_n(x_0)-f(x_0)| < o(1) + \frac{1}{2\pi}\Big(\frac{1}{1-\lambda^{-1}} + \frac{1}{\lambda-1} + 2\pi\frac{1+\lambda^{-1}}{1-\lambda^{-1}} + 8 + A\Big)\varepsilon.$$

Since ε is an arbitrary small quantity, letting it tend to zero, we get

$$S_n(x_0) - f(x_0) \to 0,$$

i.e.,

$$\lim_{\nu\to\infty} S_{n_\nu}(x_0) = f(x_0).$$

This proves the theorem.

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