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by

B. L. J. Braaksma

§ 1. Introduction

§ 1.1. Asymptotic expansions for $|z| \to \infty$ and analytic continuations will be derived for the function $H(z)$ defined by

$$(1.1) \qquad H(z) = \frac{1}{2\pi i}\int_C h(s) z^s ds$$

where z is not equal to zero and

$$(1.2) \qquad z^s = \exp\{s\,(\text{Log}\,|z| + i \arg z)\}$$

in which $\text{Log}\,|z|$ denotes the natural logarithm of $|z|$ and $\arg z$ is not necessarily the principal value. Further

$$(1.3) \qquad h(s) = \frac{\prod_1^n \Gamma(1-a_j+\alpha_j s) \prod_1^m \Gamma(b_j-\beta_j s)}{\prod_{m+1}^q \Gamma(1-b_j+\beta_j s) \prod_{n+1}^p \Gamma(a_j-\alpha_j s)},$$

where p, q, n, m are integers satisfying

$$(1.4) \qquad 0 \leqq n \leqq p,\ 1 \leqq m \leqq q,$$

$\alpha_j (j = 1, \ldots, p)$, $\beta_j (j = 1, \ldots, q)$ are positive numbers and $a_j (j = 1, \ldots, p)$, $b_j (j = 1, \ldots, q)$ are complex numbers such that

$$(1.5) \qquad \alpha_j(b_h+\nu) \neq \beta_h(a_j-1-\lambda) \text{ for } \nu, \lambda = 0, 1, \ldots; \quad h = 1, \ldots, m;\ j = 1, \ldots, n.$$

C is a contour in the complex s-plane such that the points

$$(1.6) \qquad s = (b_j+\nu)/\beta_j \qquad (j = 1, \ldots, m;\ \nu = 0, 1, \ldots)$$

resp.

$$(1.7) \qquad s = (a_j-1-\nu)/\alpha_j \qquad (j = 1, \ldots, n;\ \nu = 0, 1, \ldots)$$

lie to the right resp. left of C, while further C runs from $s = \infty - ik$ to $s = \infty + ik$. Here k is a constant with $k > |\operatorname{Im} b_j|/\beta_j$ $(j = 1, \ldots, m)$. The conditions for the contour C can be fulfilled on account of (1.5). Contours like C and also contours like C but with endpoints $s = -i\infty+\sigma$ and $s = i\infty+\sigma$ (σ real) instead of $s = \infty - ik$ and $s = \infty+ik$ are called Barnes-contours and the corresponding integrals are called Barnes-integrals.

In the following we always assume (1.4) and (1.5).

In § 6.1, theorem 1, we show that $H(z)$ makes sense in the following cases:

I. for every $z \neq 0$ if μ is positive where

$$\mu = \sum_1^q \beta_j - \sum_1^p \alpha_j, \tag{1.8}$$

II. if $\mu = 0$ and

$$0 < |z| < \beta^{-1} \tag{1.9}$$

where

$$\beta = \prod_1^p \alpha_j^{\alpha_j} \prod_1^q \beta_j^{-\beta_j}. \tag{1.10}$$

In general $H(z)$ is a multiple-valued function of z.

§ 1.2. The function $H(z)$ and special cases of it occur at various places in the literature. A first systematic study of the function $H(z)$ has been made in a recent paper by C. Fox [18]. [1]) In the case that some special relations between the constants α_j, β_j, a_j, b_j are satisfied Fox derives theorems about $H(z)$ as a symmetrical Fourier kernel and a theorem about the asymptotic behaviour of $H(z)$ for $z \to \infty$ and $z > 0$.

The function defined by (1.1) but with the contour C replaced by another contour C' has been considered by A. L. Dixon and W. L. Ferrar [12]. C' is a contour like C but with endpoints $s = -\infty i+\sigma$ and $s = \infty i+\sigma$ (σ real). Their investigation concerns the convergence of the integrals, discontinuities and analytic continuations (not for all values of z) and integrals in whose integrands the function defined by (1.1) with $C \,||\, C'$ ($||$ means: replaced by) occurs.

Special cases of the function $H(z)$ occur in papers on functional equations with multiple gamma-factors and on the average order of arithmetical functions by S. Bochner [5], [5a], [6] and K. Chandrasekharan and Raghavan Narasimhan [9]. In these papers in some cases the analytic continuation resp. an estimation for $H(z)$ has been derived.

A large number of special functions are special cases of the

[1]) Numbers between brackets refer to the bibliography. In each paragraph the footnotes are numbered anew.

function $H(z)$. In the first place the G-function and all special cases of it as for instance Bessel-, Legendre-, Whittaker-, Struve-functions, the ordinary generalized hypergeometric functions (cf. [15] pp. 216-222) and a function considered by J. Boersma [7]. The G-function is the special case of the function $H(z)$ in (1.1) with $\alpha_j = 1$ ($j = 1, \ldots, p$), $\beta_j = 1$ ($j = 1, \ldots, q$). The ordinary generalized hypergeometric function is a special case of the G-function with $m = 1$, $n = p$ among others.

Further $H(z)$ contains as special cases the function of G. Mittag-Leffler (cf. G. Sansone-J. C. H. Gerretsen [28], p. 345), the generalized Bessel-function considered by E. M. Wright [31], [35] and the generalization of the hypergeometric function considered by C. Fox [17] and E. M. Wright [32], [34].

The results about the function $H(z)$ which will be derived here contain the asymptotic expansions for $|z| \to \infty$ and the analytic continuations of the functions mentioned above. The latter expansions and continuations have been derived by various methods among others by E. W. Barnes [2], [3], [4] (cf. a correction in F. W. J. Olver [25]), G. N. Watson [29], D. Wrinch [38], [39], [40], C. Fox [17], [18], W. B. Ford [16], E. M. Wright [31]–[36], C. V. Newsom [23], [24], H. K. Hughes [19], [20], T. M. MacRobert [21], C. S. Meijer [22] and J. Boersma [7]. The most important papers in this connection are those of Barnes, Wright and Meijer.

In [3] Barnes considered the asymptotic expansion of a number of G-functions. In the first place he derived algebraic asymptotic expansions (cf. § 4.6) for a class of G-functions. These expansions are derived by means of a simple method involving Barnes-integrals and the theorem of residues. In the second place he derived exponentially small and exponentially infinite asymptotic expansions (cf. § 4.4 for a definition) for another G-function. The derivation of these expansions is difficult and complicated. The G-function is written as a suitable exponential function multiplied by a contour integral. The integrand in this integral is a series of which the analytic continuation and the residues in the singular points are derived by means of an ingenious, complicated method involving among others zeta-functions and other functions considered previously by Barnes. The contour integral mentioned before has an algebraic asymptotic expansion which can be deduced by means of the theorem of residues. The investigation in [3] yields among others the asymptotic expansions of the ordinary generalized hypergeometric functions. Barnes also obtained the analytic continuation of a special case of the G-function by means of Barnes-integrals (cf. [4]).

In [22] Meijer has derived all asymptotic expansions and analytic continuations of the G-function. The method depends upon the fact that the G-function satisfies a homogeneous linear differential equation of which the G-functions considered by Barnes constitute a fundamental system of solutions. So every G-function can be expressed linearly in these G-functions of Barnes and from the asymptotic behaviour of these functions the asymptotic behaviour of arbitrary G-functions can be derived.

In [31], [32], [34] and [35] Wright considered the asymptotic expansions of generalizations of Bessel- and hypergeometric functions. The majority of his results are derived by a method which is based on the theorem of Cauchy and an adapted and simplified version of the method of steepest descents. In [33] and [36] these methods are applied to a class of more general integral functions. However, these methods do not yield all asymptotic expansions: any exponentially small asymptotic expansion has to be established by different methods (cf. [32], [34], [35]). The results of Wright have as an advantage over the results of the other authors mentioned before that his asymptotic expansions hold uniformly on sectors which cover the entire z-plane. Further the results of Wright — and also those of H. K. Hughes [19] — contain more information about the coefficients occurring in the asymptotic expansions.

§ 1.3. A description of the methods which we use to obtain the asymptotic expansions and analytic continuations of the function $H(z)$ is given in § 2. The results cannot be derived in the same manner as in the case of the G-function in [22] because in general the functions $H(z)$ do not satisfy expansion-formulae which express $H(z)$ in terms of some special functions $H(z)$ (if this would be the case then we should have to consider in detail only these latter functions as in the case of the G-function).

The analytic continuations of $H(z)$ in the case $\mu = 0$ can be found by bending parallel to the imaginary axis the contour in the integral (1.1) in which from the integrand some suitable analytic functions have been subtracted. This method is an extension of the method of Barnes in [4].

This method can be applied also in a number of cases to the determination of the asymptotic expansion of $H(z)$ for $|z| \to \infty$ if $\mu > 0$. Then algebraic asymptotic expansions are obtained.

However, in some cases all coefficients in these expansions are equal to zero ("dummy" expansions) and in these cases $H(z)$ has an exponentially small asymptotic expansion. This expansion will be derived by approximating the integrand in (1.1) by means of lemma 3 in § 3.3. In this way the difficulties in the researches of Barnes and Wright (cf. [3], [34], [35]) about special cases of these expansions are avoided. Contrary to their proofs here the derivation of the exponentially small expansions is easier than the derivation of the exponentially infinite expansions.

The remaining asymptotic expansions of $H(z)$ in the case $\mu > 0$ are derived by splitting in (1.1) the integrand into parts so that in the integral of some of these parts the contour can be bended parallel to the imaginary axis while the integrals of the other parts can be estimated by a method similar to the method which yields the exponentially small expansions. Some aspects of this method have been borrowed from Wright [33].

In the derivation of the asymptotic expansions of $H(z)$ the estimation of the remainder-terms is the most difficult part. The method used here depends upon the lemmas in § 5 which contain analytic continuations and estimates for a class of integrals related to Barnes-integrals. This method is related to the indirect Abelian asymptotics of Laplace transforms.

The remainder-terms can also be estimated by a direct method viz. the method of steepest descents. This will be sketched in § 10. In the case of the exponentially infinite expansions of $H(z)$ this method is analogous to the method of Wright in [33].

The asymptotic expansions of $H(z)$ are given in such a way that given a certain closed sector in the z-plane this sector can be divided into a finite number of closed subsectors on each of which the expansion of $H(z)$ for $|z| \to \infty$ holds uniformly in $\arg z$. Moreover it is indicated how the coefficients in the asymptotic expansions can be found.

§ 1.4. The results concerning $H(z)$ are contained in theorem 1 in § 6.1 (behaviour near $z = 0$), theorem 2 in § 6.2 (analytic continuations and behaviour near $z = \infty$ in the case $\mu = 0$), theorem 3 in § 6.3 (algebraic behaviour near $z = \infty$ in the case $\mu > 0$), theorem 4 in § 7.3 (exponentially small expansions in the case $\mu > 0$), theorems 5 and 6 in § 9.1 (exponentially infinite expansions in the case $\mu > 0$) and theorems 7–9 (expansions in the remaining barrier-regions for $\mu > 0$). In these theorems the notations introduced in (1.8), (1.10) and the definitions I–IV in § 4 are used. The terminology of asymptotic expansions is given in § 4.4 and § 4.6. In § 9.3 we have given a survey from which one may deduce which theorem contains the asymptotic expansion for $|z| \to \infty$ of a given function $H(z)$ on a given sector. In § 10.2 and § 10.3 some supplements on the theorems in § 6 and § 9 are given.

In § 11 the results about the function $H(z)$ are applied to the G-function: see the theorems 10–17. The asymptotic expansions and analytic continuations given in [22] are derived again. An advantage is that the asymptotic expansions are formulated in such a way that they hold uniformly on closed sectors — also in transitional regions — while moreover the coefficients of the expansions can be found by means of recurrence formulae. The notations used in the theorems and a survey of the theorems are given in § 11.3.

In § 12.1 and § 12.2 the results concerning $H(z)$ are applied to the generalized hypergeometric functions considered by Wright (cf. theorems 18–22). A survey of the theorems and the notations are given at the end of § 12.1 and in § 12.2. In § 12.3 a general class of series which possess exponentially small asymptotic expansions is considered. In § 12.4 the generalized Bessel-function is considered. The results are formulated in the theorems 24–26. The notations used in these theorems are given in (12.45).

§ 2. Description of the methods

§ 2.1. In this section we sketch the method by which the algebraic asymptotic expansions for $|z| \to \infty$ resp. the analytic continuation of the function $H(z)$ in case I resp. II of § 1 will be derived. First we consider the simplest cases which are analogous to the simplest cases considered by Barnes in [3] and [4].

To that end we replace the contour C in (1.1) by two other paths L and L_1. L resp. L_1 runs from $s = w$ to $s = w+il$ resp. $w-il$ and then to $s = \infty+il$ resp. $\infty-il$, while both parts of L resp. L_1 are rectilinear. Here w and l are real numbers so that

$$(2.1) \quad w \neq \operatorname{Re}(a_j-1-\nu)/\alpha_j \qquad (j = 1, \ldots, p;\ \nu = 0, 1, 2, \ldots)$$

$$(2.2) \quad w < \operatorname{Re} b_j/\beta_j \qquad (j = 1, \ldots, m)$$

$$(2.3) \quad \begin{cases} l = 1+\max\{|\operatorname{Im} a_j/\alpha_j|(j = 1, \ldots, p), |\operatorname{Im} b_j/\beta_j|(j = 1, \ldots, q), \\ |\operatorname{Im} \alpha/\mu|\} \text{ (cf. (3.24) for } \alpha) \text{ if } \mu \text{ is positive, while for } \mu = 0 \\ l = 1+\max\{|\operatorname{Im} a_j/\alpha_j|(j = 1, \ldots, p), |\operatorname{Im} b_j/\beta_j|(j = 1, \ldots, q)\}. \end{cases}$$

Then we easily deduce from (1.1) and the theorem of residues that

$$(2.4)\qquad H(z) = Q_w(z) + \frac{1}{2\pi i}\int_L h(s)z^s ds - \frac{1}{2\pi i}\int_{L_1} h(s)z^s ds$$

where

(2.5) $Q_w(z) = \sum$ residues of $h(s)z^s$ in those points (1.7) where $\operatorname{Re} s > w$.

Formula (2.4) holds in case I and also in case II of § 1. For the consideration of the integrals in (2.4) we need approximations for the function $h(s)$. Therefore we write $h(s)$ as a product of two other functions. Using

$$(2.6)\qquad \Gamma(s) = \pi/\{\sin \pi s \Gamma(1-s)\}$$

we see that

$$(2.7)\qquad h(s) = h_0(s)h_1(s)$$

where if

$$(2.8)\qquad s \neq (a_j-1-\nu)/\alpha_j \qquad (j = 1, \ldots, p;\ \nu = 0, 1, 2, \ldots)$$

resp.

$$(2.9)\qquad s \neq (b_j+\nu)/\beta_j \qquad (j = 1, \ldots, m;\ \nu = 0, \pm 1, \pm 2, \ldots)$$

we define

$$(2.10)\qquad h_0(s) = \prod_1^p \Gamma(1-a_j+\alpha_j s)/\prod_1^q \Gamma(1-b_j+\beta_j s)$$

resp.

$$(2.11)\qquad h_1(s) = \pi^{m+n-p} \prod_{n+1}^p \sin \pi(a_j-\alpha_j s)/\prod_1^m \sin \pi(b_j-\beta_j s).$$

For $h_0(s)$ resp. $h_1(s)$ approximation formulae are formulated in the lemmas 2 and 2a in § 3.2 resp. 4a in § 4.3. From these formulae estimates for $h(s)$ can be obtained.

Now define δ_0 by

$$(2.12)\qquad \delta_0 = (\sum_1^m \beta_j - \sum_{n+1}^p \alpha_j)\pi.$$

Consider the case that

$$(2.13)\qquad \delta_0 > \tfrac{1}{2}\mu\pi.$$

Then we may derive from the estimates for $h(s)$ mentioned above that the lemmas 6–7a from § 5 can be applied to the integrals in (2.4) for certain values of z; the path of integration L resp. L_1 in

(2.4) may be replaced by the half-line from $s = w$ to $s = w+i\infty$ resp. $w-i\infty$:

$$H(z) = Q_w(z) + \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} h(s)z^s ds \tag{2.14}$$

if $\mu = 0$, (1.9) and

$$|\arg z| < \delta_0 - \tfrac{1}{2}\mu\pi \tag{2.15}$$

holds and also if μ is positive and (2.15) is satisfied. The integral in (2.14) is absolutely convergent if $\mu \geqq 0$ and (2.15) holds. If $\mu = 0$ then $H(z)$ can be continued analytically by means of (2.14) into the domain (2.15). If μ is positive then

$$H(z) = Q_w(z) + O(z^w)$$

for $|z| \to \infty$ uniformly on every closed subsector of (2.15) with the vertex in $z = 0$. Hence by definition IV in § 4.6

$$H(z) \sim Q(z) \tag{2.16}$$

for $|z| \to \infty$ uniformly on every closed subsector of (2.15) with the vertex in $z = 0$, if μ is positive. The asymptotic expansion (2.16) is algebraic.

In the case that $\mu = 0$ another application of the lemmas 6 and 6a shows that the integral in (2.14) — and so $H(z)$ — can be continued analytically for $|z| > \beta^{-1}$.

Next we drop the assumption (2.13) and we extend the method used above to obtain the analytic continuation of $H(z)$ if $\mu = 0$ and the algebraic asymptotic expansion of $H(z)$ for $|z| \to \infty$ if $\mu > 0$ in the general case. Therefore we define (cf. (2.10) for $h_0(s)$):

$$\begin{cases} P_w(z) = \sum \text{ residues of } h_0(s)z^s \text{ in those points } s \text{ for which} \\ \operatorname{Re} s > w \quad \text{as well as } s = (a_j - 1 - \nu)/\alpha_j \; (j = 1, \ldots, p; \\ \nu = 0, 1, 2, \ldots). \end{cases} \tag{2.17}$$

Let r be an arbitrary integer and let δ_j, κ, C_j and D_j be given by the definitions I and II in § 4.2. Then it easily follows from (1.1), the theorem of residues, the definition of L, L_1, $Q_w(z)$ and $P_w(z)$ (cf. (2.5) and (2.17)) and (2.7) that

$$\begin{aligned} H(z) = Q_w(z) &+ \sum_r^{\kappa} D_j P_w(ze^{i\delta_j}) - \sum_0^{r-1} C_j P_w(ze^{i\delta_j}) \\ &+ \frac{1}{2\pi i}\left(\int_L - \int_{L_1}\right) h_0(s)\left\{h_1(s) + \sum_r^{\kappa} D_j e^{i\delta_j s} - \sum_0^{r-1} C_j e^{i\delta_j s}\right\} z^s ds \end{aligned} \tag{2.18}$$

in case I and also in case II of § 1. Like in (2.4) we want to stretch the path of integration L and L_1 in (2.18) to the straight line $\operatorname{Re} s = w$. This is possible if $\mu > 0$,

$$\delta_r - \delta_{r-1} > \mu\pi \tag{2.19}$$

and

$$-\delta_r + \tfrac{1}{2}\mu\pi < \arg z < -\delta_{r-1} - \tfrac{1}{2}\mu\pi \tag{2.20}$$

hold and also if $\mu = 0$, (1.9), (2.19) and (2.20) hold. The proof depends on the lemmas 6–7a from § 5. The assumptions concerning the integrands in these lemmas can be verified with the help of the estimates in the lemmas 2, 2a and 4a from § 3.2 and § 4.3 for the factors of the integrand in (2.18). Moreover the lemmas 6–7a applied to the integrals in (2.18) furnish the analytic continuation of $H(z)$ into (2.20), if $\mu = 0$ and (2.19) holds, and the algebraic asymptotic expansion of $H(z)$ on subsectors of (2.20) if (2.19) is satisfied and μ is positive. The results are formulated in theorem 2 and theorem 3 in § 6 where also the complete proof is given. The case with (2.13) appears to be contained in theorem 2 (cf. remark 1 after theorem 2) and theorem 3.

§ **2.2.** In this section we consider the exponentially small asymptotic expansions of $H(z)$. A condition for the occurrence of these expansions is that $n = 0$. If $n = 0$ then $Q_w(z) \equiv 0$ by (2.5) and $Q(z)$ represents a formal series of zeros (cf. (4.26)). So if $n = 0$, $\mu > 0$ and (2.13) are fulfilled then by (2.14)

$$H(z) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} h(s) z^s \, ds \tag{2.21}$$

on (2.15) where the integral in (2.21) converges absolutely on (2.15), and moreover by (2.16) and the definitions in § 4.6 we have $H(z) = O(z^w)$ for $|z| \to \infty$ uniformly on closed subsectors of (2.15) with the vertex in $z = 0$ and where w is an arbitrary negative number. Hence in this case better estimates for $H(z)$ have to be obtained. It appears that $H(z)$ has an exponentially small asymptotic expansion in this case.

To derive this expansion we first treat the special case that $n = 0$, $m = q$, $\mu > 0$. Then $\delta_0 = \mu\pi$ by (1.8) and (2.12). So the sector (2.15) can be written as

$$|\arg z| < \tfrac{1}{2}\mu\pi. \tag{2.22}$$

We temporarily denote the function $H(z)$ for which the assumptions above are satisfied by $H_0(z)$. Further we denote $h(s)$ by $h_2(s)$ in this case. So by (1.3), if

$$s \neq (b_j+\nu)/\beta_j \qquad (j=1,\ldots,q;\ \nu=0,1,2,\ldots) \tag{2.23}$$

then

$$h_2(s) = \prod_1^q \Gamma(b_j-\beta_j s)/\prod_1^p \Gamma(a_j-\alpha_j s). \tag{2.24}$$

Hence if (2.22) is satisfied then by (2.21)

$$H_0(z) = \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} h_2(s) z^s ds \tag{2.25}$$

where the integral is absolutely convergent.

To the factor $h_2(s)$ of the integrand in (2.25) we want to apply lemma 3 in § 3.3. Therefore we choose an arbitrary non-negative integer N and next the real number w so that besides (2.1) and (2.2) also

$$w < (1-\mathrm{Re}\,\alpha-N)/\mu \tag{2.26}$$

is satisfied. Then we derive from lemma 3 and (2.25):

$$\begin{aligned} H_0(\beta^{-1}\mu^{-\mu}z) &= \sum_0^{N-1}(-1)^j A_j (2\pi)^{q-p-1}\frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty}\Gamma(1-\mu s-\alpha-j)z^s ds \\ &\quad -i(2\pi)^{q-p-2}\int_{w-i\infty}^{w+i\infty}\rho_N(s)\Gamma(1-\mu s-\alpha-N)z^s ds \end{aligned} \tag{2.27}$$

on (2.22); all integrals in (2.27) converge absolutely (cf. § 7.1 for details of the proof). To the first N integrals in (2.27) we apply (cf. § 7.1)

$$\frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty}\Gamma(1-\mu s-\alpha-j)z^s ds = \frac{1}{\mu} z^{(1-\alpha-j)/\mu}\exp(-z^{1/\mu}) \tag{2.28}$$

for $j=0,\ldots,N-1$ and (2.22). So the first N terms at the righthand side of (2.27) vanish exponentially on closed subsectors of (2.22) with the vertex in $z=0$.

Next we have to estimate the last integral in (2.27) which we denote by $\sigma(z)$. So if (2.22) is fulfilled

$$\begin{aligned} \sigma(z) &= \int_{w-i\infty}^{w+i\infty}\rho_N(s)\Gamma(1-\mu s-\alpha-N)z^s ds \\ &= \int_{w-i\infty}^{w+i\infty}\rho_N(s)\,\frac{\Gamma(3-\mu s-\alpha-N)}{(1-\mu s-\alpha-N)_2}\,z^s ds \end{aligned} \tag{2.29}$$

(cf. (3.11) for the notation $(\lambda)_2$). These integrals converge absolutely as this is also the case with the integrals in (2.27). We estimate $\sigma(z)$ in a crude manner with the method of indirect Abelian asymptotics (cf. G. Doetsch [13] II p. 41). An alternative approach will be sketched in § 10.1; there we use the method of steepest descents.

Here we start with the formula

$$(2.30)\quad \Gamma(3-\mu s-\alpha-N)z^s = z^{(3-\alpha-N)/\mu}\int_0^\infty t^{2-\mu s-\alpha-N}\exp(-z^{1/\mu}t)\,dt$$

for $\operatorname{Re} s = w$ and (2.22); on account of (2.26) the integral is absolutely convergent. In the last integrand in (2.29) we replace the lefthand side of (2.30) by the righthand side of (2.30) and next we revert the order of integration (justification in § 7.2); then we obtain

$$(2.31)\qquad \sigma(z) = z^{(3-\alpha-N)/\mu}\int_0^\infty \rho(t)\exp(-z^{1/\mu}t)\,dt$$

for (2.22) where for $t > 0$:

$$(2.32)\qquad \rho(t) = \int_{w-i\infty}^{w+i\infty} \rho_N(s)t^{2-\mu s-\alpha-N}\frac{ds}{(1-\mu s-\alpha-N)_2}.$$

So $\sigma(z^\mu)$ and $\rho(t)$ are related to each other by the Laplace transformation. By (3.33)

$$(2.33)\qquad \rho_N(s)/(1-\mu s-\alpha-N)_2 = O(s^{-2})$$

for $|s| \to \infty$ uniformly on $\operatorname{Re} s \leqq w$ (cf. § 7.2 for all details of the proofs of (2.33)–(2.36)). Then it is easy to deduce that

$$(2.34)\quad \begin{aligned}|\rho(t)| &\leqq \int_{w-i\infty}^{w+i\infty} |\rho_N(s)t^{2-\mu s-\alpha-N}/(1-\mu s-\alpha-N)_2|\cdot|ds| \\ &\leqq Kt^{2-\mu w-\operatorname{Re}\alpha-N}\end{aligned}$$

for $t > 0$ and some constant K independent of t. Further it appears that $\rho(t) = 0$ for $0 < t \leqq 1$. From this, (2.34) and (2.31) we derive

$$(2.35)\qquad \sigma(z) = z^{(3-\alpha-N)/\mu}\exp(-z^{1/\mu})O(1)$$

for $|z| \to \infty$ uniformly on every closed sector with vertex $z = 0$ which is contained in (2.22). From (2.27), (2.28), (2.29) and (2.35) we may derive

$$(2.36)\qquad H_0(z) = (2\pi)^{q-p}e^{(\pm\alpha-\frac{1}{2})\pi i}E_N(ze^{\pm\mu\pi i})$$

for $|z| \to \infty$ uniformly on every closed sector with vertex in $z = 0$ and which is contained in (2.22). Here N is an arbitrary non-negative integer, the lower resp. upper signs belong together and $E_N(z)$ is defined in definition III in § 4.4. From (2.36) we immediately derive the exponentially small asymptotic expansions of $H(z)$ (or $H_0(z)$) in the case considered above. This will be formulated in theorem 4 in § 7.3.

Now we consider the case that $\mu > 0$, $n = 0$, $0 < m < q$ and (2.13) hold. Then by (2.6), (2.24) and (1.3)

$$h(s) = h_2(s)\pi^{m-q} \prod_{m+1}^{q} \sin \pi(b_j - \beta_j s) \tag{2.37}$$

if (2.23) is fulfilled. The factor of $h_2(s)$ in (2.37) satisfies

$$\pi^{m-q} \prod_{m+1}^{q} \sin \pi(b_j - \beta_j s) = \sum_{0}^{M} \tau_j e^{i\omega_j s} \tag{2.38}$$

where M is a positive integer, $\omega_0, \ldots, \omega_M$ are real and independent of s with

$$\omega_0 < \omega_1 < \ldots < \omega_M, \quad \omega_M = \pi \sum_{m+1}^{q} \beta_j = \mu\pi - \delta_0 = -\omega_0 \tag{2.39}$$

(cf. (1.8) and (2.12)) while $\tau_0, \ldots, \tau_M$ are complex and independent of s with

$$\begin{aligned} \tau_0 &= (2\pi i)^{m-q} \exp\,(\pi i \sum_{m+1}^{q} b_j), \\ \tau_M &= (-2\pi i)^{m-q} \exp\,(-\pi i \sum_{m+1}^{q} b_j). \end{aligned} \tag{2.40}$$

By (2.39) we have if (2.15) holds:

$$-\tfrac{1}{2}\mu\pi < \arg z + \delta_0 - \mu\pi \leqq \arg z + \omega_j \leqq \arg z + \mu\pi - \delta_0 < \tfrac{1}{2}\mu\pi \tag{2.41}$$

for $j = 0, \ldots, M$. Since further (2.25) holds for (2.22), now also (2.25) with $z \,||\, ze^{i\omega_j}$ is valid on (2.15) by (2.41). From this, (2.21), (2.37) and (2.38) we deduce

$$H(z) = \sum_{0}^{M} \tau_j H_0(ze^{i\omega_j}). \tag{2.42}$$

This implies on account of (2.36) and (2.41)

$$H(z) = (2\pi)^{q-p} \sum_{0}^{M} \tau_j e^{(\pm\alpha-\frac{1}{2})\pi i} E_N(ze^{i(\omega_j \pm \mu\pi)}) \tag{2.43}$$

for $|z| \to \infty$ uniformly on closed subsectors of (2.15) with vertex $z = 0$ and for every non-negative integer N. In (2.43) the upper resp. lower signs in the products belong together but for different values of j the terms may be taken with different signs. With the help of lemma 5 from § 4.5 we can now derive the asymptotic expansion of $H(z)$ in the case we consider here. These expansions which are again exponentially small are given in theorem 4 in § 7.3.

§ 2.3. We have to consider now the methods which can be used to obtain the asymptotic expansions of $H(z)$ which are not algebraic and not exponentially small in the case $\mu > 0$. Therefore we consider a sector

$$\varepsilon_0 - \tfrac{1}{2}(\delta_r + \delta_{r+1}) \leqq \arg z \leqq \varepsilon_0 - \tfrac{1}{2}(\delta_{r-1} + \delta_r) \tag{2.44}$$

where r is an integer, δ_j is defined in definition I in § 4.2 and ε_0 is a positive number independent of z.

Let N be a non-negative integer and w a real number satisfying (2.1), (2.2), (2.26) and

$$w \neq -(\nu + \operatorname{Re} \alpha)/\mu \qquad (\nu = 0, \pm 1, \pm 2, \ldots) \tag{2.45}$$

while l is defined by (2.3). Then we have to approximate the integrals in (2.4) on the sector (2.44). This will be done by using (2.7) for $h(s)$ and approximating $h_1(s)$. However, in the general case it appears that we have to use different approximations for $h_1(s)$ on L and on L_1 contrary to the case where (2.19) holds and where we could use the same approximation on L and on L_1 (cf. § 2.1: (2.18)). Here we introduce integers λ and ν so that

$$\begin{cases} \delta_{\nu+1} \geqq \tfrac{1}{2}(\mu\pi + \delta_r + \delta_{r+1}),\ \delta_{\lambda-1} \leqq \tfrac{1}{2}(-\mu\pi + \delta_r + \delta_{r-1}) - 2\varepsilon_0, \\ \lambda \leqq 0 \leqq \nu,\ \lambda \leqq \kappa \leqq \nu,\ \lambda < r \leqq \nu. \end{cases} \tag{2.46}$$

Here κ is given by definition II in § 4.2 and r and ε_0 are the same as in (2.44). Then we may deduce from lemma 7 and lemma 7a from § 5, (4.8), (4.9) and lemma 2 in § 3.2 that

$$\begin{cases} \displaystyle\int_L h_0(s)\{h_1(s) + \sum_{\nu+1}^{\kappa} D_j e^{i\delta_j s} - \sum_0^{\nu} C_j e^{i\delta_j s}\} z^s ds = O(z^w) \\ \displaystyle\int_{L_1} h_0(s)\{h_1(s) + \sum_{\lambda}^{\kappa} D_j e^{i\delta_j s} - \sum_0^{\lambda-1} C_j e^{i\delta_j s}\} z^s ds = O(z^w) \end{cases} \tag{2.47}$$

for $|z| \to \infty$ uniformly on (2.44).

Now define for $z \neq 0$ and $\mu > 0$:

$$F(z) = \frac{1}{2\pi i} \int_L h_0(s) z^s ds. \tag{2.48}$$

Then by (2.17) and the definition of L and L_1:

$$(2.49) \qquad F(z) = -P_w(z) + \frac{1}{2\pi i}\int_{L_1} h_0(s)z^s ds.$$

From (2.4) and (2.47)–(2.49) we may deduce

$$(2.50) \qquad H(z) = Q_w(z) + \sum_{\lambda}^{\kappa} D_j P_w(ze^{i\delta_j}) + \sum_{\lambda}^{\nu} (C_j + D_j) F(ze^{i\delta_j}) + O(z^w)$$

for $|z| \to \infty$ uniformly on (2.44). Hence it is sufficient to derive estimates for $F(z)$ for then we deduce by means of (2.50) estimates for $H(z)$.

To derive the estimates for $F(z)$ we choose a constant ε such that $0 < \varepsilon < \frac{1}{2}\mu\pi$. Then by (2.48), lemma 7 in § 5.2 and lemma 2 in § 3.2 we have

$$(2.51) \qquad F(z) = O(z^w)$$

for $|z| \to \infty$ uniformly on (5.14). In the same way using (2.49) and lemma 7a in § 5.3 we obtain

$$(2.52) \qquad F(z) = -P_w(z) + O(z^w)$$

for $|z| \to \infty$ uniformly on (5.29).

For the consideration of $F(z)$ on the sector

$$(2.53) \qquad |\arg z| \leqq \tfrac{1}{2}\mu\pi + \varepsilon$$

we use the property

(2.54) $\quad |e^{\mp\mu\pi i s}/\sin \pi(\mu s + \alpha)|$ is bounded for $\pm \operatorname{Im} s \geqq l$,

where the upper resp. lower signs belong together. Using lemma 7 from § 5.2, the property (2.54) and lemma 2 from § 3.2 we may deduce

$$(2.55) \qquad \int_L h_0(s) z^s e^{\mu\pi i s} \frac{ds}{\sin \pi(\mu s + \alpha)} = O(z^w)$$

and

$$(2.56) \qquad \int_{L_1} h_0(s) z^s e^{-\mu\pi i s} \frac{ds}{\sin \pi(\mu s + \alpha)} = O(z^w)$$

for $|z| \to \infty$ uniformly on (2.53). In view of

$$(2.57) \qquad \frac{1}{2i}\left(e^{\pi i(\mu s + \alpha)} - e^{-\pi i(\mu s + \alpha)}\right)/\sin \pi(\mu s + \alpha) = 1,$$

the definition of $F(z)$ in (2.48) and (2.55), (2.56) imply

$$(2.58)\quad F(z) = \frac{1}{4\pi} e^{-\pi i\alpha}\left(\int_L - \int_{L_1}\right) h_0(s)(ze^{-\mu\pi i})^s \frac{ds}{\sin \pi(\mu s+\alpha)} + O(z^w)$$

for $|z| \to \infty$ uniformly on (2.53). By (3.25) and (2.6) we may write instead of (2.58)

$$(2.59)\quad F(\beta^{-1}\mu^{-\mu}z)$$
$$= \frac{1}{4\pi^2} e^{-\pi i\alpha} \sum_0^{N-1} (-1)^j A_j \left(\int_L - \int_{L_1}\right) \Gamma(1-\mu s-\alpha-j)(ze^{-\mu\pi i})^s ds$$
$$+ \frac{1}{4\pi^2} e^{-\pi i\alpha} \tau(z) + O(z^w)$$

for $|z| \to \infty$ uniformly on (2.53) where

$$(2.60)\quad \tau(z) = \pi \left(\int_L - \int_{L_1}\right) r_N(s)(ze^{-\mu\pi i})^s \frac{ds}{\sin \pi(\mu s+\alpha)\Gamma(\mu s+\alpha+N)}$$
$$= (-1)^N \left(\int_L - \int_{L_1}\right) r_N(s)\Gamma(1-\mu s-\alpha-N)(ze^{-\mu\pi i})^s ds.$$

Using (2.59) and (7.1) we infer

$$(2.61)\quad F(\beta^{-1}\mu^{-\mu}z) = \frac{1}{2\pi i\mu} \sum_0^{N-1} A_j z^{(1-\alpha-j)/\mu} \exp(z^{1/\mu})$$
$$+ \frac{1}{4\pi^2} e^{-\pi i\alpha} \tau(z) + O(z^w)$$

for $|z| \to \infty$ uniformly on (2.53). So we have to estimate the analytic function $\tau(z)$ on (2.53).

From (2.60), (3.27), (2.54) and the lemmas 7 and 7a from § 5 we deduce that if $\arg z = \frac{1}{2}\mu\pi+\varepsilon$

$$(2.62)\quad \tau(z) = (-1)^N \int_{w-i\infty}^{w+i\infty} r_N(s) \frac{\Gamma(3-\mu s-\alpha-N)}{(1-\mu s-\alpha-N)_2} (ze^{-\mu\pi i})^s ds.$$

The last integral is of the same type as that in (2.29); the only difference is that almost all poles of $r_N(s)$ are lying to the left of $\mathrm{Re}\, s = w$ while all poles of $\rho_N(s)$ are lying to the right of $\mathrm{Re}\, s = w$. The integral in (2.62) can be rewritten using (2.30) as a multiple integral; in this multiple integral we revert the order of integration like at (2.29) and (2.31). Then we obtain for $\arg z = \frac{1}{2}\mu\pi+\varepsilon$

$$(2.63)\quad \tau(z) = (ze^{-\mu\pi i})^{(3-\alpha-N)/\mu} \int_0^\infty r(t) \exp(z^{1/\mu}t)\, dt$$

where for $t > 0$

$$(2.64)\quad r(t) = (-1)^N \int_{w-i\infty}^{w+i\infty} r_N(s) t^{2-\mu s-\alpha-N} \frac{ds}{(1-\mu s-\alpha-N)_2}.$$

In view of (3.27) we have

$$(2.65)\qquad r_N(s)/(1-\mu s-\alpha-N)_2 = O(s^{-2})$$

for $|s| \to \infty$ uniformly on $\operatorname{Re} s \geqq w$. So the integral in (2.64) converges absolutely for $t > 0$ and

$$(2.66)\qquad |r(t)| \leqq K|t|^{2-\mu w-\operatorname{Re}\alpha-N}$$

for $t > 0$; here K is independent of t. From lemma 2 and (2.66) we derive that for $t > 1$ the function $r(t)$ is equal to the sum of the residues of the integrand in (2.64) in the poles s with $\operatorname{Re} s > w$ multiplied by $2\pi i(-1)^{N+1}$. The number of these poles is finite and it follows that the function $r(t)$ for $t > 1$ can be continued analytically for $t \neq 0$. It is easy now to estimate the integral in (2.63) with the help of the lemmas 6 and 6a from § 5 and the properties of $r(t)$. The results are formulated in lemma 8 in § 8.

From the properties of $F(z)$ mentioned in lemma 8 and (2.50) we deduce in § 9 the asymptotic expansions of $H(z)$ for $|z| \to \infty$ in the case $\mu > 0$ which are not contained in the theorems 3 and 4 (though theorem 3 can also be deduced from lemma 8 and (2.50) again). In § 8 the details of the proofs of the assertions in § 2.3 are presented.

§ 3. Approximations for quotients of gamma-functions

In this paragraph approximation formulae for the functions $h_0(s)$ defined by (2.10) and $h_2(s)$ defined by (2.24) will be derived. Here and in the following paragraphs we use the notation of § 1.1 and § 2.1. Further Log z always denotes the principal value of $\log z$ and $\sum_{j=k}^{l} \ldots$ is interpreted as zero if $k > l$.

§ 3.1. In this section we derive lemma 1 on which the approximations for $h_0(s)$ and $h_2(s)$ will be based. Lemma 1 will be derived from the formula of Stirling in the following form:

Let a be a complex number, ε a constant satisfying $0 < \varepsilon < \pi$ and M a non-negative integer. Then

$$(3.1)\quad \operatorname{Log} \Gamma(s+a) = (s+a-\tfrac{1}{2}) \operatorname{Log} s - s + \tfrac{1}{2} \operatorname{Log} (2\pi) + \sum_{1}^{M-1} B_{j+1}(a) \cdot \frac{(-1)^{j+1} \cdot s^{-j}}{j(j+1)} + O(s^{-M})$$

for $|s| \to \infty$ uniformly on the sector

$$|\arg s| \leqq \pi - \varepsilon. \tag{3.2}$$

Here $B_j(a)$ is the value of the j-th Bernoulli polynomial in the point a. These polynomials are defined by

$$te^{at}/(e^t-1) = \sum_0^\infty B_j(a)t^j/j!, \qquad |t| < 2\pi. \tag{3.3}$$

That (3.1) holds uniformly on the sector (3.2) for $|s| \to \infty$ means that there exist positive constants K_1 and K_2 such that for $|s| \geqq K_1$ and (3.2):

$$|O(s^{-M})| \leqq K_2|s|^{-M} \tag{3.4}$$

where $O(s^{-M})$ is the O-term in (3.1).

(3.1) occurs in A. Erdélyi a.o. [15], p. 48 formula (12), however, without mentioning the uniformity on (3.2). A proof of (3.1) including the uniformity on (3.2) is contained in a paper by C. H. Rowe [27]. For a particular case see E. T. Whittaker and G. N. Watson [30], § 13.6.

From the formula of Stirling we derive

LEMMA 1.

Suppose g and h are positive integers, ρ_j ($j = 1, \ldots, g$) and $\sigma_j (j = 1, \ldots, h)$ are positive constants so that

$$\sum_1^g \rho_j = \sum_1^h \sigma_j \tag{3.5}$$

and c_j ($j = 1, \ldots, g$), d_j ($j = 1, \ldots, h$), λ and c are complex constants with $\lambda \neq 0$. Define $P(s)$ by

$$P(s) = \prod_1^g \Gamma(\rho_j s + c_j) / \prod_1^h \Gamma(\sigma_j s + d_j) \tag{3.6}$$

for

$$\rho_j s + c_j \neq 0, -1, -2, \ldots \qquad (j = 1, \ldots, g). \tag{3.7}$$

Further we define

$$a = \sum_1^g \rho_j \operatorname{Log} \rho_j - \sum_1^h \sigma_j \operatorname{Log} \sigma_j \tag{3.8}$$

and

$$b = \tfrac{1}{2}(h-g) + \sum_1^g c_j - \sum_1^h d_j. \tag{3.9}$$

Then there exist numbers E_0, $E_1, \ldots$ independent of s but depending on the other parameters used above, with the following property:

If N is a non-negative integer and ε is a constant with $0 < \varepsilon < \pi$ then

$$(3.10) \qquad P(s) = e^{as}s^b\{\sum_0^{N-1} E_j/(\lambda s+c)_j+O(1/(\lambda s+c)_N)\}$$

for $|s| \to \infty$ uniformly on (3.2). Here the definition of $(\lambda s+c)_j$ follows from:

$$(3.11) \quad (r)_0 = 1,\ (r)_j = r(r+1)\dots(r+j-1) \text{ for } j = 1, 2, \dots.$$

In (3.10) the value of E_0 is given by

$$(3.12) \qquad E_0 = (2\pi)^{\frac{1}{2}(g-h)} \cdot \prod_1^g \rho_j^{c_j-\frac{1}{2}} \cdot \prod_1^h \sigma_j^{\frac{1}{2}-d_j}.$$

PROOF: From (3.6) and the formula of Stirling (3.1) we derive that for $|s| \to \infty$ uniformly on (3.2)

$$(3.13) \quad \begin{aligned} \log P(s) = &\sum_1^g (\rho_j s+c_j-\tfrac{1}{2})\operatorname{Log}(\rho_j s)-\sum_1^h(\sigma_j s+d_j-\tfrac{1}{2})\operatorname{Log}(\sigma_j s) \\ &-\sum_1^g \rho_j s+\sum_1^h \sigma_j s+\tfrac{1}{2}(g-h)\operatorname{Log}(2\pi)+\sum_1^{N-1} D_j s^{-j}+O(s^{-N}). \end{aligned}$$

Here $\log P(s)$ is not necessarily the principal value and the D_j are numbers only depending on j, ρ_ν, c_ν ($\nu = 1, \dots, g$) and σ_ν, d_ν ($\nu = 1, \dots, h$) but independent of the choice of N, of ε and of s.

Regrouping the terms in the righthand side of (3.13) and using (3.5), (3.8), (3.9) and (3.12) we obtain

$$(3.14) \quad \log P(s) = as+b\operatorname{Log} s+\log E_0+\sum_1^{N-1} D_j s^{-j}+\varphi_1(s)$$

where

$$(3.15) \qquad \varphi_1(s) = O(s^{-N})$$

for $|s| \to \infty$ uniformly on (3.2)

From the relation

$$\exp\left(\sum_1^{N-1} D_j w^j\right) = 1+\sum_{\nu=1}^{\infty}\frac{1}{\nu!}\left(\sum_1^{N-1} D_j w^j\right)^\nu$$

we deduce

$$(3.16) \qquad \exp\left(\sum_1^{N-1} D_j w^j\right) = 1+\sum_1^{\infty} C_j w^j$$

where C_j ($j = 1, 2, \dots$) is a number which only depends on those numbers D_h for which $h = 1, \dots, N-1$ and $h \leqq j$. Replacing w by $1/s$ in (3.16) we obtain

$$(3.17) \qquad \exp\left(\sum_1^{N-1} D_j s^{-j}\right) = 1+\sum_1^{N-1} C_j s^{-j}+O(s^{-N})$$

for $|s| \to \infty$ uniformly on (3.2). On account of the remark after (3.13), C_j $(j=1, \ldots, N-1)$ only depends on j, ρ_ν, c_ν $(\nu=1, \ldots, g)$ and σ_ν, d_ν $(\nu=1, \ldots, h)$ and is independent of N, of ε and of s.

Since $e^w = 1+O(w)$ for $w \to 0$ it follows from (3.15) that

$$(3.18) \qquad \exp \varphi_1(s) = 1+O(\varphi_1(s)) = 1+O(s^{-N})$$

for $|s| \to \infty$ uniformly on (3.2). From (3.14), (3.17) and (3.18) we deduce

$$P(s) = e^{as}s^b E_0\{1+\sum_1^{N-1} C_j s^{-j}+O(s^{-N})\}\{\{1+O(s^{-N})\}$$

and

$$(3.19) \qquad P(s) = e^{as}s^b\{E_0+\sum_1^{N-1} E_0 C_j s^{-j}+O(s^{-N})\}$$

for $|s| \to \infty$ uniformly on (3.2). This implies (3.10) for $N=0$ and $N=1$.

Next we derive (3.10) from (3.19) for $N \geqq 2$. First we show that if k is an integer with $2 \leqq k \leqq N$ then

$$(3.20) \qquad \sum_{j=1}^{N-1} E_0 C_j s^{-j} = \sum_{j=1}^{k-1} E_j/(\lambda s+c)_j+\sum_{j=k}^{N-1} c_{k,j} s^{-j}+O(s^{-N})$$

for $|s| \to \infty$ uniformly on (3.2). Here E_j $(j=1, \ldots, N-1)$ resp. $c_{k,j}$ $(k=2, \ldots, N-1; j=k, \ldots, N-1)$ are numbers only depending on λ, c, $C_1, \ldots, C_j$, E_0 resp. k, λ, c, $C_1, \ldots, C_j$, E_0.

From the expansion of $1/(\lambda s+c)_k$ near $s=\infty$ it follows that for $2 \leqq k \leqq N-1$:

$$(3.21) \qquad 1/(\lambda s+c)_k = (\lambda s)^{-k}+\sum_{j=k+1}^{N-1} d_{k,j} s^{-j}+O(s^{-N})$$

for $|s| \to \infty$ uniformly on (3.2). Here the numbers $d_{k,j}$ only depend on k, j, λ and c. (3.21) implies

$$(3.22) \qquad s^{-k} = \lambda^k/(\lambda s+c)_k-\lambda^k \sum_{j=k+1}^{N-1} d_{k,j} s^{-j}+O(s^{-N})$$

for $|s| \to \infty$ uniformly on (3.2). From (3.22) with $k=1$ we deduce (3.20) for $k=2$. Further if $N>2$ and (3.20) holds for some integer k with $2 \leqq k < N$ then from (3.20) and (3.22) we derive (3.20) with k replaced by $k+1$. Hence (3.20) holds generally for $2 \leqq k \leqq N$.

Using (3.20) with $k=N$, (3.19) and

$$O(s^{-N}) = O(1/(\lambda s+c)_N) \tag{3.23}$$

for $|s| \to \infty$ uniformly on (3.2) we obtain (3.10) for $|s| \to \infty$ uniformly on (3.2).

REMARK: Lemma 1 also occurs in a more elementary form in the book of W. B. Ford [16] p. 74 and in H. K. Hughes [20], p. 458.

§ 3.2. From lemma 1 we deduce approximations for $h_0(s)$.

LEMMA 2.

We use the notation of § 1 (*cf.* (1.8) *and* (1.10)). $h_0(s)$ *is defined by* (2.10) *if* (2.8) *holds.* N *is an arbitrary non-negative integer and* ε *is a constant so that* $0 < \varepsilon < \pi$. *Suppose* μ *is positive and define*

$$\alpha = \sum_1^p a_j - \sum_1^q b_j + \tfrac{1}{2}(q-p+1). \tag{3.24}$$

Then there exist numbers A_0, $A_1, \ldots$ *independent of* s, *of* N *and of* ε *and only depending on* p, q, α_j, a_j, β_j, b_j *with the following property*:

If $r_N(s)$ *is defined by*

$$h_0(s)(\beta\mu^\mu)^{-s} = \sum_0^{N-1} A_j/\Gamma(\mu s+\alpha+j) + r_N(s)/\Gamma(\mu s+\alpha+N) \tag{3.25}$$

if (2.8) *and*

$$s \neq -(\alpha+N+\nu)/\mu \qquad (\nu = 0, 1, 2, \ldots) \tag{3.26}$$

hold and if $(\beta\mu^\mu)^{-s}$ *has the principal value then* $r_N(s)$ *is analytic in* s *in its domain of definition and*

$$r_N(s) = O(1) \tag{3.27}$$

for $|s| \to \infty$ *uniformly on* (3.2). *In particular*

$$A_0 = (2\pi)^{\frac{1}{2}(p-q+1)}\mu^{\alpha-\frac{1}{2}} \prod_1^p \alpha_j^{\frac{1}{2}-a_j} \prod_1^q \beta_j^{b_j-\frac{1}{2}}. \tag{3.28}$$

PROOF: That $r_N(s)$ is analytic in s in the domain where (2.8) and (3.26) hold will be clear.

To prove (3.27) we apply lemma 1 with $\lambda = \mu$, $c = \alpha$,

$$P(s) = h_0(s)\Gamma(\mu s+\alpha). \tag{3.29}$$

Then by (2.10) we have $g = p+1$, $h = q$, $\rho_j = \alpha_j$ $(j = 1, \ldots, p)$, $\rho_{p+1} = \mu$, $c_j = 1-a_j$ $(j = 1, \ldots, p)$, $c_{p+1} = \alpha$, $\sigma_j = \beta_j$ and $d_j = 1-b_j$ $(j = 1, \ldots, q)$. From this and (1.8) we derive (3.5).

Further by (3.9) and (3.24) we have $b = 0$ while by (3.8), (1.8) and (1.10): $e^a = \beta\mu^\mu$. Hence by (3.10)

$$(3.30) \quad h_0(s)\Gamma(\mu s+\alpha) = (\beta\mu^\mu)^s\{\sum_0^{N-1} A_j/(\mu s+\alpha)_j+O(1/(\mu s+\alpha)_N)\}$$

for $|s| \to \infty$ uniformly on (3.2). The numbers A_j play the part of the numbers E_j in (3.10). So A_j depends on j, p, q, α_ν, a_ν ($\nu = 1, \ldots, p$), β_ν, b_ν ($\nu = 1, \ldots, q$) only. (3.25) and (3.30) imply (3.27) for $|s| \to \infty$ uniformly on (3.2). Finally (3.28) follows from (3.12).

REMARK: Lemma 2 occurs in a slightly different form in E. M. Wright [32] p. 287 and a special case of lemma 2 occurs in H. K. Hughes [20] p. 459. For $\alpha_j = 1$ ($j = 1, \ldots, p$) and $\beta_j = 1$ ($j = 1, \ldots, q$) this lemma is contained in papers by a.o. T. D. Riney [26] p. 245 and p. 246, J. G. van der Corput [11] p. 337 and E. M. Wright [37] p. 38. There recurrence formulae for the numbers $A_0, A_1, \ldots$ are derived.

The analogue of lemma 2 for the case $\mu = 0$ is:

LEMMA 2a.

Let the assumptions of lemma 2 be satisfied save that now $\mu = 0$ instead of μ positive. Then

$$(3.31) \qquad h_0(s) = \beta^s s^{\frac{1}{2}-\alpha}O(1)$$

for $|s| \to \infty$ uniformly on (3.2). In (3.31) β^s and $s^{\frac{1}{2}-\alpha}$ have the principal values.

PROOF: On account of (1.4) we have $q \geqq 1$. As the numbers β_j are positive μ would be positive if $p = 0$ (cf. (1.8)). Hence $p \geqq 1$. We now apply lemma 1 with $g = p$, $h = q$, $\rho_j = \alpha_j$ and $c_j = 1-a_j$ for $j = 1, \ldots, g$ while $\sigma_j = \beta_j$ and $d_j = 1-b_j$ for $j = 1, \ldots, h$. Then (3.5) is fulfilled on account of $\mu = 0$ and (1.8). Further $a = \operatorname{Log} \beta$ in view of (3.8) and (1.10), while $b = \frac{1}{2}-\alpha$ according to (3.9) and (3.24). By choosing $\lambda = 1$, $c = 0$ and $N = 0$ in (3.10) we obtain (3.31).

§ 3.3. The following lemma contains approximations for the function $h_2(s)$ defined by (2.24).

LEMMA 3.

We use the notation of § 1. $h_2(s)$ is defined by (2.24) if (2.23) holds and α, $A_0, A_1, \ldots$ are given by lemma 2. N will be a non-negative integer and ε a constant so that $0 < \varepsilon < \pi$. Suppose μ is positive.

Then we define

$$\rho_N(s) = (2\pi)^{p+1-q} \frac{(\beta\mu^\mu)^{-s}}{\Gamma(1-\mu s-\alpha-N)} h_2(s) - \sum_0^{N-1} (-1)^j A_j \frac{\Gamma(1-\mu s-\alpha-j)}{\Gamma(1-\mu s-\alpha-N)} \tag{3.32}$$

if (2.23) *holds.* $(\beta\mu^\mu)^{-s}$ *has the principal value.*

Then $\rho_N(s)$ *is analytic in* s *in the domain where* (2.23) *holds and further*

$$\rho_N(s) = O(1) \tag{3.33}$$

for $|s| \to \infty$ *uniformly on*

$$|\arg(-s)| \leqq \pi-\varepsilon. \tag{3.34}$$

(3.32) *can be written also as*

$$h_2(s) = (2\pi)^{q-p-1}(\beta\mu^\mu)^s \cdot \left\{\sum_0^{N-1} (-1)^j A_j \Gamma(1-\mu s-\alpha-j)+\rho_N(s)\Gamma(1-\mu s-\alpha-N)\right\} \tag{3.35}$$

for (2.23) *and*

$$s \neq (1-\alpha-N+\nu)/\mu \qquad (\nu = 0, 1, 2, \ldots). \tag{3.36}$$

PROOF: From (3.32) and (2.24) we easily derive that $\rho_N(s)$ is analytic in s in the domain where (2.23) holds.

To prove (3.33) we apply lemma 1 with

$$P(s) = h_2(-s)/\Gamma(\mu s+1-\alpha) = \prod_1^q \Gamma(\beta_j s+b_j)/\left\{\Gamma(\mu s+1-\alpha) \prod_1^p \Gamma(\alpha_j s+a_j)\right\}, \tag{3.37}$$

$g = q$, $h = p+1$, $\rho_j = \beta_j$ and $c_j = b_j$ $(j = 1, \ldots, q)$, $\sigma_j = \alpha_j$ and $d_j = a_j$ $(j = 1, \ldots, p)$, $\sigma_h = \mu$, $d_h = 1-\alpha$, $\lambda = -\mu$, $c = \alpha$ and ε the same constant as in lemma 3. Then (3.5) holds by (1.8), $e^{-a} = \beta\mu^\mu$ by (3.8) and (1.10), $b = 0$ by (3.9) and (3.24). So (3.10) implies

$$\frac{h_2(-s)}{\Gamma(\mu s+1-\alpha)} = (\beta\mu^\mu)^{-s} \left\{\sum_0^{N-1} \frac{E_j}{(-\mu s+\alpha)_j} + O\left(\frac{1}{(-\mu s+\alpha)_N}\right)\right\} \tag{3.38}$$

for $|s| \to \infty$ uniformly on (3.2). The numbers E_j are independent of s.

We easily verify that

$$(3.39)\quad (-\mu s+\alpha)_j = (-1)^j\Gamma(\mu s+1-\alpha)/\Gamma(\mu s+1-\alpha-j) \qquad (j = 0, 1, \ldots)$$

where the last two gamma-functions exist if (3.2) holds and $|s|$ is sufficiently large. For sufficiently large $|s|$ and (3.2) we substitute the righthand side of (3.39) for the terms $(-\mu s+\alpha)_j$ in (3.38) and next we multiply both sides of the resulting formula by $\Gamma(\mu s+1-\alpha)$. Finally we replace s by $-s$ and obtain

$$(3.40)\quad h_2(s) = (\beta\mu^\mu)^s \cdot\Big\{\sum_0^{N-1}(-1)^jE_j\Gamma(1-\mu s-\alpha-j)+O(\Gamma(1-\mu s-\alpha-N))\Big\}$$

for $|s| \to \infty$ uniformly on (3.34).

For the connection between the numbers E_j in (3.40) and the numbers A_j in (3.25) we now consider only values of s with $\arg s = \pi/2$, so $\arg(-s) = -\pi/2$. Using (2.6) we may derive from (2.10) and (2.24) that

$$(3.41)\quad h_0(s) = \pi^{p-q}h_2(s)\prod_1^q \sin\pi(b_j-\beta_j s)/\prod_1^p \sin\pi(a_j-\alpha_j s)$$

for sufficiently large $|s|$ and $\arg s = \pi/2$. So by (3.40)

$$(3.42)\quad h_0(s) = \prod_1^q \sin\pi(b_j-\beta_j s) \cdot\Big\{\sin\pi(\mu s+\alpha)\prod_1^p \sin\pi(a_j-\alpha_j s)\Big\}^{-1}\pi^{p-q+1} \cdot(\beta\mu^\mu)^s\Big\{\sum_0^{N-1}E_j/\Gamma(\mu s+\alpha+j)+O(1/\Gamma(\mu s+\alpha+N))\Big\}$$

for $|s| \to \infty$ and $\arg s = \pi/2$.

To the sine-factors in (3.42) we apply $\sin z = e^{iz}(1-e^{-2iz})/2i$. Then we obtain in view of (1.8)

$$(3.43)\quad \prod_1^q \sin\pi(b_j-\beta_j s)\Big\{\sin\pi(-\mu s-\alpha)\prod_1^p \sin\pi(a_j-\alpha_j s)\Big\}^{-1} = (2i)^{p+1-q}\exp\Big[\pi i\Big(\sum_1^q b_j-\sum_1^p a_j+\alpha\Big)\Big]\cdot\prod_1^q\big(1-e^{2\pi i(\beta_j s-b_j)}\big) \cdot\big(1-e^{2\pi i(\mu s+\alpha)}\big)^{-1}\prod_1^p\big(1-e^{2\pi i(\alpha_j s-a_j)}\big)^{-1},$$

if $|s|$ is sufficiently large and $\arg s = \pi/2$. Now for $|s| \to \infty$ and $\arg s = \pi/2$ we have

$$1-\exp 2\pi i(\beta_j s-b_j) = 1+O(\exp-2\pi\beta_j|s|) \qquad (j = 1, \ldots, q)$$
$$1-\exp 2\pi i(\alpha_j s-a_j) = 1+O(\exp-2\pi\alpha_j|s|) \qquad (j = 1, \ldots, p)$$

and so as $(1+w)^{-1} = 1+O(w)$ for $w \to 0$ and because α_j and μ are positive:

$$(1-\exp 2\pi i(\alpha_j s-a_j))^{-1} = 1+O(\exp -2\pi\alpha_j |s|) \qquad (j = 1, \ldots, p)$$
$$(1-\exp 2\pi i(\mu s+\alpha))^{-1} = 1+O(\exp -2\pi\mu |s|)$$

for $|s| \to \infty$ and $\arg s = \pi/2$. We use these relations and (3.24) in (3.43). Then we obtain for $|s| \to \infty$ and $\arg s = \pi/2$:

$$(3.44) \quad \prod_1^q \sin \pi(b_j-\beta_j s)\{\sin \pi(-\mu s-\alpha) \prod_1^p \sin \pi(a_j-\alpha_j s)\}^{-1}$$
$$= -2^{p+1-q} \prod_1^q (1+O(e^{-2\pi\beta_j|s|})) \prod_1^p (1+O(e^{-2\pi\alpha_j|s|}))$$
$$\cdot (1+O(e^{-2\pi\mu|s|})) = -2^{p+1-q}(1+O(e^{-k|s|}))$$

where k is the minimum of the numbers $2\pi\mu$, $2\pi\alpha_j (j = 1, \ldots, p)$ and $2\pi\beta_j$ $(j = 1, \ldots, q)$. So k is positive.

From (3.42) and (3.44) we derive

$$(3.45) \quad h_0(s) = (2\pi)^{p-q+1}(\beta\mu^\mu)^s(1+O(e^{-k|s|}))\{\sum_0^{N-1} E_j/\Gamma(\mu s+\alpha+j)$$
$$+ O(1/\Gamma(\mu s+\alpha+N))\}$$

for $|s| \to \infty$ and $\arg s = \pi/2$.

If $j = 0, 1, \ldots, N-1$ then for $|s| \to \infty$ and $\arg s = \pi/2$

$$O(e^{-k|s|})/\Gamma(\mu s+\alpha+j) = O(e^{-k|s|}) \frac{(\mu s+\alpha+j) \ldots (\mu s+\alpha+N-1)}{\Gamma(\mu s+\alpha+N)}$$
$$= O(1/\Gamma(\mu s+\alpha+N)).$$

Hence (3.45) may be reduced to

$$(3.46) \quad h_0(s) = (2\pi)^{p-q+1}(\beta\mu^\mu)^s$$
$$\{\sum_0^{N-1} E_j/\Gamma(\mu s+\alpha+j)+O(1)/\Gamma(\mu s+\alpha+N)\}$$

for $|s| \to \infty$ and $\arg s = \pi/2$. However, for $|s| \to \infty$ and $\arg s = \pi/2$ according to lemma 2 also (3.25) and (3.27) hold. A comparison with (3.46) shows that $(2\pi)^{p-q+1}E_j = A_j$ for $j = 0, \ldots, N-1$. From (3.40) now (3.35) with $\rho_N(s)\Gamma(1-\mu s-\alpha-N)$ replaced by $O(\Gamma(1-\mu s-\alpha-N))$ follows for $|s| \to \infty$ uniformly on (3.34). This is equivalent to (3.32) and (3.33) for $|s| \to \infty$ uniformly on (3.34).

§ 4. Some lemmas and definitions

§ 4.1. First a lemma will be proved which gives generalized Fourier expansions for the function $h_1(s)$ defined by (2.11) if (2.9) holds.

LEMMA 4.

We use the notation of § 1.1. $h_1(s)$ *is defined by* (2.11) *if* (2.9) *holds.* l *is defined by* (2.3).

ASSERTIONS: *For* $\operatorname{Im} s \geqq l$ *we have*

$$(4.1) \qquad h_1(s) = c_0 e^{i\gamma_0 s} \prod_{n+1}^{p} (1-e^{2\pi i(\alpha_j s-a_j)}) \prod_{1}^{m} \sum_{\nu=0}^{\infty} e^{2\nu\pi i(\beta_j s-b_j)}$$

while for $\operatorname{Im} s \leqq -l$

$$(4.2) \qquad h_1(s) = d_0 e^{-i\gamma_0 s} \prod_{n+1}^{p} (1-e^{-2\pi i(\alpha_j s-a_j)}) \prod_{1}^{m} \sum_{\nu=0}^{\infty} e^{-2\nu\pi i(\beta_j s-b_j)}$$

where

$$(4.3) \qquad \begin{cases} \gamma_0 = (\sum_{1}^{m} \beta_j - \sum_{n+1}^{p} \alpha_j)\pi = \delta_0 \quad (cf.\ (2.12)), \\ c_0 \;= (2\pi i)^{m+n-p} \exp (\sum_{n+1}^{p} a_j - \sum_{1}^{m} b_j)\pi i, \\ d_0 = (-2\pi i)^{m+n-p} \exp (\sum_{1}^{m} b_j - \sum_{n+1}^{p} a_j)\pi i. \end{cases}$$

After carrying out the multiplications and a regrouping of the terms in the rtghthand side of (4.1) *resp.* (4.2) *these formulae may be written as*

$$(4.4) \qquad h_1(s) = \sum_{0}^{\infty} c_j e^{i\gamma_j s} \text{ resp. } h_1(s) = \sum_{0}^{\infty} d_j e^{-i\gamma_j s}$$

for $\operatorname{Im} s \geqq l$ *resp.* $\operatorname{Im} s \leqq -l$ *where the series are absolutely convergent. Here* $\{\gamma_j\}$ $(j = 0, 1, 2, \ldots)$ *is an increasing sequence of real numbers independent of* s; c_j *and* d_j $(j = 0, 1, \ldots)$ *are complex numbers independent of* s. γ_0, c_0 *and* d_0 *are given by* (4.3). *The numbers* $(\gamma_j-\gamma_0)/2\pi$ $(j = 1, 2, \ldots)$ *are linear combinations of the numbers* $\alpha_{n+1}, \ldots, \alpha_p, \beta_1, \ldots, \beta_m$ *with non-negative integral coefficients.*

PROOF: In (2.11) we apply $\sin z = e^{iz}(1-e^{-2iz})/2i$. Then we obtain

$$(4.5) \quad h_1(s) = c_0 e^{i\gamma_0 s} \prod_{n+1}^{p} (1-e^{2\pi i(\alpha_j s-a_j)}) \prod_{1}^{m} (1-e^{2\pi i(\beta_j s-b_j)})^{-1},$$

if c_0 and γ_0 are given by (4.3).

For $j = 1, \ldots, m$ and $\operatorname{Im} s \geqq l$ we have on account of (2.3) and because β_j is positive:

$$\operatorname{Re}\{2\pi i(\beta_j s - b_j)\} = 2\pi\beta_j \operatorname{Im}(-s + b_j/\beta_j) \leqq 2\pi\beta_j(-l + \operatorname{Im} b_j/\beta_j) < 0$$

and so $|\exp 2\pi i(\beta_j s - b_j)| < 1$. Hence

$$(1 - e^{2\pi i(\beta_j s - b_j)})^{-1} = \sum_{\nu=0}^{\infty} e^{2\nu\pi i(\beta_j s - b_j)}$$

for $j = 1, \ldots, m$ and $\operatorname{Im} s \geqq l$ where the series in the righthand side is absolutely convergent. Using this formula with (4.5) we obtain (4.1) for $\operatorname{Im} s \geqq l$.

Next we carry out the multiplications in the righthand side of (4.1) and then we regroup the terms such that they are finally written down in order of increasing powers of exp (is). It will be clear that the first term then becomes $c_0 \exp(i\gamma_0 s)$ for α_j and β_j are positive. Thus we get the first part of (4.4). These operations on the series in (4.1) are legitimate: for a product of absolutely convergent series remains absolutely convergent after carrying out the multiplications; also after a regrouping of the terms in this series the resulting series is absolutely convergent. So the series in the first part of (4.4) is absolutely convergent for $\operatorname{Im} s \geqq l$.

(4.2) and the second part of (4.4) can be obtained by using $\sin z = -e^{-iz}(1 - e^{2iz})/2i$ in (2.11). Then we get

$$(4.6) \quad h_1(s) = d_0 e^{-i\gamma_0 s} \prod_{n+1}^{p} (1 - e^{-2\pi i(\alpha_j s - a_j)}) \prod_{1}^{m} (1 - e^{-2\pi i(\beta_j s - b_j)})^{-1},$$

with d_0 and γ_0 given by (4.3). Next we use

$$(1 - e^{-2\pi i(\beta_j s - b_j)})^{-1} = \sum_{\nu=0}^{\infty} e^{-2\nu\pi i(\beta_j s - b_j)}$$

for $\operatorname{Im} s \leqq -l$ and $j = 1, \ldots, m$. Application of this formula in (4.6) leads to (4.2) for $\operatorname{Im} s \leqq -l$. From (4.2) we may derive the second part of (4.4) for $\operatorname{Im} s \leqq -l$. The details of the proofs are omitted as they are similar to those in the preceding case with (4.1) and the first part of (4.4). That in the second part of (4.4) terms $\exp(-i\gamma_j s)$ occur will be clear from a comparison of (4.1) and (4.2). Further we see that $\gamma_j - \gamma_0$ $(j = 1, 2, \ldots)$ is a linear combination of the numbers $\alpha_{n+1}, \ldots, \alpha_p, \beta_1, \ldots, \beta_m$ with non-negative integral coefficients.

§ **4.2.** In the formulation of the theorems about $H(z)$ we shall use numbers which are derived from the γ_j, c_j and d_j of lemma 4. They are given by the following two definitions:

DEFINITION I: *$\{\delta_j\}$ $(j = 0, \pm 1, \pm 2, \ldots)$ is the monotonic increasing sequence which arises if we write down the set of numbers γ_g and $-\gamma_h$ $(g, h = 0, 1, \ldots)$ in order of increasing magnitude such that if there are contingently two equal numbers in this set we only write down this number once, while further $\delta_0 = \gamma_0$ (cf. (4.3)). (So in this sequence $\delta_j < \delta_h$ if $j < h$).*

For an alternative definition see § 10.3.

If r is an arbitrary integer we may distinguish three different cases for δ_r:

a) there exists a non-negative integer g such that $\delta_r = \gamma_g$ while $\delta_r \neq -\gamma_j$ for $j = 0, 1, \ldots$.

b) there exists a non-negative integer h such that $\delta_r = -\gamma_h$ while $\delta_r \neq \gamma_j$ for $j = 0, 1, \ldots$.

c) there exist two non-negative integers g and h such that $\delta_r = \gamma_g = -\gamma_h$.

DEFINITION II: *With the preceding notation we define the integer κ by $\delta_\kappa = -\gamma_0 = -\delta_0$. Further if r is an arbitrary integer we define in case* a): *$C_r = c_g$, $D_r = 0$; in case* b): *$C_r = 0$, $D_r = -d_h$; in case* c): *$C_r = c_g$, $D_r = -d_h$.*

§ **4.3.** Using the preceding definitions and lemmas we derive:

LEMMA 4a.

We use the notation of § 1, (2.3), (2.11) *and the definitions* I *and* II. *r is an integer.*

ASSERTIONS: *In the first place*

$$\kappa \geqq -1,\ \kappa = -1 \text{ if } \delta_0 > 0,\ C_r = 0 \text{ if } r < 0,\ D_r = 0 \text{ if } r > \kappa. \tag{4.7}$$

Further there exists a positive number K independent of s so that [1])

$$|(h_1(s) + \sum_r^{\kappa} D_j e^{i\delta_j s} - \sum_0^{r-1} C_j e^{i\delta_j s}) e^{-i\delta_r s}| \leqq K \tag{4.8}$$

for $\operatorname{Im} s \geqq l$ *while for* $\operatorname{Im} s \leqq -l$:

$$|(h_1(s) + \sum_r^{\kappa} D_j e^{i\delta_j s} - \sum_0^{r-1} C_j e^{i\delta_j s}) e^{-i\delta_{r-1} s}| \leqq K. \tag{4.9}$$

PROOF: Suppose first $-\gamma_0 \geqq \gamma_0$. Then $\delta_\kappa \geqq \delta_0$, $\delta_0 \leqq 0$ and so $\kappa \geqq 0$. Suppose next $-\gamma_0 < \gamma_0$. Then $-\gamma_j \leqq -\gamma_0 < \gamma_0 \leqq \gamma_g$ for $j, g = 0, 1, \ldots$ since $\{\gamma_j\}$ $(j = 0, 1, \ldots)$ is monotonic in-

[1]) In the special case $\delta_0 > 0$, $\kappa = -1$, $r = 1$ the formulae (4.8) and (4.9) are related to the second amalgamation lemma in C. Fox [18] p. 419. Cf. the beginning of § 3 for the notation.

creasing by lemma 4. From the construction of the δ_j it follows now that $\delta_{-1} = -\gamma_0$, $\delta_0 > 0$ and $\kappa = -1$. Hence in all cases $\kappa \geqq -1$.

If $r < 0$ then, since $\delta_0 = \gamma_0$, $\delta_r < \gamma_0 \leqq \gamma_j (j = 0, 1, \ldots)$ and so $\delta_r \neq \gamma_j$ $(j = 0, 1, \ldots)$. So case a) and case c) of § 4.2 cannot occur and therefore $C_r = 0$. In the same way one may prove $D_r = 0$ for $r > \kappa$. Thus (4.7) follows.

Next we may consider (4.8) and (4.9). From (4.4), definition II and (4.7) it follows that

$$
h_1(s) = \sum_0^\infty C_j e^{i\delta_j s} \text{ resp. } h_1(s) = -\sum_{-\infty}^{\kappa} D_j e^{i\delta_j s} \tag{4.10}
$$

for $\operatorname{Im} s \geqq l$ resp. $\operatorname{Im} s \leqq -l$. The first resp. second series in (4.10) is absolutely convergent for $\operatorname{Im} s \geqq l$ resp. $\operatorname{Im} s \leqq -l$. Thus in view of (4.10) and (4.7) we have for $\operatorname{Im} s \geqq l$:

$$
\begin{aligned}
&|(h_1(s) + \sum_r^\kappa D_j e^{i\delta_j s} - \sum_0^{r-1} C_j e^{i\delta_j s})e^{-i\delta_r s}| \\
&= |(\sum_r^\kappa D_j e^{i\delta_j s} + \sum_r^\infty C_j e^{i\delta_j s})e^{-i\delta_r s}| \leqq \sum_r^\kappa |D_j| e^{(\delta_r - \delta_j)\operatorname{Im} s} \\
&\quad + \sum_r^\infty |C_j| e^{(\delta_r - \delta_j)\operatorname{Im} s} \leqq \sum_r^\kappa |D_j| e^{l(\delta_r - \delta_j)} + \sum_r^\infty |C_j| e^{l(\delta_r - \delta_j)}.
\end{aligned}
$$

This shows that there exists a number K independent of s such that (4.8) holds for $\operatorname{Im} s \geqq l$.

In the same way using the second part of (4.10) we may deduce (4.9) for $\operatorname{Im} s \leqq -l$ with a suitable K independent of s. We choose K so large that in (4.8) and (4.9) the same K can be used.

§ 4.4. In the formulation of theorems about the asymptotic behaviour of $H(z)$ we use:

DEFINITION III: *Suppose N is a non-negative integer and μ — defined by* (1.8) — *is positive. Let β resp. α, A_0, $A_1, \ldots$ be defined by* (1.10) *resp. in lemma* 2 *in* § 3.2. *Ω will be a subset of the Riemann surface of log z containing some subsector of this surface. c resp. γ will be a complex resp. real constant. Finally if $d > 0$ and $z \neq 0$ then*

$$
(dze^{i\gamma})^\delta = \exp \delta \{\operatorname{Log} |dz| + i (\arg z + \gamma)\}. \tag{4.11}
$$

Then functions which are equal to

$$
\frac{1}{2\pi i \mu}\{c \sum_0^{N-1} A_j (\beta\mu^\mu z e^{i\gamma})^{(1-\alpha-j)/\mu} + O(z^{(1-\alpha-N)/\mu})\} \exp (\beta\mu^\mu z e^{i\gamma})^{1/\mu} \tag{4.12}
$$

for $|z| \to \infty$ *uniformly on* Ω *will be abbreviated by* $cE_N(ze^{i\gamma})$; *stated more precisely*:

If an equality holds for $|z| \to \infty$ *uniformly on* Ω *and this equality contains the expression* (4.12) *then by definition this equality in which the expression* (4.12) *is replaced by* $cE_N(ze^{i\gamma})$ *holds for* $|z| \to \infty$ *uniformly on* Ω *and conversely.*

Further we define $E(z)$ *by a formal series*:

$$E(z) = \frac{1}{2\pi i\mu}\sum_0^\infty A_j(\beta\mu^\mu z)^{(1-\alpha-j)/\mu} \exp{(\beta\mu^\mu z)^{1/\mu}}. \tag{4.13}$$

In addition to definition III we say that if $\varphi(z)$ is an asymptotic series, h is a non-negative integer, $l_0, \ldots, l_h$ resp. $s_0, \ldots, s_h$ are complex resp. real constants then the assertion

$$f(z) \sim \sum_0^h l_j E(ze^{is_j}) + \varphi(z)$$

for $|z| \to \infty$ uniformly on Ω means: $f(z)$ is defined for the points z on Ω with sufficiently large $|z|$ and there exists a function $\psi(z)$ also defined for the points z on Ω with sufficiently large $|z|$ so that $\psi(z) \sim \varphi(z)$ for $|z| \to \infty$ uniformly on Ω and so that for every non-negative integer N

$$f(z) = \sum_0^h l_j E_N(ze^{is_j}) + \psi(z)$$

for $|z| \to \infty$ uniformly on Ω. Of course it is sufficient that the last relation holds for arbitrary large N.

The assertion

$$f(z) \sim \sum_0^h l_j E(ze^{is_j}) \tag{4.14}$$

for $|z| \to \infty$ uniformly on Ω means that $f(z)$ is defined for the points z on Ω with sufficiently large $|z|$ and that for arbitrary non-negative integers N

$$f(z) = \sum_0^h l_j E_N(ze^{is_j})$$

for $|z| \to \infty$ uniformly on Ω. Of course it is sufficient that the last relation holds for sufficiently large integers N. If (4.14) holds for $|z| \to \infty$ uniformly on Ω and moreover $|\arg z + s_j| \leqq \frac{1}{2}\mu\pi - \varepsilon$ for z on Ω, $j = 0, \ldots, h$ and ε a constant with $0 < \varepsilon < \frac{1}{2}\mu\pi$ then we say that $f(z)$ has an *exponentially infinite asymptotic expansion* for $|z| \to \infty$ on Ω: The terms in the formal series for $E(z \exp is_j)$ (cf. (4.13)) contain the factor $\exp{(\beta\mu^\mu ze^{is_j})^{1/\mu}}$ which tends to infinity for $|z| \to \infty$ uniformly on Ω because

$\mathrm{Re}(\beta\mu^\mu z e^{is_j})^{1/\mu} = |\beta\mu^\mu z|^{1/\mu} \cos\{(\arg z+s_j)/\mu\} \geqq |\beta\mu^\mu z|^{1/\mu} \sin(\varepsilon/\mu)$.

If (4.14) holds for $|z| \to \infty$ uniformly on Ω and

$$\tfrac{1}{2}\mu\pi+\varepsilon \leqq |\arg z+s_j| \leqq \tfrac{3}{2}\mu\pi-\varepsilon \text{ for } z \text{ on } \Omega, \qquad j = 0, \ldots, h$$

and ε a constant such that $0 < \varepsilon < \frac{1}{2}\mu\pi$ then we say that $f(z)$ has an *exponentially small asymptotic expansion* for $|z| \to \infty$ on Ω: The terms of the formal series for $E(z \exp is_j)$ (cf. (4.13)) contain the factor $\exp(\beta\mu^\mu z e^{is_j})^{1/\mu}$ which tends to zero for $|z| \to \infty$ uniformly on Ω.

§ **4.5.** Here two lemmas concerning the function $E_N(z)$ defined above will be proved.

LEMMA 5.

N, μ and Ω satisfy the assumptions of definition III. *ε is a constant such that* $0 < \varepsilon < 2\mu\pi$, *$k_1$ and k_2 are complex constants and t_1 and t_2 are real constants so that for z on Ω:*

$$|\arg z+t_1| \leqq |\arg z+t_2|-\varepsilon \tag{4.15}$$

and

$$|\arg z+t_1|+|\arg z+t_2| \leqq 2\mu\pi-\varepsilon. \tag{4.16}$$

Suppose finally that

$$f(z) = k_1 E_N(ze^{it_1})+k_2 E_N(ze^{it_2}) \tag{4.17}$$

for $|z| \to \infty$ uniformly on Ω.

Then also for $|z| \to \infty$ uniformly on Ω

$$f(z) = k_1 E_N(ze^{it_1}). \tag{4.18}$$

PROOF: For $h = 1, 2$ we put $\varphi_h = \arg z+t_h$. If z belongs to Ω, then on account of (4.15) and (4.16):

$$|\varphi_1| \leqq |\varphi_2|-\varepsilon \quad \text{and} \quad |\varphi_1|+|\varphi_2| \leqq 2\mu\pi-\varepsilon, \quad \text{hence}$$
$$\varepsilon \leqq |\varphi_2|-|\varphi_1| \leqq 2\mu\pi-\varepsilon,\ \varepsilon \leqq |\varphi_1|+|\varphi_2| \leqq 2\mu\pi-\varepsilon.$$

From this and

$$\cos(\varphi_2/\mu)-\cos(\varphi_1/\mu) = -2\sin\{(|\varphi_2|-|\varphi_1|)/2\mu\}\sin\{(|\varphi_2|+|\varphi_1|)/2\mu\}$$

we deduce that for z on Ω:

$$\cos(\varphi_2/\mu)-\cos(\varphi_1/\mu) \leqq -2\sin^2(\varepsilon/2\mu) < 0.$$

This last formula implies that for $j = 0, \ldots, N$; z on Ω and $|z| \geqq 1$:

$$(4.19)\quad \begin{aligned} &|(ze^{it_2})^{(1-\alpha-j)/\mu} \exp (\beta\mu^{\mu} ze^{it_2})^{1/\mu} z^{-(1-\alpha-N)/\mu} \exp -(\beta\mu^{\mu} ze^{it_1})^{1/\mu}| \\ &\leqq |z|^{(N-j)/\mu} \exp (t_2 \operatorname{Im} \alpha/\mu) \\ &\cdot \exp \{(\beta\mu^{\mu}|z|)^{1/\mu} (\cos (\varphi_2/\mu) - \cos (\varphi_1/\mu))\} \\ &\leqq |z|^{N/\mu} \exp \{t_2 \operatorname{Im} \alpha/\mu - 2(\beta\mu^{\mu}|z|)^{1/\mu} \sin^2(\varepsilon/2\mu)\}. \end{aligned}$$

The last side of (4.19) tends to zero for $|z| \to \infty$ and so the lefthand side of (4.19) is $o(1)$ for $|z| \to \infty$ uniformly on Ω. Hence for $j = 0, \ldots, N$:

$$(ze^{it_2})^{(1-\alpha-j)/\mu} \exp (\beta\mu^{\mu} ze^{it_2})^{1/\mu} = O(z^{(1-\alpha-N)/\mu}) \exp (\beta\mu^{\mu} ze^{it_1})^{1/\mu}$$

for $|z| \to \infty$ uniformly on Ω. From this and the definition for $E_N(z)$ by means of (4.12) we easily derive

$$k_2 E_N(ze^{it_2}) = O(z^{(1-\alpha-N)/\mu}) \exp (\beta\mu^{\mu} ze^{it_1})^{1/\mu}$$

for $|z| \to \infty$ uniformly on Ω. This property, the definition of $E_N(z)$ and (4.17) imply (4.18) for $|z| \to \infty$ uniformly on Ω.

LEMMA 5a.

N, μ and Ω satisfy the assumptions of definition III. *w is a real number satisfying* (2.1) *and* (2.2). *$Q_w(z)$ and $P_w(z)$ are defined by* (2.5) *and* (2.17). *h is an integer, $h \geqq -1$, t, $s_0, \ldots, s_h$ resp. $l_0, \ldots, l_h$ k_1, k_2, k_3 are real resp. complex constants, ε, t_1, t_2 are real constants so that for z on Ω*:

$$(4.20)\qquad |\arg z + t_1| \leqq \tfrac{1}{2}\mu\pi - \varepsilon,$$

$$(4.21)\qquad |\arg z + t_1| + \varepsilon \leqq |\arg z + t_2| \leqq \mu\pi.$$

Suppose

$$(4.22)\qquad \begin{aligned} f(z) &= k_1 E_N(ze^{it_1}) + k_2 E_N(ze^{it_2}) + k_3 Q_w(z) \\ &\quad + \sum_0^h l_j P_w(ze^{is_j}) + O(z^t) \end{aligned}$$

for $|z| \to \infty$ uniformly on Ω.

Then also (4.18) *holds for $|z| \to \infty$ uniformly on Ω.*

If in (4.22) *the term $k_2 E_N(z \exp it_2)$ is omitted and if the condition* (4.21) *is deleted then the assertion* (4.18) *remains valid.*

PROOF: If z belongs to Ω then (4.15) and (4.16) are fulfilled according to (4.20) and (4.21). Hence by lemma 5:

$$(4.23)\quad f(z) = k_1 E_N(ze^{it_1}) + k_3 Q_w(z) + \sum_0^h l_j P_w(ze^{is_j}) + O(z^t)$$

for $|z| \to \infty$ uniformly on Ω. Further there exists a positive constant K independent of z such that if z belongs to Ω then

$$(4.24)\quad |(ze^{it_1})^{(1-\alpha-N)/\mu} \exp\,(\beta\mu^\mu ze^{it_1})^{1/\mu}| \geqq K|z|^{(1-\operatorname{Re}\alpha-N)/\mu} \cdot \exp\,\{(\beta\mu^\mu|z|)^{1/\mu} \sin\,(\varepsilon/\mu)\},$$

on account of (4.20). From the definitions of $Q_w(z)$ and $P_w(z)$ in (2.5) and (2.17) we easily deduce that there exists a real number k independent of z such that $Q_w(z) = O(z^k)$, $P_w(ze^{is_j}) = O(z^k)$ for $|z| \to \infty$ uniformly on Ω and for $j = 0, \ldots, h$. Hence by (4.24)

$$(4.25)\quad k_3 Q_w(z) + \sum_0^h l_j P_w(ze^{is_j}) + O(z^t) = O(1) z^{(1-\alpha-N)/\mu} \cdot \exp\,(\beta\mu^\mu ze^{it_1})^{1/\mu}$$

for $|z| \to \infty$ uniformly on Ω. From (4.25), (4.23) and the definition of $E_N(z \exp it_1)$ (cf. (4.12)) we deduce (4.18) for $|z| \to \infty$ uniformly on Ω.

If the term $k_2 E_N(z \exp it_2)$ in (4.22) does not appear and condition (4.21) is deleted then (4.23) holds for $|z| \to \infty$ uniformly on Ω and (4.18) can be deduced from (4.23) in the same way as above.

§ 4.6. Finally we use in the theorems about $H(z)$:

DEFINITION IV: *$Q(z)$ resp. $P(z)$ is defined by a formal series:*

$$(4.26)\quad Q(z) = \sum \textit{residues of } h(s)z^s \textit{ in the points } (1.7),$$

$$(4.27)\quad P(z) = \sum \textit{residues of } h_0(s)z^s \textit{ in the points}$$

$$(4.28)\quad s = (a_j - 1 - \nu)/\alpha_j \qquad (j = 1, \ldots, p;\ \nu = 0, 1, \ldots).$$

Here $h_0(s)$ and $h(s)$ are defined by (2.10) *and* (1.3).

If Ω is a sector on the Riemann surface of $\log z$, h is an integer with $h \geqq -1$ and $l_0, \ldots, l_h$ resp. $s_0, \ldots, s_h$ are complex resp. real constants then

$$(4.29)\quad f(z) \sim Q(z) + \sum_0^h l_j P(ze^{is_j})$$

for $|z| \to \infty$ uniformly on Ω means that for every real number w satisfying (2.1) and (2.2)

$$(4.30)\quad f(z) = Q_w(z) + \sum_0^h l_j P_w(ze^{is_j}) + O(z^w)$$

for $|z| \to \infty$ uniformly on Ω (cf. (2.5) and (2.17)). It will be clear that it is sufficient to require (4.30) for a sequence of numbers w tending to $-\infty$ while moreover the term $O(z^w)$ in (4.30) may be replaced by $O(z^{w+t})$ where t is a constant independent of z and of w.

The terms in the series in (4.26) and (4.27) are of the form cz^k with c and k independent of z save in the case that there are multiple poles among the points (1.7) or (4.28); then there occur terms of the form cz^k times a polynomial in $\log z$.

An expansion of the type (4.29) is called an *algebraic asymptotic expansion* (so also in the case that the formal series in (4.26) and (4.27) contain logarithmic terms).

§ 5. Estimates for some classes of integrals

This paragraph contains some lemmas which give with some extensions the estimates which Barnes used for the determination of the analytic continuations and algebraic asymptotic expansions in the simplest cases of the G-function and the generalized hypergeometric function (cf. § 2.1 and [3], [4]). The proofs we give of these lemmas are similar to those of Barnes.

In the following z^s is always defined by (1.2) for $z \neq 0$ where $\arg z$ is not necessarily the principal value. If we put any condition on $\arg z$ we always exclude $z = 0$. We always assume that the path of integration in integrals of the type $\int_w^{w+i\infty}$, $\int_w^{w-i\infty}$ and $\int_{w-i\infty}^{w+i\infty}$ are rectilinear. Finally we use the notation $||$ which means replaced by.

§ 5.1. The first lemma reads

LEMMA 6.

Let w, l and σ be real constants with $l \geqq 0$. L will be the contour in the complex s-plane from $s = w$ to $s = w+il$ and then to $s = il+\infty$ where the parts of L between these points are rectilinear. w, l and L are not necessarily the same as in § 2. S is the set of points s for which $\operatorname{Re} s \geqq w$ *as well as* $\operatorname{Im} s \geqq l$.

Suppose $f(s)$ is defined and continuous on L and on S, $f(s)$ is analytic in the interior of S and

$$f(s) = O(s^\sigma) \tag{5.1}$$

for $|s| \to \infty$ uniformly on S.

Then the integral

$$\int_L f(s) z^s ds \tag{5.2}$$

resp.

$$(5.3) \qquad \int_w^{w+i\infty} f(s)z^s\,ds$$

is absolutely convergent and represents an analytic function of z *on the Riemann surface of* $\log z$ *for* $0 < |z| < 1$ *resp.* $\arg z > 0$. *The function in* (5.3) *resp.* (5.2) *is the analytic continuation of the function in* (5.2) *resp.* (5.3) *for* $\arg z > 0$, $|z| \geqq 1$ *resp.* $\arg z \leqq 0$, $0 < |z| < 1$.

Corresponding to an arbitrary positive constant ε *there exists a positive constant* K *independent of* z *so that* [1])

$$(5.4) \qquad \int_w^{w+i\infty} |f(s)z^s||ds| \leqq K\,|z|^w$$

for

$$(5.5) \qquad \arg z \geqq \varepsilon.$$

In particular one may choose $K = \int_w^{w+i\infty}|f(s)||e^{-\varepsilon \operatorname{Im} s}|ds|$ *(this integral is convergent).*

If $\sigma < -1$ *then the integral* (5.2) *resp.* (5.3) *converges absolutely and is continuous in* z *for* $0 < |z| \leqq 1$ *resp.*

$$(5.6) \qquad \arg z \geqq 0.$$

If $\sigma < -1$ *then corresponding to an arbitrary positive constant* k_0 *there exists a constant* K_0 *independent of* z *so that* (5.4) *with* $K||K_0$ *holds for* (5.6) *and so that*

$$(5.7) \qquad \int_L |f(s)z^s||ds| \leqq K_0|z|^w$$

for

$$(5.8) \qquad 0 < |z| \leqq 1, \ |\arg z| \leqq k_0\pi.$$

Proof: By (1.2) we have for $z \neq 0$

$$(5.9) \qquad \begin{aligned} |z|^s &= |z^w \exp\{(s-w)\log z\}| \\ &= |z|^w \exp\{\operatorname{Re}(s-w)\operatorname{Log}|z| - \operatorname{Im} s \arg z\}. \end{aligned}$$

Now let D_1 be a simply-connected domain on the Riemann surface of $\log z$, such that $0 < |z| < 1$ on D_1 and $\operatorname{Log}|z|$ resp. $|\arg z|$ have negative resp. positive upper bounds on D_1 which we denote by $-M_1$ resp. M_2. So by (5.9) we have for z on D_1, $\operatorname{Im} s = l$ and $\operatorname{Re} s \geqq w$:

$$(5.10) \qquad |z^s| \leqq |z|^w \exp\{-M_1 \operatorname{Re}(s-w) + M_2 l\}.$$

From this and (5.1) it follows that the integral (5.2) converges

[1]) cf. G. Sansone–J. C. H. Gerretsen [28] p. 53 for this notation.

uniformly and absolutely in D_1. Consequently [2]) this integral is analytic in z in D_1. On account of the choice of D_1 the integral (5.1) also converges absolutely and is analytic in z for $0 < |z| < 1$ on the Riemann surface of $\log z$.

Suppose now $\sigma < -1$. Let D_2 be a closed simply-connected domain on the Riemann surface of $\log z$ such that $0 < |z| \leqq 1$ and so $\text{Log}\,|z| \leqq 0$ on D_2 and such that $|\arg z|$ has a finite upper bound, denoted by M_3, on D_2. Then by (5.9) for z on D_2, $\text{Im}\, s = l$, $\text{Re}\, s \geqq w$ again (5.10) holds with $M_1||0$, $M_2||M_3$. From this and (5.1) with $\sigma < -1$ it follows that the integral (5.2) is uniformly and absolutely convergent on D_2. Hence the integral (5.2) converges absolutely and is continuous in z for $0 < |z| \leqq 1$ on the Riemann surface of $\log z$, if $\sigma < -1$.

Further if (5.8) holds then $\text{Log}\,|z| \leqq 0$ and so by (5.9) formula (5.10) holds with $M_1||0$, $M_2||k_0\pi$ for s on L and (5.8). From this we deduce that the lefthand side of (5.7) is at most $|z|^w \exp(k_0 l\pi)$ $\int_L |f(s)||ds|$ for (5.8) and $\sigma < -1$. Here the last integral is convergent by (5.1) with $\sigma < -1$. Now (5.7) follows for a suitable K_0 independent of z and for (5.8) and $\sigma < -1$.

Now we drop the condition $\sigma < -1$. Choose a positive constant ε and let D_3 be a simply-connected domain on the Riemann surface of $\log z$ such that $0 < |z| < 1$ and (5.5) hold on D_3 and such that in D_3 $\text{Log}\,|z|$ has a negative upper bound denoted by $-M_4$. Further suppose $\text{Im}\, s \geqq 0$, $\text{Re}\, s \geqq w$ but $s \neq w$, $\theta = \arg(s-w)$ with $0 \leqq \theta \leqq \frac{1}{2}\pi$. Then by (5.9) for z in D_3:

$$(5.11) \qquad |z^s| \leqq |z|^w \exp\{-|s-w|(M_4 \cos\theta + \varepsilon \sin\theta)\}.$$

Since the function $M_4 \cos\theta + \varepsilon \sin\theta$ is positive and continuous for $0 \leqq \theta \leqq \frac{1}{2}\pi$ this function has a positive lower bound M_5 only depending on M_4 and ε for $0 \leqq \theta \leqq \frac{1}{2}\pi$. So

$$|z^s| \leqq |z|^w \exp(-M_5|s-w|)$$

for z in D_3 and $\text{Re}\, s \geqq w$, $\text{Im}\, s \geqq 0$ (now $s = w$ is allowed). From this and (5.1) we may deduce that the integral $\int f(s) z^s ds$ where the path of integration is taken along the smallest part of the circle $|s-w| = R (R > l)$ between L and the half line $\text{Re}\, s = w$, $\text{Im}\, s \geqq l$, traversed from the right to the left, tends to zero for $R \to \infty$ and z in D_3. By the theorem of Cauchy and the assumptions concerning $f(s)$ this integral is equal to the integral (5.3) minus the integral (5.2) with the modification that the contours

[2]) cf. E. T. Copson [10] p. 110.

in (5.3) and (5.2) are replaced by the parts of these contours inside the circle $|s-w| = R$. So the difference between these modified integrals tends to zero for $R \to \infty$ and any fixed z in D_3. Because for z in D_3 we have $0 < |z| < 1$ and consequently the complete integral (5.2) converges, it follows that for z in D_3 the complete integral (5.3) converges and is equal to the complete integral (5.2).

However, if we only require (5.5) for z and $\operatorname{Re} s = w$, $\operatorname{Im} s \geqq 0$ then by (5.9)

$$|z^s| \leqq |z|^w \exp(-\varepsilon \operatorname{Im} s). \tag{5.12}$$

From this and (5.1) (we do not assume $\sigma < -1$) we infer that the integral (5.3) converges absolutely for (5.5) and the convergence is uniform on every bounded closed subdomain of (5.5). So (cf. footnote 2) the integral (5.3) is analytic in z for $\arg z > \varepsilon$ and as ε is arbitrary positive also for $\arg z > 0$. Because in D_3 the integrals (5.2) and (5.3) are equal to each other and because they are analytic for $0 < |z| < 1$ resp. $\arg z > 0$ (the last two domains contain D_3), the function in (5.2) resp. (5.3) is the analytic continuation of that in (5.3) resp. (5.2) for $0 < |z| < 1$ resp. $\arg z > 0$.

From (5.12) we deduce that the lefthand side of (5.4) is at most equal to $|z|^w \int_w^{w+i\infty} |f(s)| e^{-\varepsilon \operatorname{Im} s} |ds|$ for (5.5); the last integral converges on account of (5.1) (with no extra-condition on σ). From this (5.4) follows for (5.5) and a suitable constant K independent of z, for example $K = \int_w^{w+i\infty} |f(s)| e^{-\varepsilon \operatorname{Im} s} |ds|$.

Finally suppose $\sigma < -1$ and (5.6) is satisfied. Then by (5.9) for $\operatorname{Re} s = w$, $\operatorname{Im} s \geqq 0$ formula (5.12) with $\varepsilon = 0$ holds. From this formula and (5.1) we infer that the integral (5.3) converges uniformly and absolutely on every bounded closed subdomain of (5.6) for $\sigma < -1$; so this integral is continuous in z and absolutely convergent for (5.6) and $\sigma < -1$. Further from (5.12) with $\varepsilon = 0$ we deduce that the lefthand side of (5.4) is at most equal to $\int_w^{w+i\infty} |f(s)| |ds| |z|^w$ for (5.6), where the last integral converges by (5.1) with $\sigma < -1$. This implies (5.4) for (5.6) with K replaced by a suitable constant K_0 independent of z. By choosing K_0 large enough we may take the same K_0 in (5.7) and (5.4) with $K||K_0$ in the case $\sigma < -1$ and (5.8) resp. (5.6).

§ 5.2. The second lemma of the Barnes' type is:

Lemma 7.

Let w, l, L, S and $f(s)$ be defined as in lemma 6 *only* (5.1) *is replaced by*

$$f(s) = O(1)/\Gamma(\mu s+c) \tag{5.13}$$

for $|s| \to \infty$ *uniformly on* S. *Here* μ *is a positive constant* (μ *need not satisfy* (1.8)) *and* c *is a complex constant.*

Then the integral (5.2) *resp.* (5.3) *converges absolutely and represents an analytic function of* z *on the Riemann surface of* $\log z$ *resp. on the part of this surface where* $\arg z > \frac{1}{2}\mu\pi$. *The function in* (5.2) *is the analytic continuation of the function in* (5.3) *for* $\arg z \leqq \frac{1}{2}\mu\pi$.

Corresponding to a positive constant ε *there exists a positive constant* K *independent of* z *so that* (5.4) *holds if*

$$\arg z \geqq \tfrac{1}{2}\mu\pi+\varepsilon, \tag{5.14}$$

and so the function defined originally by (5.2) *or* (5.3) *is* $O(z^w)$ *for* $|z| \to \infty$ *uniformly on* (5.14).

PROOF: Let R be an arbitrary positive constant. We want to apply lemma 6 with

$$f(s)||R^s e^{\frac{1}{2}\mu\pi i s}f(s),\ z||zR^{-1}e^{-\frac{1}{2}\mu\pi i},\ \arg z||\arg z-\tfrac{1}{2}\mu\pi. \tag{5.15}$$

Here the second $f(s)$ is the function occurring in lemma 7. We have to verify (5.1) now. For sufficiently large $|s|$ and s on S we may write

$$1/\Gamma(\mu s+c) = \exp -\operatorname{Log}\Gamma(\mu(s-w)+\mu w+c). \tag{5.16}$$

Applying the formula of Stirling (3.1) with $M = 0$, $s||\mu(s-w)$, $a||\mu w+c$ we obtain for $|s| \to \infty$ uniformly on S:

$$1/\Gamma(\mu s+c) = O(1)\,|\exp[-\{\mu(s-w)+\mu w+c-\tfrac{1}{2}\} \cdot \operatorname{Log}\{\mu(s-w)\}+\mu(s-w)]|. \tag{5.17}$$

If we put $\theta = \arg(s-w)$ with $0 \leqq \theta \leqq \frac{1}{2}\pi$ for s on S and $s \neq w$ then

$$\operatorname{Re}(s-w) = |s-w|\cos\theta,\quad \operatorname{Im} s = |s-w|\sin\theta,$$
$$\operatorname{Log}\{\mu(s-w)\} = \operatorname{Log}|\mu(s-w)|+i\theta.$$

Using this and (5.17) we deduce that for $|s| \to \infty$ uniformly on S

$$1/\Gamma(\mu s+c) = O(1)|s-w|^{\mu w+\operatorname{Re} c-\frac{1}{2}} \cdot \exp[-\mu|s-w|\{(\operatorname{Log}|\mu(s-w)|-1)\cos\theta-\theta\sin\theta\}+\theta\operatorname{Im} c]. \tag{5.18}$$

Using (5.13) and (5.15) we obtain

(5.19)

$$f(s) = O(1)R^w|s-w|^{\mu w+\mathrm{Re}\,c-\frac{1}{2}}$$

$$\exp\left[-\mu|s-w|\left\{\left(\mathrm{Log}\,|\mu(s-w)|-1-\frac{1}{\mu}\log R\right)\cos\theta+\left(\frac{\pi}{2}-\theta\right)\sin\theta\right\}\right]$$

for $|s| \to \infty$ uniformly on S. As $\mathrm{Log}\,|\mu(s-w)| \to \infty$ for $|s| \to \infty$ on S and as $0 \leqq \theta \leqq \frac{1}{2}\pi$ on S, the expression $\{\dots\}$ in (5.19) is non-negative if s belongs to S and moreover $|s|$ is sufficiently large. So in view of (5.19)

$$f(s) = O(1)|s-w|^{\mu w+\mathrm{Re}\,c-\frac{1}{2}} \tag{5.20}$$

for $|s| \to \infty$ uniformly on S. Because $|s-w|/|s|$ and $|s|/|s-w|$ are $O(1)$ for $|s| \to \infty$ uniformly on S, (5.20) implies (5.1) with $\sigma = \mu w+\mathrm{Re}\,c-\frac{1}{2}$ for $|s| \to \infty$ uniformly on S. Herewith the assumptions of lemma 6 are verified.

By lemma 6 and (5.15) the integral in (5.2) resp. (5.3) (where $f(s)$ is the function in lemma 7) converges absolutely and represents an analytic function of z on the Riemann surface of $\log z$ for $0 < |z| < R$ resp $\arg z > \frac{1}{2}\mu\pi$. As R is arbitrary positive the integral in (5.2) converges and represents an analytic function of z on the Riemann surface of $\log z$. The function in (5.2) is the analytic continuation of the function in (5.3) for $\arg z \leqq \frac{1}{2}\mu\pi$.

Finally if $\mathrm{Re}\, s = w$, $\mathrm{Im}\, s \geqq 0$ and (5.14) holds then by (5.9)

$$|z^s| \leqq |z|^w \exp\{-(\tfrac{1}{2}\mu\pi+\varepsilon)|s-w|\} \tag{5.21}$$

From (5.21) we deduce that if (5.14) holds then

$$\int_w^{w+i\infty} |f(s)z^s||ds| \leqq |z|^w \int_w^{w+i\infty} |f(s)| \exp\{-(\tfrac{1}{2}\mu\pi+\varepsilon)|s-w|\}|ds|. \tag{5.22}$$

The last integral is convergent because by (5.18) with $\theta = \frac{1}{2}\pi$ and (5.13)

$$f(s) = O(1)|s-w|^{\mu w+\mathrm{Re}\,c-\frac{1}{2}} \exp(\tfrac{1}{2}\mu\pi|s-w|).$$

From (5.22) now (5.4) follows for (5.14) and a suitable K independent of z where in (5.4) $f(s)$ is the function occurring in lemma 7. ((5.4) also follows from lemma 6 with (5.15)).

§ 5.3. The following lemmas 6a and 7a are similar to the lemmas 6 and 7. The proofs of these lemmas can be given in the same way as in § 5.1 and § 5.2. Therefore we omit the proofs. It is also possible to deduce the lemmas 6a and 7a from the lemmas 6 and 7 using complex conjugates and exercise 5 on p. 42 in L. V. Ahlfors [1].

LEMMA 6a.

Suppose l, w and σ are real numbers with $l \geqq 0$. L_1 will be the contour in the complex s-plane from $s = w$ to $s = w - il$ and then to $s = -il + \infty$ such that the parts of L_1 between these points are rectilinear. L_1, w and l are not necessarily the same as in § 2. *S_1 is the set consisting of the points s with* $\operatorname{Re} s \geqq w$, $\operatorname{Im} s \leqq -l$.

Suppose $f(s)$ is defined and continuous on S_1 and on L_1, $f(s)$ is analytic in the interior of S_1 and (5.1) *holds for $|s| \to \infty$ uniformly on S_1.*

Then the integral

$$\int_{L_1} f(s) z^s \, ds \tag{5.23}$$

resp.

$$\int_{w}^{w-i\infty} f(s) z^s \, ds \tag{5.24}$$

is absolutely convergent and represents an analytic function of z on the part of the Riemann surface of log z where $0 < |z| < 1$ resp. $\arg z < 0$. *The function in* (5.23) *resp.* (5.24) *is the analytic continuation of the integral* (5.24) *resp.* (5.23) *for* $0 < |z| < 1$, $\arg z \geqq 0$ *resp.* $|z| \geqq 1$, $\arg z < 0$.

Corresponding to an arbitrary positive constant ε there exists a positive number K independent of z so that

$$\int_{w}^{w-i\infty} |f(s) z^s| |ds| \leqq K \, |z|^w \tag{5.25}$$

for

$$\arg z \leqq -\varepsilon. \tag{5.26}$$

One may choose $K = \int_{w}^{w-i\infty} |f(s)| e^{\varepsilon \operatorname{Im} s} |ds|$ *(the last integral converges).*

If $\sigma < -1$ then the integral (5.23) *resp.* (5.24) *converges absolutely and is continuous in z for* $0 < |z| \leqq 1$ *resp.*

$$\arg z \leqq 0. \tag{5.27}$$

If $\sigma < -1$ then to every positive constant k_0 there exists a number K_0 independent of z so that (5.25) *with $K || K_0$ holds if* (5.27) *is satisfied and so that if* (5.8) *holds then*

$$\int_{L_1} |f(s) z^s| |ds| \leqq K_0 |z|^w. \tag{5.28}$$

LEMMA 7a.

Let w, l, L_1, S_1 and $f(s)$ satisfy the same assumptions as in lemma 6a save that instead of (5.1) *now* (5.13) *holds for* $|s| \to \infty$

uniformly on S_1. Here μ is a positive constant (μ need not satisfy (1.8)) and c is a complex constant.

Then the integral (5.23) resp. (5.24) is absolutely convergent and represents an analytic function of z on the Riemann surface of $\log z$ *resp. on the part of this surface where* $\arg z < -\frac{1}{2}\mu\pi$. *The function in (5.23) is the analytic continuation of the function in (5.24) for* $\arg z \geqq -\frac{1}{2}\mu\pi$. *Corresponding to a positive constant ε there exists a positive constant K independent of z so that (5.25) holds for*

$$\arg z \leqq -\tfrac{1}{2}\mu\pi-\varepsilon \tag{5.29}$$

and so the function defined originally by (5.23) or (5.24) is $O(z^w)$ for $|z| \to \infty$ uniformly on (5.29).

§ 6. Existence, analytic continuation and algebraic asymptotic expansions of $H(z)$

§ 6.1. First we derive the domain of definition of $H(z)$ by means of lemma 2 and lemma 2a.

THEOREM 1.

We use the notation of § 1. Then the function $H(z)$ makes sense and defines an analytic function of z in the following two cases:

I. $\mu > 0$, $z \neq 0$,

II. $\mu = 0$ *and* (1.9) *holds.*

In these cases

$$H(z) = -\sum \text{residues of } h(s)z^s \text{ in the points (1.6).} \tag{6.1}$$

$H(z)$ does not depend on the choice of C. Further $H(z)$ is in general multiple-valued but one-valued on the Riemann surface of $\log z$.

PROOF: From the definition of C in § 1 it follows that C has a positive distance to the points (1.6). From this and the periodicity of the sine-factors in (2.11) we may deduce that $h_1(s)$ is bounded on C.

Suppose first μ is positive. Then we apply lemma 2: (3.25) and (3.27) with $N = 0$. From this, the boundedness of $h_1(s)$ on C and (2.7) we infer to

$$h(s)z^s = O(1)\,(\beta\mu^\mu z)^s/\Gamma(\mu s+\alpha) \tag{6.2}$$

for $|s| \to \infty$ and s on C. Using this relation we may deduce that the integral in (1.1) converges and represents an analytic function of z for $z \neq 0$. The proof of this assertion runs in the same way as the proof of the corresponding assertion for the integral (5.2) in lemma 7 and will be omitted therefore.

Now suppose $\mu = 0$. Then we apply lemma 2a in § 3.2. (3.31), (2.7) and the boundedness of $h_1(s)$ on C imply

$$h(s)z^s = O(s^{\frac{1}{2}-\alpha})(\beta z)^s \tag{6.3}$$

for $|s| \to \infty$ on C. It now easily follows that the integral in (1.1) converges and represents an analytic function of z if (1.9) holds. The proof of this is similar to the corresponding assertion for the integral (5.2) in lemma 6 and it will be omitted therefore.

From the occurrence of the factor z^s in the integrand in (1.1) it follows that $H(z)$ is multiple-valued in general but one-valued on the Riemann surface of $\log z$.

That in case I and case II (6.1) holds can be easily derived from the theorem of residues and the estimates (6.2) resp. (6.3) which also hold for $|s| \to \infty$ uniformly on sets which have a positive distance to the points (1.6) and which do not contain points to the left of C. We omit the calculations. From (6.1) it also follows that $H(z)$ is independent of the choice of C.

REMARKS: If

$$\beta_h(b_j+\lambda) \neq \beta_j(b_h+\nu) \quad \text{for } j \neq h;\ j, h = 1, \ldots, m;\ \lambda, \nu = 0, 1, 2, \ldots \tag{6.4}$$

then (6.1) may be written as

$$H(z) = \sum_{h=1}^{m} \sum_{\nu=0}^{\infty} \frac{\prod_{1}^{m}{}' \Gamma(b_j-\beta_j(b_h+\nu)/\beta_h) \prod_{1}^{n} \Gamma(1-a_j+\alpha_j(b_h+\nu)/\beta_h)}{\prod_{m+1}^{q} \Gamma(1-b_j+\beta_j(b_h+\nu)/\beta_h) \prod_{n+1}^{p} \Gamma(a_j-\alpha_j(b_h+\nu)/\beta_h)} \cdot \frac{(-1)^\nu z^{(b_h+\nu)/\beta_h}}{\nu! \cdot \beta_h} \tag{6.5}$$

in case I and case II. $\prod_{1}^{m}{}'$ means: the product of the factors with $j = 1, \ldots, j = m$ save $j = h$.

If $\mu = 0$ then in some cases $H(z)$ also makes sense for $|z| = \beta^{-1}$.

If only a finite number of the points (1.6) are poles of $h(s)$ then $H(z)$ exists for $z \neq 0$.

§ 6.2. In the case that $\mu = 0$ we derive the analytic continuations of $H(z)$ in the manner indicated in § 2.1. The result is formulated in theorem 2 in which use is made of the notations of § 1 and of the definitions I, II and IV of § 4.

THEOREM 2.

Suppose $\mu = 0$ and r is an integer. Let D be a contour in the complex s-plane which runs from $s = -\infty i+\sigma$ to $s = \infty i+\sigma$ (σ is an arbitrary real number) so that the points (4.28) lie to the left of D and so that those points (1.6) which do not occur among the points (4.28) lie to the right of D. Further let $V(z)$ be the sum of the residues of $h(s)z^s$ at the points where simultaneously (1.6) and (4.28) holds. $h_0(s)$ resp. $h_1(s)$ are defined in (2.10) resp. (2.11).

Then $H(z)$ can be continued analytically by

$$(6.6)\quad H(z) = \frac{1}{2\pi i}\int_D h_0(s)\{h_1(s)+\sum_r^\kappa D_j e^{i\delta_j s}-\sum_0^{r-1} C_j e^{i\delta_j s}\}z^s ds - V(z)$$

into the sector

$$(6.7)\qquad -\delta_r < \arg z < -\delta_{r-1}.$$

$V(z)$ is equal to zero if

$$(6.8)\quad \alpha_h(b_j+\nu) \neq \beta_j(a_h-1-\lambda)$$
$$\text{for } j = 1, \ldots, m;\ h = n+1, \ldots, p;\ \nu, \lambda = 0, 1, 2, \ldots$$

Further the function $H(z)$ can be continued analytically from the sector (6.7) into the domain $|z| > \beta^{-1}$ by

$$(6.9)\qquad H(z) = Q(z)+\sum_r^\kappa D_j P(ze^{i\delta_j})-\sum_0^{r-1} C_j P(ze^{i\delta_j})$$

Here the formal series for $Q(z)$ and $P(z)$ (cf. (4.26) and (4.27)) are convergent for $|z| > \beta^{-1}$.

PROOF: Let w and l be real numbers which satisfy (2.1), (2.2) and (2.3). L and L_1 are the contours defined in § 2.1. Using (3.31) we see that the integrals $\int h_0(s)z^s ds$ taken along the contours L and L_1 converge for (1.9). Using (2.17) and the theorem of residues we deduce that if (1.9) holds:

$$P_w(z) = \frac{-1}{2\pi i}\left(\int_L - \int_{L_1}\right) h_0(s)z^s ds.$$

From this and (2.4) we deduce (2.18) if (1.9) holds. To the two integrals in (2.18) we next apply lemma 6 and lemma 6a of § 5. First we apply lemma 6 with the same w and l as in § 2.1 and with

$$(6.10)\quad f(s) = h_0(s)\beta^{-s}\{h_1(s)+\sum_r^\kappa D_j e^{i\delta_j s}-\sum_0^{r-1} C_j e^{i\delta_j s}\}e^{-i\delta_r s},$$
$$z \,||\, \beta z e^{i\delta_r},\ \arg z \,||\, \arg z+\delta_r.$$

The singular points (2.8) and (1.6) of $h_0(s)$, $h_0(s)h_1(s)$ or $h(s)$ (cf. (2.7)) do not belong to S and L (defined in lemma 6) on account of (2.1), (2.2) and (2.3). Further from (6.10), (3.31) and (4.8) we derive (5.1) with $\sigma = \frac{1}{2} -$ Re α for $|s| \to \infty$ uniformly on S. Since $\arg z + \delta_r > 0$ on (6.7), lemma 6 and (6.10) imply that the first integral in (2.18) can be continued analytically on (6.7) by replacing the path of integration L by the rectilinear path from $s = w$ to $s = w + i\infty$.

The second integral in (2.18) can be treated in the same way as the first integral. Instead of lemma 6 we use lemma 6a with the same w and l as above and with

$$(6.11)\quad \begin{cases} f(s) = h_0(s)\beta^{-s}\{h_1(s) + \sum\limits_{r}^{\kappa} D_j e^{i\delta_j s} - \sum\limits_{0}^{r-1} C_j e^{i\delta_j s}\} e^{-i\delta_{r-1} s}, \\ z\,||\,\beta z e^{i\delta_{r-1}}, \arg z \,||\, \arg z + \delta_{r-1}. \end{cases}$$

Further instead of (4.8) we now use (4.9). From lemma 6a and (6.11) we may deduce that the second integral in (2.18) can be continued analytically on (6.7) by replacing the path of integration L_1 in this integral by the rectilinear path from $s = w$ to $s = w - i\infty$. Hence the righthand side of (2.18) — and so $H(z)$ — can be continued analytically by

$$(6.12)\quad H(z) = Q_w(z) + \sum_{r}^{\kappa} D_j P_w(ze^{i\delta_j}) - \sum_{0}^{r-1} C_j P_w(ze^{i\delta_j})$$
$$+ \frac{1}{2\pi i}\int_{w-i\infty}^{w+i\infty} h_0(s)\{h_1(s) + \sum_{r}^{\kappa} D_j e^{i\delta_j s} - \sum_{0}^{r-1} C_j e^{i\delta_j s}\} z^s\, ds$$

into the sector (6.7). The integral in (6.12) is absolutely convergent for (6.7).

Next we consider (6.6). Suppose temporarily that w is so small that Re $s > w$ for s on D. Then it follows from (3.31), (4.8) and (4.9) that the integral in (6.6) is convergent on (6.7) and that it is equal to the integral in (6.12) increased by $2\pi i$ times the sum of the residues of the integrand in (6.6) in the poles between D and Re $s = w$. From this, (2.7), (2.10), (2.11), (2.5), (2.17) and the definition of D and $V(z)$ we now deduce (6.6) if (6.7) holds. From the definition of $V(z)$ and from (1.5) it follows that $V(z) \equiv 0$ if (6.8) holds.

Next we consider the integral in (6.12). It may be written as

(6.13)

$$\left(-\int_{-w}^{-w-i\infty} + \int_{-w}^{-w+i\infty}\right) h_0(-s)\{h_1(-s) + \sum_{r}^{\kappa} D_j e^{-i\delta_j s} - \sum_{0}^{r-1} C_j e^{-i\delta_j s}\} z^{-s}\, ds$$

if (6.7) holds. The last integrals will be continued analytically using the lemmas 6 and 6a. Therefore we estimate $h_0(-s)$. If

$$(6.14) \qquad s \neq -(a_j+\nu)/\alpha_j \qquad (j = 1, \ldots, p;\ \nu = 0, \pm 1, \pm 2, \ldots)$$

and

$$(6.15) \qquad s \neq -(b_j+\nu)/\beta_j \qquad (j = 1, \ldots, q;\ \nu = 0, \pm 1, \pm 2, \ldots)$$

hold then we may write (cf. (2.10) and (2.6))

$$(6.16) \qquad h_0(-s) = h_3(s)h_4(s)$$

where if (6.15) holds we define

$$(6.17) \qquad h_3(s) = \prod_1^q \Gamma(b_j+\beta_j s)/\prod_1^p \Gamma(a_j+\alpha_j s)$$

while if (6.14) holds we define

$$(6.18) \quad h_4(s) = \pi^{p-q} \prod_1^q \sin \pi(1-b_j-\beta_j s)/\prod_1^p \sin \pi(1-a_j-\alpha_j s).$$

Then $h_3(s)$ resp. $h_4(s)$ is equal to the function $h_0(s)$ resp. $h_1(s)$ (cf. (2.10) resp. (2.11)) with

$$(6.19) \quad p||q,\ q||p,\ n||0,\ m||p,\ \alpha_j||\beta_j,\ a_j||1-b_j,\ \beta_j||\alpha_j,\ b_j||1-a_j.$$

The numbers μ and α defined by (1.8) and (3.24) then remain the same as before, so $\mu = 0$; the number β given by (1.10) now changes into β^{-1}. Consequently we may apply lemma 2a to $h_3(s)$ and so by (3.31)

$$(6.20) \qquad h_3(s) = \beta^{-s}s^{\frac{1}{2}-\alpha}O(1)$$

for $|s| \to \infty$ uniformly on for example $\operatorname{Re} s \geqq -w$.

Further we apply lemma 4a to $h_4(s)$. By (2.12) and (6.19) the new number δ_0 is equal to $(\sum_1^p \alpha_j - \sum_1^q \beta_j)\pi = -\mu\pi = 0$ (cf. (1.8)). From definition II we now derive $\kappa = 0$. Further l satisfies the new formula (2.3). Hence by (4.8) with $r = 0$ and (4.9) with $r = 1$ $|h_4(s)|$ is bounded on $|\operatorname{Im} s| \geqq l$. This, (6.20) and (6.16) imply that

$$(6.21) \qquad h_0(-s) = \beta^{-s}s^{\frac{1}{2}-\alpha}O(1)$$

for $|s| \to \infty$ uniformly on $\operatorname{Re} s \geqq -w$.

Now to the first integral in (6.13) we apply lemma 6a with $w||-w$, the same l like in (2.3) and with

$$f(s) = h_0(-s)\beta^s\{h_1(-s)+\sum_r^\kappa D_j e^{-i\delta_j s}-\sum_0^{r-1} C_j e^{-i\delta_j s}\}e^{i\delta_r s},\ z||\beta^{-1}z^{-1}e^{-i\delta_r},$$
$$\arg z||-\arg z-\delta_r.$$

From (4.8) for $\operatorname{Im} s \geqq l$ and (6.21) we easily derive that the assumptions of lemma 6a are satisfied. So the first integral in (6.13) can be continued analytically for $|z| > \beta^{-1}$ by replacing the path of integration by the path from $s = -w$ to $s = -w-il$ and then to $s = \infty-il$. In an analogous way applying lemma 6 we continue the second integral in (6.13) analytically for $|z| > \beta^{-1}$ by replacing the path of integration by the path from $s = -w$ to $s = -w+il$ and then to $s = \infty+il$.

Combining these properties with (6.13) and (6.12) we see that $H(z)$ can be continued analytically from the sector (6.7) into the domain $|z| > \beta^{-1}$ by means of

$$(6.22) \quad H(z) = Q_w(z) + \sum^{\kappa} D_j P_w(ze^{i\delta_j}) - \sum_0^{r-1} C_j P_w(ze^{i\delta_j}) + \frac{1}{2\pi i}\int_{\infty-il}^{\infty+il} h_0(-s)\{h_1(-s) + \sum_r^{\kappa} D_j e^{-i\delta_j s} - \sum_0^{r-1} C_j e^{-i\delta_j s}\} z^{-s} ds$$

where the path of integration runs from $s = \infty-il$ to $s = -w-il$, then to $s = -w+il$ and finally to $s = \infty+il$, while the integral converges for $|z| > \beta^{-1}$.

From the behaviour of $h_0(-s)$ for $\operatorname{Re} s \geqq -w$ (cf. (6.21)) and the behaviour of $h_1(-s)$ for $|\operatorname{Im} s| \leqq l$ (cf. (2.11)) we may deduce that the integral in (6.22) can be calculated by means of the theorem of residues for $|z| > \beta^{-1}$. Then we obtain (6.9) for $|z| > \beta^{-1}$ in view of definition IV of § 4.6 where now the series for $Q(z)$ and $P(z)$ are convergent for $|z| > \beta^{-1}$.

REMARK 1: If $\mu = 0$ and δ_0 is positive then $\delta_{-1} = -\delta_0$ and $\kappa = -1$ by the definitions I and II of § 4.2. If moreover $r = 0$ then the expression $\{\ldots\}$ in (6.6) reduces to $h_1(s)$. Hence if $\mu = 0$, $\delta_0 > 0$ then $H(z) = Q(z)$ resp.

$$(6.23) \qquad H(z) = \frac{1}{2\pi i}\int_D h(s) z^s ds - V(z) = \frac{1}{2\pi i}\int_G h(s) z^s ds$$

for $|z| > \beta^{-1}$ resp. $|\arg z| < \delta_0$ on account of (6.6), (6.7), (6.9), (2.7) and the definition of D and $V(z)$. Here δ_0 is given by (2.12) and G is a contour in the complex s-plane from $s = -\infty i+\sigma$ to $s = \infty i+\sigma$ (σ an arbitrary real number) so that the points (1.6) resp. (1.7) lie to the right resp. left of G. Moreover if $\mu = r = 0$, $\delta_0 > 0$ then (2.18) and (6.12) (which have been proved above for (1.9) resp. (6.7)) reduce to (2.4) and (2.14).

REMARK 2: (6.9) may be rewritten as (cf. definition IV of § 4.6)

$$(6.24)\quad H(z) = \frac{1}{2\pi i}\int_{G_0} h(-s)z^{-s}ds + \sum_{r}^{\kappa} D_j \frac{1}{2\pi i}\int_{G_1} h_0(-s)(ze^{i\delta_j})^{-s}ds$$
$$-\sum_{0}^{r-1} C_j \frac{1}{2\pi i}\int_{G_1} h_0(-s)(ze^{i\delta_j})^{-s}ds$$

where G_0 and G_1 are contours of the same shape as C in § 1 but with the difference that the points

$$s = (\nu+1-a_j)/\alpha_j \qquad (j = 1, \ldots, n;\ \nu = 0, 1, 2, \ldots)$$

resp.

$$s = -(b_j+\nu)/\beta_j \qquad (j = 1, \ldots, m;\ \nu = 0, 1, 2, \ldots)$$

lie to the right resp. left of G_0 and the points

$$s = (\nu+1-a_j)/\alpha_j \qquad (j = 1, \ldots, p;\ \nu = 0, 1, 2, \ldots)$$

lie to the right of G_1. The integrals in (6.24) are special cases of the function $H(z)$ in § 1 with special values of the parameters p, q, m, n, $a_j, \ldots$ that differ from those in § 1. So $H(z)$ can be expressed as a sum of other functions $H(z^{-1}e^{-i\delta_j})$ for $|z| > \beta^{-1}$ if $\mu = 0$.

§ 6.3. We next derive the algebraic asymptotic expansions for $|z| \to \infty$ in the case that μ is positive. The notation of the definitions I, II and IV of § 4 are used again. The method of proof has been indicated in § 2.1.

THEOREM 3.

Suppose μ is positive, r is an integer and (2.19) *holds. Let ε be a constant satisfying*

$$(6.25)\qquad 0 < \varepsilon < \tfrac{1}{2}(\delta_r - \delta_{r-1} - \mu\pi).$$

Then (6.6) *holds for* (2.20) *if D and $V(z)$ have the same meaning as in theorem 2 and $h_0(s)$ resp. $h_1(s)$ are defined in* (2.10) *resp.* (2.11). *Further the algebraic asymptotic expansion*

$$(6.26)\qquad H(z) \sim Q(z) + \sum_{r}^{\kappa} D_j P(ze^{i\delta_j}) - \sum_{0}^{r-1} C_j P(ze^{i\delta_j})$$

holds for $|z| \to \infty$ uniformly on

$$(6.27)\qquad \tfrac{1}{2}\mu\pi - \delta_r + \varepsilon \leqq \arg z \leqq -\tfrac{1}{2}\mu\pi - \delta_{r-1} - \varepsilon.$$

If $\mu > 0$, $r = 0$ and (2.13) *holds then the assertions* (6.6) *on* (2.20) *and* (6.26) *on* (6.27) *reduce to* (6.23) *on* (2.15) *and* (2.16) *for $|z| \to \infty$ uniformly on*

(6.28) $$|\arg z| \leqq \delta_0 - \tfrac{1}{2}\mu\pi - \varepsilon.$$

Here we assume that ε is constant satisfying $0 < \varepsilon < \delta_0 - \frac{1}{2}\mu\pi$. If $r = n = 0$, $\mu > 0$ and (2.13) holds then the righthand side of (2.16) is a formal series of zeros and then better estimates for $H(z)$ are contained in theorem 4.

PROOF: The proof is similar to the first part of the proof of theorem 2; only instead of the lemmas 6, 6a and 2a now the lemmas 7, 7a and 2 are used.

So we start again from (2.18). (2.18) can be proved for $z \neq 0$ in an analogous way as in the proof of theorem 2. We apply lemma 7 with (6.10) where $\beta || \beta\mu^{\mu}$ to the first integral in (2.18). On account of (3.25) and (3.27) with $N = 0$, and (4.8) the condition (5.13) with $c = \alpha$ and μ given by (1.8) is satisfied on the set S of lemma 7. Further $f(s)$ is continuous on S and L and analytic in the interior of S (cf. the proof of theorem 2). So by lemma 7 and (6.10) with $\beta || \beta\mu^{\mu}$ we may replace the contour L in the first integral in (2.18) by the rectilinear path from $s = w$ to $s = w + \infty i$ if (2.20) holds. Moreover

(6.29) $$\int_w^{w+\infty i} h_0(s) \left\{ h_1(s) + \sum_r^K D_j e^{i\delta_j s} - \sum_0^{r-1} C_j e^{i\delta_j s} \right\} z^s ds = O(z^w)$$

for $|z| \to \infty$ uniformly on (6.27) because by lemma 7 (5.4) holds on (5.14) (cf. also (6.10) with $\beta || \beta\mu^{\mu}$). The integral in (6.29) converges absolutely on (2.20).

In the same way using lemma 7a with (6.11) where $\beta || \beta\mu^{\mu}$, (3.25) and (3.27) with $N = 0$, and (4.9) we may show that if (2.20) holds in the second integral in (2.18) we may replace L_1 by the rectilinear path from $s = w$ to $s = w - \infty i$, that (6.29) with $w + \infty i || w - \infty i$ holds for $|z| \to \infty$ uniformly on (6.27) and that the integral in (6.29) with $w + \infty i || w - \infty i$ is absolutely convergent on (2.20).

From these properties concerning the integrals in (2.18) we deduce that (6.12) holds if (2.20) is satisfied and that

(6.30) $$H(z) = Q_w(z) + \sum_r^K D_j P_w(ze^{i\delta_j}) - \sum_0^{r-1} C_j P_w(ze^{i\delta_j}) + O(z^w)$$

for $|z| \to \infty$ uniformly on (6.27). Further the integral in (6.12) is absolutely convergent on (2.20). In view of the definitions in § 4.6 formula (6.30) implies the algebraic asymptotic expansion (6.26) for $|z| \to \infty$ uniformly on (6.27). From (6.12) we deduce in the same way as in the proof of theorem 2 that (6.6) holds on (2.20).

If (2.13) holds then δ_0 is positive and so $\delta_{-1} = -\delta_0$ and $\kappa = -1$ by the definitions I and II of § 4.2. So if $r = 0$ and (2.13) holds then (6.6) and (2.20) reduce to (6.28) and (2.15) (cf. remark 1 after theorem 2), while (6.26) reduces to (2.16) on (6.28). If moreover $n = 0$ then $Q(z)$ represents a formal series of zeros (cf. (4.26)).

Remark 1: In § 10.2 special cases in which the series in the righthand side of (6.26) only contain a finite number of non-zero terms are considered.

Remark 2: In the course of the proof we have seen that (6.12) holds on (2.20) and that the integral in (6.12) converges absolutely on (2.20). If $r = 0$, $\mu > 0$ and (2.13) holds this reduces to (2.14) for (2.15) where the integral in (2.14) converges absolutely on (2.15).

§ 7. The exponentially small asymptotic expansions of $H(z)$

In this paragraph we give the details of the derivation of the exponentially small asymptotic expansions of $H(z)$ which has been sketched in § 2.2.

§ 7.1. Here the proofs of the assertions concerning (2.27) and (2.28) are completed. We assume that $\mu > 0$, $n = 0$, $m = q$ and (2.22) hold.

If $\operatorname{Re} s \leqq w$ then (2.23) holds on account of (2.2) and so by lemma 3 $\rho_N(s)$ is defined and analytic in all points s with $\operatorname{Re} s \leqq w$. If $\operatorname{Re} s \leqq w$ and $j = 0, \ldots, N$ then by (2.26): $\operatorname{Re}(1-\mu s-\alpha-j) > 0$ and so $\Gamma(1-\mu s-\alpha-j)$ is defined and analytic in s for $\operatorname{Re} s \leqq w$. Consequently the righthand side of (3.35) is defined and analytic in s for $\operatorname{Re} s \leqq w$. So by (3.35) the change from (2.25) to (2.27) will be proved if the convergence of the integrals in (2.27) has been verified for (2.22).

First we consider (2.28). This formula can be derived by means of the inversion formulae for the Mellin transformation (cf. G. Doetsch [13], I p. 212) from (2.30) with $N||j+2$ ($j = 0, \ldots, N-1$). Here we deduce (2.28) from (6.5) and (2.25) by choosing for $H(z)$ in these formulae the function given by (1.1) and (1.3) with $h(s) = \Gamma(1-\mu s-\alpha-j)$ $(j = 0, \ldots, N-1)$. Then $m = q = 1$, $n = p = 0$, $b_1 = 1-\alpha-j$, $\beta_1 = \mu$. The numbers μ resp. α of this special function $H(z)$ are equal to μ resp. $\alpha+j$ (cf. (1.8) and (3.24)) where the last μ and α are the same as those we always use here. By (2.26) and (2.3) for the old μ and α the numbers w and l satisfy (2.1), (2.2) and (2.3) for the new μ, α, b_1 and β_1. From (6.5) and (2.4) with (2.5) we derive that this special function $H(z)$ is equal to

$$(7.1) \quad \frac{1}{2\pi i}\left(\int_L - \int_{L_1}\right) \Gamma(1-\mu s-\alpha-j)z^s ds = \sum_{h=0}^{\infty} \frac{(-1)^h}{h!\cdot\mu} z^{(1-\alpha-j+h)/\mu}$$
$$= \mu^{-1} z^{(1-\alpha-j)/\mu} \exp(-z^{1/\mu})$$

for $z \neq 0$ and $j = 0, \ldots, N-1$. If moreover (2.22) holds then all conditions for (2.25) are satisfied and from (2.25) and (7.1) now (2.28) follows if (2.22) is satisfied and $j = 0, \ldots, N-1$. As the integral in (2.25) converges absolutely for (2.22) this is the case also with the integrals in (2.28) for $j = 0, \ldots, N-1$ and consequently with the first N integrals in (2.27). From this and the absolute convergence of the integral in (2.25) the absolute convergence of the last integral in (2.27) follows for (2.22). The proof of (2.27) is complete now.

§ 7.2. In this section the proofs of (2.31), of the properties of $\rho(t)$ (cf. (2.32)) and of (2.36) are completed. We assume again $\mu > 0$, $n = 0$, $m = q$ and (2.22) hold.

First we consider the properties of $\rho(t)$ defined by (2.32). On account of (2.26) the zeros of $(1-\mu s-\alpha-N)_2$, i.e. $(1-\mu s-\alpha-N)\cdot(2-\mu s-\alpha-N)$ are lying to the right of the line $\operatorname{Re} s = w$. Further by § 7.1 $\rho_N(s)$ is defined and analytic in s if $\operatorname{Re} s \leqq w$. Hence the integrand in (2.32) is defined and analytic in the points s with $\operatorname{Re} s \leqq w$ while by (3.33) the estimate (2.33) holds for $|s| \to \infty$ uniformly on $\operatorname{Re} s \leqq w$. From this we can deduce the properties of $\rho(t)$ using lemma 6 and lemma 6a but here we give a straightforward derivation of these properties.

The convergence of the integral in (2.32) and (2.34) for $t > 0$ follows from (2.33). So (2.34) holds with

$$(7.2) \qquad K = \int_{w-i\infty}^{w+i\infty} |\rho_N(s)/(1-\mu s-\alpha-N)_2||ds|.$$

It is easily seen that the integral in (2.32) is continuous for $t > 0$. Further for every positive number R the part between $w-iR$ and $w+iR$ of the path of integration in (2.32) may be replaced by the lefthand part of the circle $|s-w| = R$ between these points. The integrand in (2.32) is $t^{2-\mu w-\operatorname{Re}\alpha-N}O(s^{-2})$ for $|s| \to \infty$ uniformly on $\operatorname{Re} s \leqq w$ for every fixed value of t with $0 < t \leqq 1$ on account of (2.33). Therefore the contributions of the circular part and of the rectilinear part of the path of integration of the modified integral (2.32) are $O(R^{-1})$ for $R \to \infty$ if t is fixed and $0 < t \leqq 1$. So $\rho(t) = 0$ for $0 < t \leqq 1$.

Next we prove (2.31) for (2.22). On account of (2.34)

$$\text{(7.3)} \qquad \int_0^\infty dt\, |\exp(-z^{1/\mu}t)| \int_{w-i\infty}^{w+i\infty} |ds| \left| \frac{\rho_N(s)}{(1-\mu s-\alpha-N)_2} t^{2-\mu s-\alpha-N} \right| \leqq K \int_0^\infty \exp(-t \operatorname{Re} z^{1/\mu}) t^{2-\mu w-\operatorname{Re}\alpha-N} dt$$

if (2.22) holds, for the last integral — and consequently the repeated integral — converges as $\operatorname{Re} z^{1/\mu} > 0$ on (2.22) and as (2.26) holds. Further the integrals in (2.29) resp. (2.30) are absolutely convergent for (2.22) resp. for (2.22) and $\operatorname{Re} s = w$. A theorem of Bromwich ([8] p. 504) now implies that for (2.22)

$$\text{(7.4)} \qquad \int_0^\infty dt \exp(-z^{1/\mu}t) \int_{w-i\infty}^{w+i\infty} ds\, t^{2-\mu s-\alpha-N} \rho_N(s)/(1-\mu s-\alpha-N)_2 = \int_{w-i\infty}^{w+i\infty} ds \frac{\rho_N(s)}{(1-\mu s-\alpha-N)_2} \int_0^\infty dt\, t^{2-\mu s-\alpha-N} \exp(-z^{1/\mu}t).$$

From (7.4), (2.30) and (2.29) we infer to (2.31) for (2.22).

Combining (2.31) with (2.34) and the property that $\rho(t) = 0$ for $0 < t \leqq 1$ we obtain

$$\text{(7.5)} \quad |\sigma(z)| \leqq K|z^{(3-\alpha-N)/\mu}| \int_1^\infty t^{2-\mu w-\operatorname{Re}\alpha-N} \exp(-t \operatorname{Re} z^{1/\mu}) dt \leqq K|z^{(3-\alpha-N)/\mu}| \exp(-\operatorname{Re} z^{1/\mu}) \int_1^\infty t^{2-\mu w-\operatorname{Re}\alpha-N} \exp\{-(t-1)\eta\} dt$$

if

$$\text{(7.6)} \qquad |\arg z| < \tfrac{1}{2}\mu\pi, \ \operatorname{Re} z^{1/\mu} \geqq \eta$$

where η is an arbitrary positive constant. As the last integral in (7.5) is independent of z, (7.5) implies (2.35) for $|z| \to \infty$ uniformly on (7.6).

In view of (2.35), (2.27), (2.28) and (2.29) we have

$$\text{(7.7)} \qquad H_0(z\beta^{-1}\mu^{-\mu}) = \mu^{-1}(2\pi)^{q-p-1} \exp(-z^{1/\mu}) \cdot \left\{ \sum_0^{N-1} (-1)^j A_j z^{(1-\alpha-j)/\mu} + O(z^{(3-\alpha-N)/\mu}) \right\}$$

for $|z| \to \infty$ uniformly on (7.6). Here N is an arbitrary non-negative integer. Next we use that if $j = N$ or $j = N+1$ then

$$z^{(1-\alpha-j)/\mu} = O(z^{(1-\alpha-N)/\mu}) \tag{7.8}$$

for $|z| \to \infty$ uniformly on (2.22). From this and (7.7) where we replace N by $N+2$ we infer to

$$H_0(z\beta^{-1}\mu^{-\mu}) = \mu^{-1}(2\pi)^{q-p-1}\exp(-z^{1/\mu}) \cdot \left\{\sum_0^{N-1}(-1)^j A_j z^{(1-\alpha-j)/\mu} + O(z^{(1-\alpha-N)/\mu})\right\}$$

for $|z| \to \infty$ uniformly on (7.6). This may be written as (2.36) on account of definition III of § 4.4. It will be clear that (2.36) then also holds for $|z| \to \infty$ uniformly on

$$|\arg z| \leqq \tfrac{1}{2}\mu\pi - \varepsilon \tag{7.9}$$

if ε is a constant satisfying $0 < \varepsilon < \frac{1}{2}\mu\pi$.

§ **7.3.** The exponentially small asymptotic expansions of $H(z)$ are formulated in the following theorem. The proof depends on (2.39) and (2.43).

THEOREM 4.

Suppose $n = 0$, $\mu > 0$ and (2.13) holds (cf. (2.12) for δ_0). ε and η will be positive constants so that

$$0 < \varepsilon < \tfrac{1}{2}(\delta_0 - \tfrac{1}{2}\mu\pi). \tag{7.10}$$

c_0 and d_0 are defined by (4.3) and $E(z)$ is given by definition III *of* § 4.4.

ASSERTIONS: *If $m = q$ then*

$$H(z) \sim c_0 E(ze^{i\delta_0}) \tag{7.11}$$

and also

$$H(z) \sim -d_0 E(ze^{-i\delta_0}) \tag{7.12}$$

for $|z| \to \infty$ uniformly on (7.9). If $m < q$ then (7.12) holds for $|z| \to \infty$ uniformly on

$$\varepsilon \leqq \arg z \leqq \delta_0 - \tfrac{1}{2}\mu\pi - \varepsilon \tag{7.13}$$

and (7.11) holds for $|z| \to \infty$ uniformly on

$$\tfrac{1}{2}\mu\pi - \delta_0 + \varepsilon \leqq \arg z \leqq -\varepsilon. \tag{7.14}$$

Further if $m < q$ then

(7.15) $$H(z) \sim c_0 E(ze^{i\delta_0}) - d_0 E(ze^{-i\delta_0})$$

for $|z| \to \infty$ *uniformly on*

(7.16) $$|\arg z| \leqq \varepsilon.$$

The asymptotic expansions formulated above are exponentially small (cf. § 4.4).

If $m = q$ *then* (7.11) *and* (7.12) *also hold for* $|z| \to \infty$ *uniformly on* (7.6). *If* $m < q$ *then* (7.12) *resp.* (7.11) *also holds for* $|z| \to \infty$ *uniformly on the set*

(7.17) $$\mathrm{Re}\,(ze^{-i\delta_0})^{1/\mu} \leqq -\eta,\ \varepsilon \leqq \arg z < \delta_0 - \tfrac{1}{2}\mu\pi$$

resp.

(7.18) $$\mathrm{Re}\,(ze^{i\delta_0})^{1/\mu} \leqq -\eta,\ \tfrac{1}{2}\mu\pi - \delta_0 < \arg z \leqq -\varepsilon.$$ [1]

Proof: If $n = 0$ and $m = q$ then $\delta_0 = \mu\pi$ (cf. (2.12) and (1.8)) and on account of (3.24) and (4.3)

$$(2\pi)^{q-p} e^{(\alpha-\frac{1}{2})\pi i} = c_0,\ (2\pi)^{q-p} e^{(-\alpha-\frac{1}{2})\pi i} = -d_0.$$

Because further (2.36) has been proved for $|z| \to \infty$ uniformly on (7.6) and (7.9) in § 7.2 it now follows from definition III of § 4.4 that (7.11) and (7.12) hold for $|z| \to \infty$ uniformly on (7.6) and (7.9) if $m = q$ (if $m = q$, $H(z)$ has been denoted by $H_0(z)$ in § 2.2).

Now suppose $m < q$. Then $\delta_0 < \mu\pi$ by (2.12), (1.8) and because $n = 0$. First we assume that ε satisfies besides (7.10) also

(7.19) $$\varepsilon \leqq \omega_j - \omega_h \text{ for } 0 \leqq h < j \leqq M.$$

Here we use the notation of (2.38) and (2.39).

By (2.41) we have

$$\tfrac{1}{2}\mu\pi < \arg z + \omega_j + \mu\pi = |\arg z + \omega_j + \mu\pi| \text{ for } j = 0, \ldots, M.$$

Hence if

(7.20) $$\tfrac{1}{2}\mu\pi - \delta_0 < \arg z \leqq 0,\ \mathrm{Re}\,(ze^{i\delta_0})^{1/\mu} \leqq -\eta$$

then using (7.19) and (2.39) we obtain for $j = 1, \ldots, M-1$:

(7.21) $$\begin{aligned} |\arg z + \omega_0 + \mu\pi| + |\arg z + \omega_j + \mu\pi| &= 2\arg z + \omega_0 + \omega_j + 2\mu\pi \\ &\leqq 2\arg z + \omega_0 + \omega_M - \varepsilon + 2\mu\pi \leqq 2\mu\pi - \varepsilon \end{aligned}$$

and for $j = 1, \ldots, M$:

[1]) An estimate for $H(z)$ in the case that $p = 0$, $m = q$ and z large positive has been obtained by Bochner [5] p. 351.

$$(7.22) \qquad |\arg z+\omega_0+\mu\pi| = \arg z+\omega_0+\mu\pi \leqq \arg z+\omega_j+\mu\pi-\varepsilon = |\arg z+\omega_j+\mu\pi|-\varepsilon.$$

By (7.21) and (7.22) we may apply lemma 5 of § 4 with $t_1 = \omega_0+\mu\pi$ and $t_2 = \omega_j+\mu\pi$, $k_1 = \tau_0$, $k_2 = \tau_j$ ($j = 1, \ldots, M-1$; cf. (2.38)) and (7.20) for the set Ω. Then we obtain using also (2.43) with the upper sign for $j = 0, \ldots, M-1$ and with the lower sign for $j = M$:

$$(7.23) \qquad H(z) = (2\pi)^{q-p}\{e^{(\alpha-\frac{1}{2})\pi i}\tau_0 E_N(ze^{i(\omega_0+\mu\pi)}) + e^{(-\alpha-\frac{1}{2})\pi i}\tau_M E_N(ze^{i(\omega_M-\mu\pi)})\}$$

for $|z| \to \infty$ uniformly on (7.20). In view of (3.24), (2.39), (2.40) and (4.3) this is equivalent to

$$(7.24) \qquad H(z) = c_0 E_N(ze^{i\delta_0})-d_0 E_N(ze^{-i\delta_0})$$

for $|z| \to \infty$ uniformly on (7.20). N is always an arbitrary non-negative integer.

By (2.41) we have

$$\arg z+\omega_j-\mu\pi = -|\arg z+\omega_j-\mu\pi| \text{ for } j = 0, \ldots, M.$$

Hence if

$$(7.25) \qquad 0 \leqq \arg z < \delta_0-\tfrac{1}{2}\mu\pi, \; \operatorname{Re}(ze^{-i\delta_0})^{1/\mu} \leqq -\eta$$

then on account of (7.19) and (2.39):

$$(7.26) \qquad |\arg z+\omega_M-\mu\pi|+|\arg z+\omega_j-\mu\pi| = -2\arg z-\omega_M -\omega_j+2\mu\pi \leqq -2\arg z-\omega_M-\omega_0-\varepsilon+2\mu\pi \leqq 2\mu\pi-\varepsilon$$

for $j = 1, \ldots, M-1$ while for $j = 0, \ldots, M-1$:

$$(7.27) \qquad |\arg z+\omega_M-\mu\pi| = -\arg z-\omega_M+\mu\pi \leqq -\arg z-\omega_j-\varepsilon+\mu\pi = |\arg z+\omega_j-\mu\pi|-\varepsilon.$$

By (7.26) and (7.27) we may apply lemma 5 of § 4 with $t_1 = \omega_M -\mu\pi$, $t_2 = \omega_j-\mu\pi$, $k_1 = \tau_M$, $k_2 = \tau_j$ ($j = 1, \ldots, M-1$) and with (7.25) for the set Ω. From this and (2.43) where we take the lower sign for $j = 1, \ldots, M$ and the upper sign for $j = 0$ we infer to (7.23) and consequently (7.24) for $|z| \to \infty$ uniformly on (7.25). Combining (7.20) and (7.25) we see that in particular (7.24) holds for $|z| \to \infty$ uniformly on (6.28). The condition (7.19) for ε can be dropped now because the smaller ε the larger the sector (6.28) is. Further we see that (7.15) holds for $|z| \to \infty$ uniformly on (6.28) by definition III of § 4.4 and (7.24). Here (6.28) may be replaced by the smaller sector (7.16) (cf. (7.10)).

We now prove (7.12) for $|z| \to \infty$ uniformly on (7.17) and on (7.13) and (7.11) for $|z| \to \infty$ uniformly on (7.18) and on (7.14). If (7.17) or (7.18) holds then $\arg z - \delta_0 < 0$ and $\arg z + \delta_0 > 0$ and so

$$(7.28) \quad |\arg z+\delta_0|+|\arg z-\delta_0| = \arg z+\delta_0 - \arg z+\delta_0 < 2\mu\pi-\varepsilon$$

if ε satisfies

$$(7.29) \qquad \varepsilon < \mu\pi - \delta_0$$

besides (7.10). If (7.17) is satisfied then

$$(7.30) \quad |\arg z-\delta_0| = \delta_0 - \arg z \leqq \arg z+\delta_0-2\varepsilon < |\arg z+\delta_0|-\varepsilon.$$

Hence we may apply lemma 5 of § 4 with $t_1 = -\delta_0$, $t_2 = \delta_0$, $k_1 = -d_0$, $k_2 = c_0$ and with (7.17) for the set Ω. We then obtain from (7.24): $H(z) = -d_0 E_N(ze^{-i\delta_0})$ and consequently (7.12) for $|z| \to \infty$ uniformly on (7.17). Here (7.17) may be replaced by (7.13) because for large $|z|$ the points belonging to (7.13) also belong to (7.17). The condition (7.29) may be omitted because the larger ε the smaller the sets (7.17) and (7.13) are.

If (7.18) holds then

$$|\arg z+\delta_0| = \arg z+\delta_0 \leqq -\arg z+\delta_0-2\varepsilon < |\arg z-\delta_0|-\varepsilon.$$

From this and (7.28) (where temporarily we assume again (7.29)) it follows that we may apply lemma 5 of § 4 with $t_1 = \delta_0$, $t_2 = -\delta_0$, $k_1 = c_0$, $k_2 = -d_0$ and with (7.18) for the set Ω. Then (7.24) reduces to $H(z) = c_0 E_N(ze^{i\delta_0})$, so (7.11), for $|z| \to \infty$ uniformly on (7.18). We may replace (7.18) by (7.14) and we may omit the condition (7.29) for analogous reasons as above at (7.17).

The asymptotic expansions (7.11), (7.12) and (7.15) are exponentially small on (7.9) if $m = q$ and (7.14) if $m < q$, resp. (7.9) if $m = q$ and (7.13) if $m < q$ resp. (7.16) if $m < q$ because these sectors are subsectors of the sector (6.28) and because for z on the sector (6.28):

$$\tfrac{1}{2}\mu\pi+\varepsilon \leqq \arg z+\delta_0 \leqq 2\delta_0-\tfrac{1}{2}\mu\pi-\varepsilon,$$
$$-2\delta_0+\tfrac{1}{2}\mu\pi+\varepsilon \leqq \arg z-\delta_0 \leqq -\tfrac{1}{2}\mu\pi-\varepsilon$$

where $2\delta_0 \leqq 2\mu\pi$ by (2.12) with $n = 0$ and (1.8). So the definition of exponentially small asymptotic expansions in § 4.4 can be applied.

§ 8. Estimates for some auxiliary functions

In this paragraph the proofs of some approximation-formulae from § 2.3 will be completed and estimates for the auxiliary-function $F(z)$ defined by (2.48) will be derived. We assume always $\mu > 0$ and (2.46).

First we prove (2.47). The first part of (2.47) follows from lemma 7 in § 5 where we choose w and l like in § 2.3 and

$$(8.1)\quad \begin{cases} f(s) = h_0(s)(\beta\mu^\mu)^{-s}\left\{h_1(s)+\sum\limits_{\nu+1}^{\kappa} D_j e^{i\delta_j s} - \sum\limits_{0}^{\nu} C_j e^{i\delta_j s}\right\} e^{-i\delta_{\nu+1} s}, \\ z || \beta\mu^\mu z e^{i\delta_{\nu+1}}, \arg z || \arg z + \delta_{\nu+1}. \end{cases}$$

From (8.1), (3.25) and (3.27) with $N = 0$ and (4.8) with $r||\nu+1$ it follows that (5.13) is satisfied with $c = \alpha$ and μ given by (1.8). In view of lemma 2 and the choice of w and l also the other assumptions of lemma 7 are fulfilled. Since $\arg z+\delta_{\nu+1} \geqq \frac{1}{2}\mu\pi+\varepsilon_0$ on (2.44) by (2.46) we see that lemma 7 implies the first part of (2.47) for $|z| \to \infty$ uniformly on (2.44). In an analogous way using lemma 7a and (4.9) instead of lemma 7 and (4.8) the second part of (2.47) follows for $|z| \to \infty$ uniformly on (2.44). In (2.47) the sums $\sum_{\nu+1}^{\kappa}$ and $\sum_0^{\lambda-1}$ may be omitted because $\nu+1 > \kappa$ and $\lambda-1 < 0$ by (2.46).

Next we consider (2.50). From (2.4), (2.7) and (2.47) we deduce

$$H(z) = Q_w(z) + \frac{1}{2\pi i}\int_L h_0(s)\left\{\sum_0^{\nu} C_j e^{i\delta_j s}\right\} z^s\, ds$$
$$+ \frac{1}{2\pi i}\int_{L_1} h_0(s)\left\{\sum_\lambda^{\kappa} D_j e^{i\delta_j s}\right\} z^s\, ds + O(z^w)$$

for $|z| \to \infty$ uniformly on (2.44). Hence by (2.48) and (2.49)

$$(8.2)\quad H(z) = Q_w(z)+\sum_0^{\nu} C_j F(ze^{i\delta_j})$$
$$+\sum_\lambda^{\kappa} D_j\{F(ze^{i\delta_j})+P_w(ze^{i\delta_j})\}+O(z^w)$$

for $|z| \to \infty$ uniformly on (2.44). Because $C_j = 0$ if $\lambda \leqq j < 0$ and $D_j = 0$ if $\kappa < j \leqq \nu$ (cf. (4.7)) formula (8.2) may be written in the form (2.50).

The property (2.54) can be verified easily using (2.57) and (2.3). Further the proof of (2.51) resp. (2.55) can be given with lemma 7 where w and l are the same as above and

$f(s) || h_0(s)(\beta\mu^\mu)^{-s}$ resp. $h_0(s)(\beta\mu^\mu)^{-s}e^{-\mu\pi i s}/\sin \pi(\mu s+\alpha)$,
$z || \beta\mu^\mu z$ resp. $\beta\mu^\mu e^{2\pi i\mu} z$.

The assumptions of lemma 7 are satisfied now on account of lemma 2 with (3.25), (3.27) and $N = 0$, the choice of w and l and (2.54). The assertions of lemma 7 now imply (2.51) resp. (2.55) for $|z| \to \infty$ uniformly on (5.14) resp. (2.53). The proof of (2.52) and (2.56) proceeds in the same way; instead of (2.48) and lemma 7 here (2.49) and lemma 7a have to be used.

The deduction of (2.62) from (2.60) for $\arg z = \frac{1}{2}\mu\pi+\varepsilon$ is similar to the proofs given above: again the lemmas 7, 7a and 2 and (2.54) are used.

The proof of (2.63) with (2.64) for $\arg z = \frac{1}{2}\mu\pi+\varepsilon$ can be given in exactly the same way as in § 7.2 where (2.31) with (2.32) has been derived from (2.29) and (2.30).

Next we consider the properties of $r(t)$ (cf. (2.64)) for $t > 1$. By lemma 2 the integrand in (2.64) has a finite number of poles s with $\operatorname{Re} s \geqq w$. By (2.65) the integral in (2.64) tends to zero for $t > 1$ and $R \to \infty$ if the path of integration between $w-iR$ and $w+iR$ is replaced by the right part of the circle $|z-w| = R$ between these points. Hence if $t > 1$ then $r(t) = r^*(t)$ where

$$(8.3) \qquad r^*(t) = (-1)^N \int_A r_N(s) t^{2-\mu s-\alpha-N} \frac{ds}{(1-\mu s-\alpha-N)_2}.$$

Here A is a finite contour in the halfplane $\operatorname{Re} s > w$ which encloses the poles s with $\operatorname{Re} s > w$ of the integrand in (8.3) while these poles are lying to the right of A. We see that $r^*(t)$ is analytic in t for $t \neq 0$, that $r^*(t)$ is in general multiple-valued and that

$$r^*(t) = t^{2-\mu w-\operatorname{Re}\alpha-N} O(1)$$

for $|t| \to \infty$ uniformly on $|\arg t| \leqq \pi$.

Now we can deduce estimates for the function $\tau(z)$ defined by (2.60). Put

$$(8.4) \qquad \begin{cases} \tau_1(z) = \int_0^1 r(t) \exp(z^{1/\mu} t) dt - \int_{\frac{1}{2}}^1 r^*(t) \exp(z^{1/\mu} t) dt, \\ \tau_2(z) = \int_{\frac{1}{2}}^\infty r^*(t) \exp(z^{1/\mu} t) dt \end{cases}$$

if $\arg z = \frac{1}{2}\mu\pi+\varepsilon$. Then by (2.63)

$$(8.5) \qquad \tau(z) = (ze^{-\mu\pi i})^{(3-\alpha-N)/\mu} \{\tau_1(z)+\tau_2(z)\}$$

if $\arg z = \frac{1}{2}\mu\pi+\varepsilon$. On account of (2.66) and (2.26) the function $\tau_1(z)$ can be continued analytically for $z \neq 0$ and

(8.6) $$\tau_1(z) = O(\exp z^{1/\mu})+O(1)$$

for $|z| \to \infty$ uniformly on (2.53). Because $\tau(z)$ is analytic for $z \neq 0$ (cf. (2.60)), also $\tau_2(z)$ can be continued analytically for $z \neq 0$.
Now to $\tau_2(z)$ we apply lemma 6 (cf. § 5) with

$$w = \tfrac{1}{2},\ l = 0,\ f(s) = r^*(s),\ z\|\exp z^{1/\mu},\ \arg z\|\operatorname{Im} z^{1/\mu}.$$

In view of the properties of $r^*(s)$ the assumptions of lemma 6 are satisfied. So if

(8.7) $$\tfrac{1}{2}\mu\pi-\varepsilon \leqq \arg z \leqq \tfrac{1}{2}\mu\pi+\varepsilon$$

then $\operatorname{Im} z^{1/\mu} \geqq |z^{1/\mu}| \cos(\varepsilon/\mu)$ and by lemma 6

(8.8) $$\begin{aligned}\tau_2(z) &= \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} r^*(t)\exp(z^{1/\mu}t)dt\\ &= -\int_{-\frac{1}{2}}^{-\frac{1}{2}-i\infty} r^*(-t)\exp(-z^{1/\mu}t)dt.\end{aligned}$$

Further by (5.4)

(8.9) $$\tau_2(z) = O(\exp \tfrac{1}{2}z^{1/\mu})$$

for $|z| \to \infty$ uniformly on (8.7).
Next we apply lemma 6a to the last integral in (8.8) choosing

(8.10) $$w = -\tfrac{1}{2},\ l = \tfrac{1}{2}tg(\varepsilon/\mu),\ f(s) = -r^*(-s),\ z\|\exp(-z^{1/\mu}).$$

Then the integral (5.24) is equal to $\tau_2(z)$. The properties of $r^*(s)$ guarantee that the assumptions of lemma 6a are satisfied. So

(8.11) $$\tau_2(z) = -\int_{L_1} r^*(-t)\exp(-z^{1/\mu}t)dt$$

if

(8.12) $$|\arg z| \leqq \tfrac{1}{2}\mu\pi-\varepsilon$$

since then $|\exp(-z^{1/\mu})| < 1$. On the vertical part of L_1 we have $\operatorname{Re} s = -\frac{1}{2}$, $0 \geqq \operatorname{Im} s \geqq -\frac{1}{2}\operatorname{tg}(\varepsilon/\mu)$. So if (8.12) holds then on this part of L_1

$$\operatorname{Re}(-z^{1/\mu}s) \leqq \tfrac{1}{2}\operatorname{Re} z^{1/\mu}+\tfrac{1}{2}|\operatorname{Im} z^{1/\mu}|\operatorname{tg}(\varepsilon/\mu) \leqq \operatorname{Re} z^{1/\mu}.$$

It is now easy to deduce from (8.11) that

(8.13) $$\tau_2(z) = O(\exp z^{1/\mu})$$

for $|z| \to \infty$ uniformly on (8.12).
Finally we estimate $\tau_2(z)$ on the remaining part of the sector (2.53). On account of (8.11) we have

(8.14) $$\tau_2(z) = \left(\int_L - \int_{L_1}\right) r^*(-t) \exp(-z^{1/\mu}t)dt - \int_L r^*(-t) \exp(-z^{1/\mu}t)dt$$

if (8.12) holds. Here L is defined in lemma 6 with (8.10). According to the formula of Hankel for the gamma-function we have

$$\left(\int_L - \int_{L_1}\right) (-t)^\zeta \exp(-z^{1/\mu}t)dt = \frac{2\pi i}{\Gamma(-\zeta)} z^{(-\zeta-1)/\mu}$$

if (8.12) is fulfilled. This combined with (8.14) and (8.3) leads to

$$\tau_2(z) = 2\pi i(-1)^N \int_A r_N(s) z^{s+(\alpha+N-3)/\mu} \frac{ds}{\Gamma(\mu s+\alpha+N)} - \int_L r^*(-s) \exp(-z^{1/\mu}s)ds.$$

The first term in the righthand side can be calculated using the definition of A, (3.25) and (2.17). The second integral can be estimated using lemma 6 with (8.10) in the same way as has been done above. Using (5.4) on

(8.15) $$-\tfrac{1}{2}\mu\pi-\varepsilon \leqq \arg z \leqq -\tfrac{1}{2}\mu\pi+\varepsilon$$

we obtain

(8.16) $$\tau_2(z) = 4\pi^2(-1)^N z^{(\alpha+N-3)/\mu} P_w(\beta^{-1}\mu^{-\mu}z)+O(\exp \tfrac{1}{2}z^{1/\mu})$$

for $|z| \to \infty$ uniformly on (8.15).

Using the properties derived above we prove

LEMMA 8.

Suppose the number μ defined by (1.8) *is positive. Let ε be a constant so that $0 < \varepsilon < \frac{1}{2}\mu\pi$ and let N be a non-negative integer and w be a real number so that* (2.1) *holds and*

(8.17) $$\begin{cases} N > -1-\operatorname{Re}\alpha-\mu \operatorname{Re} b_j/\beta_j & (j = 1, \ldots, m), \\ (-2-\operatorname{Re}\alpha-N)/\mu < w < (-1-\operatorname{Re}\alpha-N)/\mu. \end{cases}$$

$F(z)$ and $P_w(z)$ are defined by (2.48) *and* (2.17). *Further we use definition* III (*cf.* § 4.4). *Then*

(8.18) $$F(z) = E_N(z)$$

for $|z| \to \infty$ *uniformly on* (8.12),

(8.19) $$F(z) = E_N(z)+O(z^{w+3/\mu})$$

for $|z| \to \infty$ *uniformly on* (8.7),

$$F(z) = E_N(z) - P_w(z) + O(z^{w+3/\mu}) \tag{8.20}$$

for $|z| \to \infty$ *uniformly on* (8.15), *and* (2.51) *resp.* (2.52) *holds for* $|z| \to \infty$ *uniformly on* (5.14) *resp.* (5.29).

PROOF: According to (2.61), (8.5), (8.6), (8.9) and (8.13) we have

$$F(\beta^{-1}\mu^{-\mu}z) = \frac{1}{2\pi i\mu}\exp(z^{1/\mu})\left\{\sum_0^{N-1} A_j z^{(1-\alpha-j)/\mu} + O(z^{(3-\alpha-N)/\mu})\right\} + O(z^w) + O(z^{(3-\alpha-N)/\mu}) \tag{8.21}$$

for $|z| \to \infty$ uniformly on (8.7) and on (8.12). Here w has to satisfy only (2.1), (2.2), (2.26) and (2.45) while further N is an arbitrary non-negative integer. Now we replace N by $N+2$ in (8.21) and we assume that w and N satisfy (2.1) and (8.17). Then (2.2), (2.45) and (2.26) with $N||N+2$ are fulfilled. Further we apply (7.8) with $j = N$ and $j = N+1$; this formula of course holds also on (2.53). Then we obtain from (8.21)

$$F(\beta^{-1}\mu^{-\mu}z) = \frac{1}{2\pi i\mu}\exp(z^{1/\mu})\left\{\sum_0^{N-1} A_j z^{(1-\alpha-j)/\mu} + O(z^{(1-\alpha-N)/\mu})\right\} + O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on (8.7) and on (8.12). In view of definition III this can be written as (8.19) for $|z| \to \infty$ uniformly on (8.7) and on (8.12). If (8.12) holds then it is easy to see with lemma 5a from § 4.5 that instead of (8.19) also (8.18) holds for $|z| \to \infty$ uniformly on (8.12).

In an analogous way as above we deduce from (2.61), (8.5), (8.6) and (8.16) that for $|z| \to \infty$ uniformly on (8.15) we have (8.20). The proofs of (2.51) and (2.52) have been given above.

§ 9. The exponentially infinite asymptotic expansions and the asymptotic expansions in the transitional regions for $H(z)$

§ 9.1. In this section the exponentially infinite asymptotic expansions of $H(z)$ will be deduced from (2.50) and lemma 8 in § 8. We assume that the conditions for μ, N and w in § 8 are satisfied. Further in the proofs of the theorems we apply (2.50) on (2.44) where r is an integer, ε_0 is a positive constant such that the sectors referred to in the theorem are subsectors of (2.44), and λ and ν are constant integers satisfying (2.46).

We always use the notation of the definitions I—IV in § 4.

Theorem 5.

If μ is positive and ε is a constant so that

$$0 < \varepsilon < \tfrac{1}{4}\min(\mu\pi,\ \delta_r - \delta_{r-1},\ \delta_{r+1} - \delta_r) \tag{9.1}$$

then the exponentially infinite asymptotic expansion

$$H(z) \sim (C_r + D_r)E(ze^{i\delta_r}) \tag{9.2}$$

holds for $|z| \to \infty$ uniformly on [1])

$$\varepsilon - \tfrac{1}{2}\min(\mu\pi,\ \delta_{r+1} - \delta_r) \leqq \arg z + \delta_r \leqq -\varepsilon + \tfrac{1}{2}\min(\mu\pi,\ \delta_r - \delta_{r-1}). \tag{9.3}$$

Proof: Since $\delta_j < \delta_h$ if $j < h$ (cf. definition I in § 4.2) the last side of (9.1) is positive and consequently (9.1) can be fulfilled. Further by (9.3)

$$|\arg z + \delta_r| \leqq \tfrac{1}{2}\mu\pi - \varepsilon. \tag{9.4}$$

Let j be an integer such that $\lambda \leqq j \leqq \nu$, $j \neq r$. Then we show first that

$$|\arg z + \delta_j| \geqq |\arg z + \delta_r| + 2\varepsilon. \tag{9.5}$$

If $j < r$ and $\arg z + \delta_r \leqq 0$ then by (9.1)

$$\arg z + \delta_j \leqq \arg z + \delta_{r-1} < \arg z + \delta_r - 2\varepsilon = -|\arg z + \delta_r| - 2\varepsilon$$

and (9.5) follows. If $j < r$ and $\arg z + \delta_r > 0$, then $|\arg z + \delta_r| = \arg z + \delta_r$ and $2(\arg z + \delta_r) \leqq -2\varepsilon + \delta_r - \delta_{r-1}$ (this is a consequence of (9.3)), therefore

$$\begin{aligned}\arg z + \delta_j \leqq \arg z + \delta_{r-1} &\leqq \delta_{r-1} - \delta_r + 2(\arg z + \delta_r) - (\arg z + \delta_r)\\ &\leqq -2\varepsilon - |\arg z + \delta_r|\end{aligned}$$

and (9.5) follows. So (9.5) is proved for $j < r$. For $j > r$ the proof runs in the same manner.

A corollary of (9.5) is that $|\arg z + \delta_j| \geqq 2\varepsilon$ for $j \neq r$. So if j is an integer the sector (9.3) can be divided into subsectors $\Omega_1, \ldots, \Omega_4$ with vertex $z = 0$, which are independent of z but dependent of j and which cover (9.3) while on Ω_1:

$$2\varepsilon \leqq \arg z + \delta_j \leqq \mu\pi - 2\varepsilon, \tag{9.6}$$

on Ω_2:

[1]) For convenience we use notations like (9.3) to indicate a sector in the z-plane.

$$\text{(9.7)} \qquad \arg z+\delta_j \geqq \mu\pi-2\varepsilon,$$

on Ω_3:

$$\text{(9.8)} \qquad -\mu\pi+2\varepsilon \leqq \arg z+\delta_j \leqq -2\varepsilon,$$

while on Ω_4:

$$\text{(9.9)} \qquad \arg z+\delta_j \leqq -\mu\pi+2\varepsilon.$$

However, it is possible that for example Ω_4 does not occur and that Ω_1, Ω_2 and Ω_3 already cover the sector (9.3). Now we apply lemma 8 with $\varepsilon || \frac{1}{2}\mu\pi-2\varepsilon$ to

$$(C_j+D_j)F(ze^{i\delta_j});$$

on Ω_1, Ω_2, Ω_3, Ω_4 we use (8.19), (2.51), (8.20), (2.52) while if $j||r$ we have (8.18) on (9.3). So

$$\text{(9.10)} \quad (C_r+D_r)F(ze^{i\delta_r})+(C_j+D_j)F(ze^{i\delta_j})$$
$$= (C_r+D_r)E_N(ze^{i\delta_r})+(C_j+D_j)E_N(ze^{i\delta_j})+O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on Ω_1,

$$\text{(9.11)} \quad (C_r+D_r)F(ze^{i\delta_r})+(C_j+D_j)F(ze^{i\delta_j})$$
$$= (C_r+D_r)E_N(ze^{i\delta_r})+O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on Ω_2,

$$\text{(9.12)} \quad (C_r+D_r)F(ze^{i\delta_r})+(C_j+D_j)F(ze^{i\delta_j}) = (C_r+D_r)E_N(ze^{i\delta_r})$$
$$+(C_j+D_j)E_N(ze^{i\delta_j})-(C_j+D_j)P_w(ze^{i\delta_j})+O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on Ω_3 and

$$\text{(9.13)} \quad (C_r+D_r)F(ze^{i\delta_r})+(C_j+D_j)F(ze^{i\delta_j})$$
$$= (C_r+D_r)E_N(ze^{i\delta_r})-(C_j+D_j)P_w(ze^{i\delta_j})+O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on Ω_4. From the last four formulae, (9.4) and (9.5) we easily deduce with lemma 5a in § 4.5 that

$$\text{(9.14)} \quad (C_r+D_r)F(ze^{i\delta_r})+(C_j+D_j)F(ze^{i\delta_j}) = (C_r+D_r)E_N(ze^{i\delta_r})$$

for $|z| \to \infty$ uniformly on $\Omega_1, \ldots, \Omega_4$ and henceforth on (9.3). Combining this with (2.50) on (2.44), $\varepsilon_0 = \varepsilon$ we see that

$$\text{(9.15)} \quad H(z) = Q_w(z)+\sum_{\lambda}^{\kappa} D_j P_w(ze^{i\delta_j})+(C_r+D_r)E_N(ze^{i\delta_r})+O(z^w)$$

for $|z| \to \infty$ uniformly on (9.3). Since (9.4) holds we may apply lemma 5a in § 4.5 to (9.15). Then we obtain

$$H(z) = (C_r+D_r)E_N(ze^{i\delta_r})$$

for $|z| \to \infty$ uniformly on (9.3). As N is an arbitrary large integer this implies (9.2) (cf. definition III in § 4.4). Further from (9.4) it follows that the asymptotic expansion (9.2) is exponentially infinite on (9.3) (cf. § 4.4).

THEOREM 6.

Suppose μ is positive, $\delta_r-\delta_{r-1} < \mu\pi$ and ε is a constant so that

$$(9.16)\quad 0<\varepsilon<\tfrac{1}{4}\min(\delta_{r-1}-\delta_{r-2},\ \delta_r-\delta_{r-1},\ \delta_{r+1}-\delta_r,\ \mu\pi-\delta_r+\delta_{r-1}).$$

Then the exponentially infinite asymptotic expansion

$$(9.17)\qquad H(z) \sim (C_r+D_r)E(ze^{i\delta_r})+(C_{r-1}+D_{r-1})E(ze^{i\delta_{r-1}})$$

holds for $|z| \to \infty$ uniformly on

$$(9.18)\qquad \tfrac{1}{2}(\delta_r-\delta_{r-1})-\varepsilon \leqq \arg z+\delta_r \leqq \tfrac{1}{2}(\delta_r-\delta_{r-1})+\varepsilon.$$

PROOF: In (2.44) we take $\varepsilon_0 = \varepsilon$. Then (9.18) implies (2.44). From (9.16), (9.18) and $\delta_r-\delta_{r-1} < \mu\pi$ we deduce

$$(9.19)\qquad 0 < \arg z+\delta_r \leqq \tfrac{1}{2}\mu\pi-\varepsilon$$

and

$$(9.20)\qquad -\tfrac{1}{2}\mu\pi+\varepsilon \leqq -\tfrac{1}{2}(\delta_r-\delta_{r-1})-\varepsilon \leqq \arg z+\delta_{r-1} \leqq -\tfrac{1}{2}(\delta_r-\delta_{r-1})+\varepsilon < 0.$$

So if $j = r$ or $j = r-1$ then by lemma 8, especially (8.18),

$$(9.21)\qquad F(ze^{i\delta_j}) = E_N(ze^{i\delta_j})$$

for $|z| \to \infty$ uniformly on (9.18).

Suppose there exists an integer j such that $\lambda \leqq j \leqq \nu$, $j \neq r$, $j \neq r-1$. Then we prove (9.5) for such a number j. If $j \leqq r-2$ then by (9.16) and (9.20)

$$\arg z+\delta_j \leqq \arg z+\delta_{r-2} \leqq \arg z+\delta_{r-1}-4\varepsilon \leqq -\tfrac{1}{2}(\delta_r-\delta_{r-1})-3\varepsilon,$$

so by (9.18) and (9.19)

$$\arg z+\delta_j \leqq -(\arg z+\delta_r)-2\varepsilon = -|\arg z+\delta_r|-2\varepsilon$$

and (9.5) follows. If $j \geqq r+1$ then by (9.16) and (9.19)

$$\arg z+\delta_j \geqq \arg z+\delta_{r+1} \geqq \arg z+\delta_r+4\varepsilon = |\arg z+\delta_r|+4\varepsilon$$

and (9.5) follows.

As (9.4) holds (cf. (9.19)) and (9.5) is satisfied for the value of j we consider we may derive in the same way as in the proof of theorem 5 that (9.14) holds for $|z| \to \infty$ uniformly on (9.18) by constructing subsectors $\Omega_1, \ldots, \Omega_4$ of (9.18) and using lemma 8 of § 8 and lemma 5a of § 4.5 (cf. (9.6)—(9.13)). Application of (9.14) for all values of j with $\lambda \leqq j \leqq \nu$, $j \neq r$, $j \neq r-1$ and (2.50) shows that

$$(9.22) \quad H(z)-(C_{r-1}+D_{r-1})F(ze^{i\delta_{r-1}}) \\ = Q_w(z)+\sum_{\lambda}^{\kappa} D_j P_w(ze^{i\delta_j})+(C_r+D_r)E_N(ze^{i\delta_r})+O(z^w)$$

for $|z| \to \infty$ uniformly on (9.18). If there do not exist integers j such that $\lambda \leqq j \leqq \nu$, $j \neq r$, $j \neq r-1$ then $\lambda = r-1$ and $\nu = r$ (cf. (2.46)) and in this case (9.22) follows from (2.50) and (9.21). So (9.22) holds in all cases.

Like at (9.15) we can deduce from lemma 5a and (9.19) that

$$H(z)-(C_{r-1}+D_{r-1})F(ze^{i\delta_{r-1}}) = (C_r+D_r)E_N(ze^{i\delta_r})$$

for $|z| \to \infty$ uniformly on (9.18). Combining this with (9.21) and definition III in § 4.4 we obtain (9.17) for $|z| \to \infty$ uniformly on (9.18). By (9.19) and (9.20) the asymptotic expansion (9.17) is exponentially infinite on (9.18) (cf. § 4.4).

§ **9.2.** We now consider the asymptotic expansions in the remaining sectors. In most cases these sectors are transitional regions: the righthand side of the asymptotic expansion contains two asymptotic series, each of which represents the function asymptotically in a part of the region like in (9.17).

In the following $\sum_s^t \ldots = 0$ if $s > t$.

THEOREM 7.

If μ *is positive,* $\delta_r - \delta_{r-1} = \mu\pi$ *and* ε *is a constant so that*

$$(9.23) \qquad 0 < \varepsilon < \tfrac{1}{4}\min(\delta_{r-1}-\delta_{r-2}, \delta_{r+1}-\delta_r)$$

then

$$(9.24) \quad H(z) \sim Q(z)+\sum_{r}^{\kappa} D_j P(ze^{i\delta_j})-\sum_{0}^{r-1} C_j P(ze^{i\delta_j}) \\ +(C_r+D_r)E(ze^{i\delta_r})+(C_{r-1}+D_{r-1})E(ze^{i\delta_{r-1}})$$

for $|z| \to \infty$ *uniformly on*

$$(9.25) \qquad \tfrac{1}{2}\mu\pi-\varepsilon \leqq \arg z+\delta_r \leqq \tfrac{1}{2}\mu\pi+\varepsilon.$$

Here κ *is determined in definition* II *in* § 4.2. [2])

[2]) Special cases with $\delta_0 > 0$, $\delta_0 = \delta_{-1} = \frac{1}{2}\mu\pi$ have been considered by Fox and Chandrasekharan and Narasimhan (cf. [18] p. 417 and [9] p. 100 and p. 117). They determine the behaviour of $H(z)$ for $z > 0$ and $z \to \infty$.

Proof: We take $\varepsilon_0 = \varepsilon$ in (2.44). Then (9.25) implies (2.44) and so (2.50) may be applied. Since $\delta_r - \delta_{r-1} = \mu\pi$ and (9.25) holds we have

$$(9.26) \qquad -\tfrac{1}{2}\mu\pi - \varepsilon \leqq \arg z + \delta_{r-1} \leqq -\tfrac{1}{2}\mu\pi + \varepsilon.$$

Suppose now $j \leqq r-2$. Then by (9.23) and (9.26)

$$\arg z + \delta_j \leqq \arg z + \delta_{r-2} \leqq \arg z + \delta_{r-1} - 2\varepsilon \leqq -\tfrac{1}{2}\mu\pi - \varepsilon$$

and so by lemma 8, especially (2.52),

$$(9.27) \qquad F(ze^{i\delta_j}) = -P_w(ze^{i\delta_j}) + O(z^w)$$

for $|z| \to \infty$ uniformly on (9.25) if $j \leqq r-2$. If $j \geqq r+1$ then

$$\arg z + \delta_j \geqq \arg z + \delta_{r+1} \geqq \arg z + \delta_r + 2\varepsilon \geqq \tfrac{1}{2}\mu\pi + \varepsilon$$

in view of (9.23) and (9.25). Hence by lemma 8, (2.51):

$$(9.28) \qquad F(ze^{i\delta_j}) = O(z^w)$$

for $|z| \to \infty$ uniformly on (9.25). To $F(ze^{i\delta_r})$ resp. $F(ze^{i\delta_{r-1}})$ we may apply lemma 8: (8.19) and (8.20) on account of (9.25) and (9.26). From this, (9.27), (9.28) and (2.50) we obtain

$$(9.29) \quad \begin{aligned} H(z) = Q_w(z) &+ \sum_{\lambda}^{\kappa} D_j P_w(ze^{i\delta_j}) - \sum_{\lambda}^{r-2} (C_j + D_j) P_w(ze^{i\delta_j}) \\ &+ (C_r + D_r) E_N(ze^{i\delta_r}) + (C_{r-1} + D_{r-1}) E_N(ze^{i\delta_{r-1}}) \\ &- (C_{r-1} + D_{r-1}) P_w(ze^{i\delta_{r-1}}) + O(z^{w+3/\mu}) \end{aligned}$$

for $|z| \to \infty$ uniformly on (9.25). Using (4.7) and (2.46) we see that (9.29) may be written in the form

$$(9.30) \quad \begin{aligned} H(z) = Q_w(z) &+ \sum_{r}^{\kappa} D_j P_w(ze^{i\delta_j}) - \sum_{0}^{r-1} C_j P_w(ze^{i\delta_j}) \\ &+ (C_r + D_r) E_N(ze^{i\delta_r}) + (C_{r-1} + D_{r-1}) E_N(ze^{i\delta_{r-1}}) + O(z^{w+3/\mu}) \end{aligned}$$

for $|z| \to \infty$ uniformly on (9.25). In view of the definitions III and IV in § 4 this implies (9.24).

Remark 1. If $\arg z + \delta_r = \frac{1}{2}\mu\pi$ then the modules of the terms in the formal series in (9.24) can be written like $|\xi z^\eta (\log z)^k|$ where ξ and η are complex constants and k is a non-negative integer. This follows from the definitions III and IV in § 4 and from

$$|\exp(\beta\mu^\mu z e^{i\delta_j})^{1/\mu}| = 1 \ \text{ if } \ j = r, \ r-1 \ \text{ and } \ \arg z + \delta_r = \tfrac{1}{2}\mu\pi$$

(use $\delta_r - \delta_{r-1} = \mu\pi$).

An analogous remark can be made at theorem 8 and theorem 9.

REMARK 2. If in (9.24)

$$(9.31) \qquad Q(z)+\sum_{r}^{\kappa} D_j P(ze^{i\delta_j})-\sum_{0}^{r-1} C_j P(ze^{i\delta_j})$$

is a formal series of zeros then in theorem 7 we may replace (9.24) by (9.17).

PROOF: By (9.30), (9.31), definition IV in § 4.6, (2.5) and (2.17) we have

$$H(z) = (C_r+D_r)E_N(ze^{i\delta_r})+(C_{r-1}+D_{r-1})E_N(ze^{i\delta_{r-1}})+O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on (9.25). From (8.17) and definition III in § 4.4 we deduce that if $N \geqq 2$

$$O(z^{w+3/\mu})+(C_r+D_r)E_N(ze^{i\delta_r}) = (C_r+D_r)E_{N-2}(ze^{i\delta_r})$$

for $|z| \to \infty$ uniformly on $\frac{1}{2}\mu\pi-\varepsilon \leqq \arg z+\delta_r \leqq \frac{1}{2}\mu\pi$ and the same formula with $r||r-1$ for $|z| \to \infty$ uniformly on $-\frac{1}{2}\mu\pi \leqq \arg z+\delta_{r-1} \leqq -\frac{1}{2}\mu\pi+\varepsilon$ or $\frac{1}{2}\mu\pi \leqq \arg z+\delta_r \leqq \frac{1}{2}\mu\pi+\varepsilon$ (since $\delta_r-\delta_{r-1} = \mu\pi$). From the preceding formulae we deduce

$$H(z) = (C_r+D_r)E_{N-2}(ze^{i\delta_r})+(C_{r-1}+D_{r-1})E_{N-2}(ze^{i\delta_{r-1}})$$

for $|z| \to \infty$ uniformly on (9.25). Consequently (9.17) holds for $|z| \to \infty$ uniformly on (9.25).

THEOREM 8.

If μ is positive, $\delta_r-\delta_{r-1} > \mu\pi$ and ε is a constant so that

$$(9.32) \qquad 0 < \varepsilon < \tfrac{1}{4}\min(\delta_r-\delta_{r-1}-\mu\pi,\ \delta_{r+1}-\delta_r)$$

then

$$(9.33) \quad H(z) \sim Q(z)+\sum_{r}^{\kappa} D_j P(ze^{i\delta_j})-\sum_{0}^{r-1} C_j P(ze^{i\delta_j})+(C_r+D_r)E(ze^{i\delta_r})$$

for $|z| \to \infty$ uniformly on (9.25).

PROOF: In (2.44) we choose $\varepsilon_0 = \varepsilon$. Then (9.25) implies (2.44) and (2.50) may be applied on (9.25). From (9.32), (9.25) and $\delta_r-\delta_{r-1} > \mu\pi$ we deduce that if $j < r$ then

$$\arg z+\delta_j \leqq \arg z+\delta_{r-1} < \arg z+\delta_r-\mu\pi-2\varepsilon \leqq -\tfrac{1}{2}\mu\pi-\varepsilon$$

and so by lemma 8: (2.52) we have (9.27) for $|z| \to \infty$ uniformly on (9.25). If $j > r$, then by (9.32), (9.25): $\arg z+\delta_j \geqq \frac{1}{2}\mu\pi+\varepsilon$ and consequently (9.28) holds for $|z| \to \infty$ uniformly on (9.25) (cf. (2.51)). By (9.25) we may apply (8.19) to $F(ze^{i\delta_r})$. So by (2.50)

$$H(z) = Q_w(z) + \sum_{\lambda}^{\kappa} D_j P_w(ze^{i\delta_j}) - \sum_{\lambda}^{r-1} (C_j+D_j)P_w(ze^{i\delta_j})$$
$$+(C_r+D_r)E_N(ze^{i\delta_r})+O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on (9.25). On account of (4.7) and (2.46) this is equivalent to

$$H(z) = Q_w(z) + \sum_{r}^{\kappa} D_j P_w(ze^{i\delta_j})$$
$$- \sum_{0}^{r-1} C_j P_w(ze^{i\delta_j}) + (C_r+D_r)E_N(ze^{i\delta_r}) + O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on (9.25) and this implies the assertion of the theorem in view of the definitions in § 4.

Remark: If $r = n = 0$, $\delta_0 > 0$ then (9.33) may be replaced by (7.11).

Proof: By the definition II in § 4.2 and (4.7) the datum $\delta_0 > 0$ implies $\kappa = -1$, $D_0 = 0$, $C_0 = c_0$. So the righthand side of (9.33) becomes

$$H(z) \sim Q(z) + c_0 E(ze^{i\delta_0}).$$

Therefore

$$H(z) = c_0 E_N(ze^{i\delta_0}) + O(z^{(1-\alpha-N)/\mu}) \tag{9.34}$$

for $|z| \to \infty$ uniformly on (9.25) and for arbitrary integers $N \geqq 0$ (cf. definitions III and IV in § 4 and use $n = 0$). On the part of (9.25) where

$$\mathrm{Re}(ze^{i\delta_0})^{1/\mu} \geqq -1 \tag{9.35}$$

we have uniformly for $|z| \to \infty$

$$O(z^{(1-\alpha-N)/\mu}) + c_0 E_N(ze^{i\delta_0}) = c_0 E_N(ze^{i\delta_0}) \tag{9.36}$$

(cf. (4.12)). Now (9.34) and (9.36) imply (7.11) for $|z| \to \infty$ uniformly on the part of (9.25) where (9.35) holds. On the part of (9.25) where

$$\mathrm{Re}(ze^{i\delta_0})^{1/\mu} \leqq -1$$

we also have (7.11) uniformly for $|z| \to \infty$ according to theorem 4 (cf. (7.6) and (7.18)). Hence (7.11) holds for $|z| \to \infty$ uniformly on (9.25).

Theorem 9.

If μ is positive, $\delta_r - \delta_{r-1} > \mu\pi$ and ε is a constant so that

$$(9.37) \qquad 0 < \varepsilon < \tfrac{1}{4}\min(\delta_{r-1}-\delta_{r-2},\ \delta_r-\delta_{r-1}-\mu\pi)$$

then

$$(9.38) \qquad H(z) \sim Q(z)+\sum_r^{\kappa} D_j P(ze^{i\delta_j}) - \sum_0^{r-1} C_j P(ze^{i\delta_j})+(C_{r-1}+D_{r-1})E(ze^{i\delta_{r-1}})$$

for $|z| \to \infty$ *uniformly on*

$$(9.39) \qquad -\tfrac{1}{2}\mu\pi-\varepsilon \leqq \arg z+\delta_{r-1} \leqq -\tfrac{1}{2}\mu\pi+\varepsilon.$$

Proof: We choose $\varepsilon_0 = \varepsilon+\frac{1}{2}(\delta_r-\delta_{r-1}-\mu\pi)$ in (2.44). Then (9.39) implies (2.44) and so (2.50) may be applied.

If $j \leqq r-2$ then on account of (9.37) and (9.39)

$$\arg z+\delta_j \leqq \arg z+\delta_{r-1}-(\delta_{r-1}-\delta_{r-2}) \leqq -\tfrac{1}{2}\mu\pi-\varepsilon$$

and so (9.27) holds for $|z| \to \infty$ uniformly on (9.39) by lemma 8: (2.52). If $j \geqq r$ then according to (9.37) and (9.39)

$$\arg z+\delta_j \geqq \arg z+\delta_{r-1}+(\delta_r-\delta_{r-1}) \geqq \tfrac{1}{2}\mu\pi+\varepsilon$$

and so (9.28) holds for $|z| \to \infty$ uniformly on (9.39) by lemma 8: (2.51). Finally to $F(ze^{i\delta_{r-1}})$ we may apply lemma 8, especially (8.20), on (9.39). Combining these formulae with (2.50) we get

$$H(z) = Q_w(z)+\sum_{\lambda}^{\kappa} D_j P_w(ze^{i\delta_j})-\sum_{\lambda}^{r-2}(C_j+D_j)P_w(ze^{i\delta_j}) -(C_{r-1}+D_{r-1})P_w(ze^{i\delta_{r-1}})+(C_{r-1}+D_{r-1})E_N(ze^{i\delta_{r-1}})+O(z^{w+3/\mu})$$

and so by (4.7) and (2.46)

$$H(z) = Q_w(z)+\sum_r^{\kappa} D_j P_w(ze^{i\delta_j})-\sum_0^{r-1} C_j P_w(ze^{i\delta_j}) +(C_{r-1}+D_{r-1})E_N(ze^{i\delta_{r-1}})+O(z^{w+3/\mu})$$

for $|z| \to \infty$ uniformly on (9.39). This implies (9.38) for $|z| \to \infty$ uniformly on (9.39).

Remark: If $r = n = 0$, $\delta_0 > 0$ then (9.38) may be replaced by (7.12). The proof of this assertion runs in the same way as the proof of the remark after theorem 8.

§ 9.3. We now give a survey showing the theorems which have to be applied in order to obtain the asymptotic expansions of $H(z)$ in the case $\mu > 0$ on a given general sector. On

(9.40) $$0 \leqq \arg z + \delta_r \leqq \tfrac{1}{2}(\delta_r - \delta_{r-1}) + \varepsilon$$

the asymptotic behaviour of $H(z)$ in the case $\mu > 0$ is contained in

i) theorems 5 and 6 if $\delta_r - \delta_{r-1} < \mu\pi$,
ii) theorems 5 and 7 if $\delta_r - \delta_{r-1} = \mu\pi$,
iii) theorems 5, 8 and 3 if $\delta_r - \delta_{r-1} > \mu\pi$.

On

(9.41) $$-\tfrac{1}{2}(\delta_{r+1} - \delta_r) + \varepsilon \leqq \arg z + \delta_r \leqq 0$$

we have to use

iv) theorem 5 if $\delta_{r+1} - \delta_r \leqq \mu\pi$,
v) theorem 5 and the theorems 9 and 3 if $\delta_{r+1} - \delta_r > \mu\pi$.

Here in the last two theorems we have to replace r by $r+1$. Sometimes in the cases iii) and v) theorem 3 has to be replaced by theorem 4. These special cases are mentioned in the assertion of theorem 3.

In (9.40) and (9.41) ε is a positive constant independent of z while ε is smaller than a positive number depending only on r but not on z. By varying r in (9.40) and (9.41) we see that the behaviour of $H(z)$ for $|z| \to \infty$ can be written down in any sector of the z-plane if $\mu > 0$ (for $\delta_{-r} \to -\infty$ and $\delta_r \to \infty$ if $r \to \infty$ by definition I of § 4.2).

If $\mu = 0$ then the analytic continuations and the behaviour near $z = \infty$ of $H(z)$ can be read off from theorem 2. Finally we remark that all values of m, n, p and q satisfying (1.4) have been considered. The numbers δ_r, however, depend on the choice of m, n, p and q.

§ 10. Another method to obtain the asymptotic expansions of $H(z)$ and some special cases of the theorems of § 9 and of § 6

§ 10.1. The most difficult part in the derivation of the asymptotic expansions in the theorems 4—9 is the estimation of the remainder terms, especially of the last integral in (2.27) and the function $\tau(z)$ in (2.60). These functions have been estimated with the help of the lemmas in § 5. This method is related to the method of indirect Abelian asymptotics (cf. G. Doetsch [13] II p. 41).

The estimation of the functions mentioned above can be done also by means of the method of steepest descents. In the following we give a sketch of the estimation of the last integral in (2.27) —

that is the function $\sigma(z)$ defined by the first equality in (2.29) — by means of this method.

We remind of the assumptions made at (2.27): $\mu > 0$, $n = 0$, $m = q$ and (2.22). So we treat again the exponentially small asymptotic expansions of $H(z)$ in the case $\mu > 0$, $n = 0$, $m = q$. We consider again (2.25), (2.27), (2.28) and (2.29). In (2.25) $H_0(z)$ is the special case of $H(z)$ with $\mu > 0$, $n = 0$, $m = q$. Further the function $h_2(s)$ occurring in (2.25) is defined by (2.24). For w and N we choose conditions different from those in § 2.2. N will be a non-negative integer such that (cf. (3.24)):

$$N > \tfrac{1}{2} - \operatorname{Re} \alpha, \tag{10.1}$$

while w satisfies (2.2) with $m = q$ and (2.26). The function $\rho_N(s)$ occurring in (2.27) and (2.29) is defined in lemma 3 in § 3.3. It can be verified that (2.25), (2.27) and (2.28) remain valid for (2.22). From (2.27) and (2.28) it follows that in order to estimate $H_0(z)$ we have to estimate $\sigma(z)$ defined by the first equality in (2.29). This will be done here on (7.9) where ε is a constant such that $0 < \varepsilon < \frac{1}{2}\mu\pi$.

First we remark that there exists a positive constant K_0 independent of $|z|$ and of $\arg z$ such that

$$\begin{cases} \operatorname{Re} z^{1/\mu} > -\mu \operatorname{Re} b_j/\beta_j \text{ for } j = 1, \ldots, q; \\ \operatorname{Re} z^{1/\mu} > N + \operatorname{Re} \alpha - 1 \end{cases} \tag{10.2}$$

for $|z| \geqq K_0$ and (7.9). Next we choose

$$w = -\mu^{-1} \operatorname{Re} z^{1/\mu} \tag{10.3}$$

in the case $|z| \geqq K_0$ and (7.9) hold. This is allowed since in view of (10.2) now (2.2) and (2.26) are fulfilled. Now w is negative on account of (7.9) and (10.3).

From lemma 3 in § 3.3 and the formula of Stirling (3.1) we easily deduce that there exist positive constants K_1 and K_2 independent of s such that if $|s| \geqq K_1$ and $|\arg(-s)| \leqq \frac{1}{2}\pi$ then

$$\begin{cases} |\rho_N(s)| \leqq K_2, \\ |\Gamma(1-\mu s-\alpha-N)| \leqq K_2 |s|^{\frac{1}{2}-\operatorname{Re}\alpha-N} \\ \qquad \cdot \exp\{-\mu \operatorname{Re} s \operatorname{Log} |\mu s| + \mu \operatorname{Re} s + \mu \operatorname{Im} s \arg(-s)\}. \end{cases} \tag{10.4}$$

There exists a positive constant K_3 independent of $|z|$ and $\arg z$ such that $\operatorname{Re} z^{1/\mu} \geqq \mu K_1$ if $|z| \geqq K_3$ and (7.9) hold, and such that $K_3 \geqq K_0$. Hence if $|z| \geqq K_3$, (7.9) is fulfilled and $\operatorname{Re} s = w$ (cf. (10.3)), then $|s| \geqq -w \geqq K_1$ and $|\arg(-s)| \leqq \frac{1}{2}\pi$, and consequently (10.4) holds.

So if $|z| \geqq K_3$ and (7.9) are satisfied then by (2.29) and (10.4):

$$(10.5)\quad |\sigma(z)| \leqq K_2^2 \int_{w-i\infty}^{w+i\infty} |s|^{\frac{1}{2}-\mathrm{Re}\,\alpha-N}$$
$$\cdot |z|^w \exp\{-\mu w \operatorname{Log}|\mu s| + \mu w + \mu \operatorname{Im} s(\arg(-s) - \arg z^{1/\mu})\} \cdot |ds|.$$

If $\mathrm{Re}\, s = w$ then $|s|^{\frac{1}{2}-\mathrm{Re}\,\alpha-N} \leqq (-w)^{\frac{1}{2}-\mathrm{Re}\,\alpha-N}$ by (10.1). Further we substitute in the integral in (10.5): $s = w + i(v + wx)$ where (cf. (103)):

$$(10.6)\qquad v = -\frac{1}{\mu} \operatorname{Im} z^{1/\mu}, \text{ so } z^{1/\mu} = -\mu(w+iv).$$

Then we may deduce from (10.5)

$$(10.7)\qquad |\sigma(z)| \leqq K_2^2 (-w)^{\frac{3}{2}-\mathrm{Re}\,\alpha-N} \int_{-\infty}^{\infty} e^{\mu w f(v,x)}\, dx$$

if $|z| \geqq K_3$ and (7.9) hold, where

$$(10.8)\quad \begin{aligned} f(v,x) = {} & -\tfrac{1}{2} \operatorname{Log}\{\mu^2(w^2 + (v+wx)^2)\} \\ & + (x + v/w) \operatorname{arctg}(x + v/w) + 1 \\ & + \tfrac{1}{2} \operatorname{Log}\{\mu^2(w^2+v^2)\} - (x+v/w) \operatorname{arctg}(v/w). \end{aligned}$$

The function $f(v, x)$ also depends on w. From (10.8) we derive $f(v, 0) = 1$ and

$$(10.9)\qquad \frac{\partial f(v,x)}{\partial x} = \operatorname{arctg}\left(x + \frac{v}{w}\right) - \operatorname{arctg}\frac{v}{w},$$

$$(10.10)\qquad \frac{\partial^2 f(v,x)}{\partial x^2} = \{1 + (x+v/w)^2\}^{-1}.$$

With this information about $f(v, x)$ we can attack the integral in (10.7) in an analogous way as in section 2.4 of A. Erdélyi [14].

We suppose further always $|z| \geqq K_3$ and (7.9). Let $\eta = \varepsilon/\mu$. Then according to (7.9) and (10.6)

$$(10.11)\qquad |v/w| \leqq \operatorname{cotg} \eta.$$

So if $|x| \leqq 1$ then by (10.10) and (10.11)

$$(10.12)\qquad \frac{\partial^2 f(v,x)}{\partial x^2} \geqq 2k$$

where k is a positive constant independent of z and of x. From (10.8), (10.9) and (10.12) we easily derive $f(v,x) \geqq 1 + kx^2$ for $|x| \leqq 1$. Consequently

(10.13) $$\int_{-1}^{1} \exp\{\mu w f(v, x)\}\, dx \leqq \sqrt{\pi/(-\mu k w)} \exp(\mu w)$$

for $|z| \geqq K_3$ and (7.9).

Using (10.9) and (10.11) we may infer that if $x \geqq 1$

$$\frac{\partial f(v, x)}{\partial x} \geqq \operatorname{arctg}\ (1+\operatorname{cotg}\eta)-(\tfrac{1}{2}\pi-\eta) > 0$$

and that if $x \leqq -1$

$$\frac{\partial f(v, x)}{\partial x} \leqq \operatorname{arctg}\ (-1-\operatorname{cotg}\eta)+(\tfrac{1}{2}\pi-\eta) < 0.$$

From these formulae and $f(v, 0) = 1$ we may deduce

(10.14) $$\int_{-\infty}^{-1} \exp(\mu w f(v, x))\, dx + \int_{1}^{\infty} \exp(\mu w f(v, x)) dx \leqq K_4 \exp(\mu w)/(-w)$$

for $|z| \geqq K_3$ and (7.9) where K_4 is a positive constant independent of w and of z.

The estimates (10.7), (10.13) and (10.14) imply

$$\sigma(z) = O(w^{1-\alpha-N} e^{\mu w})$$

for $|z| \to \infty$ uniformly on (7.9). On account of (10.3) and (7.9) this can be written as

(10.15) $$\sigma(z) = z^{(1-\alpha-N)/\mu} \exp(-z^{1/\mu}) O(1)$$

for $|z| \to \infty$ uniformly on (7.9). This estimate is sharper than the estimate in (2.35). Now (2.36) and the assertions in theorem 4 on (7.9), (7.13), (7.14) and (7.16) follow like in § 2.2 and § 7.

In an analogous way we may apply the method of steepest descents to estimate the function $\tau(z)$ in (2.60) on (8.12). In view of (2.59) this function plays the rôle of a part of the remainder term in the asymptotic expansion of $F(z)$, the auxiliary function defined by (2.48). The function $F(z)$ has been introduced because from the behaviour of $F(z)$ near $z = \infty$ we may deduce the behaviour of $H(z)$ near $z = \infty$ in most cases with μ positive by means of (2.50). The estimation of $\tau(z)$ by means of the method of steepest descents very much resembles the method used in the lemmas 5, 7, 8, 9 and 10 of Wright [33]. Therefore we do not sketch this method and we refer to the work of Wright.

§ 10.2. Here we consider more closely a special case of the theorems 3, 8 and 9 in § 6.3 and § 9.2. It may occur in these theorems that the formal series (9.31) only contains a finite number of non-zero terms. One of these cases is mentioned in theorem 3 and in the remarks after theorem 8 and theorem 9, viz. the case that $\mu > 0$, $\delta_0 > 0$ and $r = n = 0$.

Here we treat the general case in which the formal series (9.31) only contains a finite number of non-zero terms, that $\mu > 0$ and $\delta_r - \delta_{r-1} > \mu\pi$. We give a sketch how to obtain more information about the behaviour of $H(z)$ for $|z| \to \infty$ than the information contained in the theorems 3, 8 and 9.

From definition IV in § 4.6 and (2.7) it follows that

$$(10.16)\quad \left\{\begin{array}{l} Q(z)+\sum_r^\kappa D_j P(ze^{i\delta_j})-\sum_0^{r-1} C_j P(ze^{i\delta_j}) = -\sum \text{residues of} \\ h(s)z^s \text{ in the points } s \text{ for which both (1.6) and (4.28) hold} \\ +\sum \text{ residues of } h_0(s)\{h_1(s)+\sum_r^\kappa D_j e^{i\delta_j s}-\sum_0^{r-1} C_j e^{i\delta_j s}\}z^s \\ \text{in the points (4.28).} \end{array}\right.$$

In view of the supposition about (9.31) there is only a finite number of non-zero terms on either side of (10.16). Let the sum of these terms be denoted by $T(z)$ and let these terms be the residues in the points $s = s_0, \ldots, s_\lambda$ of the sequence (4.28).

It can be verified easily using (1.1), (2.7) and (2.10) that

$$(10.17)\quad H(z) = T(z)+\frac{1}{2\pi i}\int_{C_1} h_0(s)\{h_1(s)+\sum_r^\kappa D_j e^{i\delta_j s}-\sum_0^{r-1} C_j e^{i\delta_j s}\}z^s ds$$

for $z \neq 0$, where C_1 is a contour in the complex s-plane from $s = \infty - ik$ to $s = \infty + ik$ (k is a suitable positive constant) such that the points $s = s_0, \ldots, s_\lambda$ and

$$(10.18)\qquad s = (b_j+\nu)/\beta_j \qquad (j = 1, \ldots, q;\ \nu = 0, 1, 2, \ldots)$$

are lying to the right of C_1.

From the fact that the righthand side of (10.16) only contains a finite number of non-zero terms we may deduce further

$$(10.19)\quad h_0(s)\ \{h_1(s)+\sum_r^\kappa D_j e^{i\delta_j s}-\sum_0^{r-1} C_j e^{i\delta_j s}\} = h_2(s)h_5(s)$$

where $h_2(s)$ is defined by (2.24) and where $h_5(s)$ is either identically equal to zero or

$$(10.20)\qquad h_5(s) = \sum_0^\omega k_j \exp\,(il_j s).$$

Here ω is a non-negative integer, $k_0, \ldots, k_\omega$ are complex numbers independent of s and $\{l_j\}$ ($j = 0, \ldots, \omega$) is an increasing sequence of real numbers independent of s. In the case $h_5(s) \equiv 0$ then

$H(z) = T(z)$ by (10.17) and (10.19) and the asymptotic behaviour is known.

We now consider the case $h_5(s) \not\equiv 0$. Then it appears that $\delta_r - \delta_{r-1} \leqq 2\mu\pi$ and

$$(10.21)\quad \begin{cases} l_0 = \delta_r - \mu\pi,\ l_\omega = \delta_{r-1} + \mu\pi, \\ k_0 = (C_r + D_r)(2\pi i)^{p-q} \exp\left(\sum_1^q b_j - \sum_1^p a_j\right)\pi i, \\ k_\omega = -(C_{r-1} + D_{r-1})(-2\pi i)^{p-q} \exp\left(\sum_1^p a_j - \sum_1^q b_j\right)\pi i. \end{cases}$$

We omit the proof of these assertions. From (10.17), (10.19) and (10.20) we deduce if $z \neq 0$:

$$(10.22)\qquad H(z) = T(z) + \sum_0^\omega k_j \frac{1}{2\pi i} \int_{C_1} h_2(s)(ze^{il_j})^s\, ds.$$

The coefficient of k_j in (10.22) is a special case of the function $H(z \exp il_j)$ with $q = m$, $n = 0$ (cf. (1.1), (1.3) and (2.24)). To these functions we apply theorem 4 in § 7.3 on the sector (6.27) where ε satisfies (6.25). Then we obtain for $H(z) - T(z)$ an exponentially small asymptotic expansion on (6.27). In particular if $\delta_r - \delta_{r-1} \neq 2\mu\pi$ we may deduce from (10.22), (10.21), theorem 4 and lemma 5 in § 4.5:

$$(10.23)\quad H(z) - T(z) \sim (C_r + D_r)E(ze^{i\delta_r}) + (C_{r-1} + D_{r-1})E(ze^{i\delta_{r-1}})$$

for $|z| \to \infty$ uniformly on (6.27) or more generally on

$$(10.24)\qquad \begin{cases} \frac{1}{2}\mu\pi - \delta_r < \arg z < -\frac{1}{2}\mu\pi - \delta_{r-1}, \\ \operatorname{Re}(ze^{i\delta_r}) \leqq -\varepsilon_1,\ \operatorname{Re}(ze^{i\delta_{r-1}}) \leqq -\varepsilon_1 \end{cases}$$

where ε_1 is an arbitrary positive constant. Further we may deduce that if $\delta_r - \delta_{r-1} < 2\mu\pi$:

$$(10.25)\qquad H(z) - T(z) \sim (C_{r-1} + D_{r-1})E(ze^{i\delta_{r-1}})$$

for $|z| \to \infty$ uniformly on

$$(10.26)\quad -\tfrac{1}{2}(\delta_r - \delta_{r-1}) + \varepsilon \leqq \arg z + \delta_{r-1} < -\tfrac{1}{2}\mu\pi,\ \operatorname{Re}(ze^{i\delta_{r-1}}) \leqq -\varepsilon_1$$

and

$$(10.27)\qquad H(z) - T(z) \sim (C_r + D_r)E(ze^{i\delta_r})$$

for $|z| \to \infty$ uniformly on

$$(10.28)\quad \tfrac{1}{2}\mu\pi < \arg z + \delta_r \leqq \tfrac{1}{2}(\delta_r - \delta_{r-1}) - \varepsilon,\ \operatorname{Re}(ze^{i\delta_r}) \leqq -\varepsilon_1.$$

If $\delta_r - \delta_{r-1} = 2\mu\pi$ we may deduce from (10.22) and theorem 4 that (10.25) and (10.27) hold for $|z| \to \infty$ uniformly on (10.24). These assertions supplement the assertions of theorem 3 in the case that (9.31) only contains a finite number of non-zero terms and $H(z) \not\equiv T(z)$.

From (10.25) and (10.27) and the theorems 8 and 9 of § 9.2 we may deduce that (10.27) resp. (10.25) also holds for $|z| \to \infty$ uniformly on (9.25) resp. (9.39) if (9.31) only contains a finite number of non-zero terms and $H(z) \neq T(z)$. This supplements the assertions of the theorems 8 and 9.

Analogous remarks can be made in § 11 and § 12 at the applications of the theorems on $H(z)$.

§ 10.3. The last special case we consider is the case that one or more of the coefficients $C_j + D_j$ occurring in the asymptotic expansions in the theorems 5—9 in § 9 are equal to zero. In these cases better approximations for $H(z)$ can be obtained by modifying the definition of δ_j and so of C_j and D_j.

Assume $C_0 + D_0 \neq 0$ and $C_\kappa + D_\kappa \neq 0$ but some of the other coefficients $C_j + D_j$ are equal to zero. Then we change the definition of δ_j and so of κ, C_j and D_j as follows:

Let γ_j, c_j and d_j be defined in lemma 4. Consider the formal series

$$(10.29) \qquad \sum_0^\infty c_j e^{i\gamma_j s} - \sum_0^\infty d_j e^{-i\gamma_j s}$$

and rearrange the terms in this series such that we obtain a series of increasing powers of exp is:

$$\sum_{-\infty}^{\infty} e_j \exp{(i\delta_j s)}$$

where $\delta_0 = \gamma_0$, δ_j resp. e_j are real resp. complex constants and none of the numbers e_j is equal to zero. Using these numbers δ_j we may construct the numbers κ, C_j and D_j with definition II. Then we see that now $C_j + D_j = e_j \neq 0$. Further again lemma 4a can be derived: the proof needs only slight alterations. In our considerations in § 2, §§ 6—9 we only used the properties in lemma 4a of the numbers δ_j, κ, C_j and D_j. So the theorems about $H(z)$ remain valid if we use the definitions given above for δ_j etc. instead of the definitions in § 4.2. Now the coefficients $C_j + D_j$ in the asymptotic expansions are not equal to zero and so better approximations for $H(z)$ are obtained.

In the case that $C_0 + D_0 = 0$ or $C_\kappa + D_\kappa = 0$ in the original definition the new definition has to be changed slightly.

§ 11. Applications to the G-function

§ 11.1. We consider the function $H(z)$ from § 1 in the case that

$$(11.1)\qquad \alpha_j = 1 \ (j = 1, \ldots, p),\ \beta_j = 1 \ (j = 1, \ldots, q).$$

The conditions I resp. II of § 1 for the existence of $H(z)$ now read: $q > p$, $z \neq 0$ resp. $q = p$, $0 < |z| < 1$, since by (1.8) and (1.10): $\mu = q-p$ and $\beta = 1$. In these cases $H(z)$ coincides with the G-function. So:

Suppose m, n, p and q are integers such that $0 \leqq n \leqq p \leqq q$, $1 \leqq m \leqq q$ and suppose $z \neq 0$ if $q > p$ and $0 < |z| < 1$ if $q = p$. Assume further: $a_1, \ldots, a_p, b_1, \ldots, b_q$ are complex constants such that

$$(11.2)\qquad a_j - b_h \neq 1, 2, 3, \ldots \qquad (j = 1, \ldots, n;\ h = 1, \ldots, m).$$

Then the G-function is defined by (cf. C. S. Meijer [22], p. 229)

$$(11.3)\quad G^{m,n}_{p,q}\left(z\middle|{}^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right) = \frac{1}{2\pi i}\int_C \frac{\prod\limits_1^m \Gamma(b_j-s)\prod\limits_1^n \Gamma(1-a_j+s)}{\prod\limits_{m+1}^q \Gamma(1-b_j+s)\prod\limits_{n+1}^p \Gamma(a_j-s)}\, z^s\, ds$$

where z^s is defined by (1.2) and C is a contour in the complex s-plane which runs from $\infty - i\tau$ to $\infty + i\tau$ (τ is a suitable positive number) and which encloses the poles $b_j, b_j+1, \ldots$ $(j = 1, \ldots, m)$ but none of the poles $a_j-1, a_j-2, \ldots$ $(j = 1, \ldots, n)$ of the integrand (such contours exist because of (11.2)). The suppositions made above will be assumed tacitly in the rest of § 11.

If p and q are integers with $0 \leqq p \leqq q+1$, if $a_1, \ldots, a_p$, $b_1, \ldots, b_q$ are complex numbers and if $|z| < 1$ in the case that $p = q+1$ we define the generalized hypergeometric function ${}_p\varphi_q(z)$ by

$$(11.4)\quad {}_p\varphi_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{h=0}^{\infty} \frac{z^h}{h!} \prod_1^p (a_j)_h \Big/ \prod_1^q \Gamma(b_j+h).$$

If (11.2) and

$$(11.5)\qquad b_j - b_r \neq 0, \pm 1, \pm 2, \ldots \qquad (j = 1, \ldots, m;\ r = 1, \ldots, m;\ j \neq r)$$

hold and if $0 < |z| < 1$ when $p = q$ resp. $z \neq 0$ when $p < q$ then

$$(11.6)\quad G^{m,n}_{p,q}\left(z\Big|{a_1,\ldots,a_p \atop b_1,\ldots,b_q}\right)=\pi^{m-1}\sum_{h=1}^{m}\prod_{1}^{n}\Gamma(1-a_j+b_h)$$
$$\cdot\left\{\prod_{n+1}^{p}\Gamma(a_j-b_h)\prod_{\substack{1\\ j\neq h}}^{m}\sin\pi(b_j-b_h)\right\}^{-1}$$
$$z^{b_h}\cdot {}_p\varphi_{q-1}(1+b_h-a_1,\ldots,1+b_h-a_p;$$
$$1+b_h-b_1,\ldots*\ldots,1+b_h-b_q;\,(-1)^{p+m+n}z);$$

the asterisk denotes that the number $1+b_h-b_h$ has to be omitted in the sequence $1+b_h-b_1,\ldots,1+b_h-b_q$ (cf. [22], p. 230). (11.6) can be proved by means of the theorem of residues; it is a special case of (6.5).

§ **11.2.** Before we specialize the theorems 2 up to 9 for the G-function we prove some properties of the quantities occurring in the definitions in § 4 in the case (11.1).

Lemma 9.

If $p\geqq m+n-1$ *then*

$$(11.7)\qquad \delta_j=(m+n-p+2j)\pi$$

(*cf. definition* I, § 4.2). *If* $p<m+n-1$ *then* (11.7) *also holds for* $j\geqq 0$ *while for* $j<0$:

$$(11.8)\qquad \delta_j=(p-m-n+2j+2)\pi.$$

If k is an integer, arbitrary in the case $p\geqq m+n-1$ *and satisfying* $k\geqq 0$ *or* $k\leqq p-m-n$ *in the case* $p<m+n-1$ *then*

$$(11.9)\qquad \delta_r=(m+n-p+2k)\pi$$

where r is determined by

$$(11.10)\quad \begin{cases} r=k \text{ if } p\geqq m+n-1 \text{ and } k \text{ is arbitrary and also if}\\ k\geqq 0,\ p<m+n-1;\ r=m+n-p-1+k \text{ if}\\ p<m+n-1 \text{ and } k\leqq p-m-n.\end{cases}$$

Finally

$$(11.11)\qquad \delta_j-\delta_{j-1}=2\pi$$

save in the case $j=0$, $p<m+n-1$ *for in this case*

$$(11.12)\qquad \delta_0-\delta_{-1}=2(m+n-p)\pi>2\pi.$$

Proof: By lemma 4 in § 4 and (11.1) we have

$$\gamma_j=(m+n-p+2j)\pi\qquad (j=0,1,2,\ldots)$$

and so

$$(11.13)\quad (m+n-p+2h)\pi = \begin{cases} \gamma_h \text{ if } h \geqq 0 \\ -\gamma_g \text{ if } h \leqq p-m-n \text{ and} \\ g = p-m-n-h. \end{cases}$$

Suppose first $p \geqq m+n-1$. Then any integer j satisfies $j \geqq 0$ or/and $j \leqq p-m-n$. So by (11.13) every number $(m+n-p+2h)\pi$ where h is an arbitrary integer is equal to a number γ_h or/and $-\gamma_g$ for an appropriate g. So if we write down the numbers γ_j and $-\gamma_j (j = 0, 1, 2, \ldots)$ in an ascending sequence we obtain the sequence

$$\{(m+n-p+2h)\pi\} \qquad (h = \ldots, -2, -1, 0, 1, 2, \ldots).$$

From this, definition I in § 4.2 and $\gamma_0 = (m+n-p)\pi$ (cf. (11.13)) we deduce (11.7).

Suppose next $p < m+n-1$. By (11.13) the number $(m+n-p+2h)\pi$ is only equal to a number γ_j or $-\gamma_j$ if $h \geqq 0$ or $h \leqq p-m-n$. So if we write down the numbers γ_j and $-\gamma_j$ $(j = 0, 1, 2, \ldots)$ in an ascending sequence we obtain the sequence

$$\{(m+n-p+2h)\pi\}$$
$$(h = \ldots, p-m-n-2, p-m-n-1, p-m-n, 0, 1, \ldots).$$

From this, definition I in § 4.2 and $\gamma_0 = (m+n-p)\pi$ (cf. (11.13)) we deduce (11.7) for $j \geqq 0$ and (11.8) for $j < 0$.

From (11.7) and (11.8) we infer (11.9) with (11.10) in both cases $p \geqq m+n-1$ and $p < m+n-1$. Now also (11.11) and (11.12) are easily verified.

Before proving other properties for the quantities in the definitions I—IV we give

Definition V. *If* $1 \leqq h \leqq m$ *and* (11.5) *holds then*

$$(11.14)\quad \sigma_h = \pi^{m+n-p}\{\exp(p-m-n+1)\pi i b_h\} \cdot \prod_{n+1}^{p} \sin \pi(a_j-b_h) \Big/ \prod_{\substack{1\\ j\neq h}}^{m} \sin \pi(b_j-b_h).$$

In the second place

(11.15) $S(z) = \sum$ *residues of* $h(s)z^s$ *in the points* s *satisfying simultaneously* $s = b_j+\nu$ $(j = 1, \ldots, m;\ \nu = 0, 1, \ldots)$ *and* $s = a_r-1-\rho$ $(r = n+1, \ldots, p;\ \rho = 0, 1, \ldots)$ (*cf.* (1.3) *with* (11.1)).

Thirdly if g is an integer and (11.5) *holds then we define the formal series* $R(g; z)$ *by*

(11.16) $R(g;z) = -S(z) + \sum_{h=1}^{m} \sigma_h \exp(-2g\pi i b_h)$
$\times \sum$ *residues of* $h_0(s)\{ze^{(m+n-p+2g-1)\pi i}\}^s/\sin \pi(b_h - s)$ *in the points* $s = a_r - 1 - \nu$ $(r = 1, \ldots, p;\ \nu = 0, 1, \ldots)$ (*cf.* (2.10) *with* (11.1)).

Finally if $p < m+n$ *then*

$$\text{(11.17)} \quad \begin{cases} \lambda_0 = (2\pi i)^{m+n-p} \exp\left(\sum_{n+1}^{p} a_j - \sum_{1}^{m} b_j\right)\pi i, \\ \lambda_{p-m-n} = -(-2\pi i)^{m+n-p} \exp\left(\sum_{1}^{m} b_j - \sum_{n+1}^{p} a_j\right)\pi i, \end{cases}$$

while if g *is an integer,* $g \neq 0$ *and* $g \neq p-m-n$ *in the case* $p < m+n$, *if* g *is an arbitrary integer in the case* $p \geqq m+n$, *and if* (11.5) *holds then*

$$\text{(11.18)} \qquad \lambda_g = 2i \sum_{1}^{m} \sigma_h \exp\{-(2g+1)\pi i b_h\}.$$

In the case that (11.5) *does not hold we also use* (11.16) *and* (11.18) *as definitions but then we replace the righthand side in* (11.16) *and* (11.18) *by the corresponding limit.*

Remarks: It can be verified that the limits just mentioned exist.

If

$$\text{(11.19)} \quad \begin{cases} a_r - b_j \neq 1, 2, \ldots & (r = 1, \ldots, p;\ j = 1, \ldots, q) \\ a_r - a_j \neq 0, \pm 1, \pm 2, \ldots & (r = 1, \ldots, p;\ j = 1, \ldots, p;\ r \neq j) \end{cases}$$

and (11.5) are fulfilled then

$$\text{(11.20)} \quad R(g;z) = \sum_{r=1}^{p} \tau(r;g)(ze^{(m+n-p+2g-1)\pi i})^{a_r-1}$$
$$\times \sum_{\nu=0}^{\infty} (ze^{(q-m-n)\pi i})^{-\nu} \prod_{1}^{q} \Gamma(1+b_j-a_r+\nu) \Big/ \prod_{1}^{p} \Gamma(1+a_j-a_r+\nu)$$

where for $r = 1, \ldots, p$:

$$\text{(11.21)} \quad \tau(r;g) = \pi^{m+n-1-q}\Big\{\prod_{\substack{1 \\ j \neq r}}^{p} \sin \pi(a_r - a_j)\Big\}^{-1}$$
$$\times \sum_{h=1}^{m} \{\exp(p-m-n+1-2g)\pi i b_h\}$$
$$\times \prod_{\substack{1 \\ j \neq h}}^{q} \sin \pi(a_r - b_j) \cdot \prod_{n+1}^{p} \sin \pi(a_j - b_h) \cdot \Big\{\prod_{\substack{1 \\ j \neq h}}^{m} \sin \pi(b_j - b_h)\Big\}^{-1}.$$

(11.20) can be verified by calculating the residues in (11.16). The function $S(z)$ in (11.16) is equal to zero if (11.19) is fulfilled.

Using definition V we now prove

LEMMA 10.

Suppose (11.1) *and* (11.2) *are satisfied. Let* k *be an integer such that* $k \geqq 0$ *or* $k \leqq p-m-n$ *(so if* $p \geqq m+n-1$ *then* k *is an arbitrary integer). Let* r *be determined by* (11.10). *We use the definitions in* § 4 *and* (2.11).

Then if $s-b_h$ *is not an integer for* $h = 1, \ldots, m$ *and if* (11.5) *holds*:

$$(11.22)\quad h_1(s)+\sum_{r}^{\kappa} D_j e^{i\delta_j s}-\sum_{0}^{r-1} C_j e^{i\delta_j s} = e^{(m+n-p+2k-1)\pi i s}\sum_{1}^{m} \sigma_j \exp(-2k\pi i b_j)/\sin \pi(b_j-s).$$

Further

$$(11.23)\quad Q(z)+\sum_{r}^{\kappa} D_j P(ze^{i\delta_j})-\sum_{0}^{r-1} C_j P(ze^{i\delta_j}) = R(k; z).$$

This is an equality between formal series. If $0 \leqq p < m+n$ *and* $k = 0$ *then* $r = 0$ *and the lefthand side of* (11.23) *reduces to* $Q(z)$. *Finally*

$$(11.24)\qquad C_r+D_r = \lambda_k$$

while if $p < m+n$

$$(11.25)\quad \lambda_0 E(ze^{(m+n-p)\pi i})+\lambda_{p-m-n} E(ze^{(p-m-n)\pi i}) = (C_0+D_0)E(ze^{i\delta_0})+(C_{-1}+D_{-1})E(ze^{i\delta_{-1}})$$

and for the values of k *mentioned above save for* $k=0$ *with* $p<m+n$

$$(11.26)\quad \lambda_k E(ze^{(m+n-p+2k)\pi i})+\lambda_{k-1} E(ze^{(m+n-p+2k-2)\pi i}) = (C_r+D_r)E(ze^{i\delta_r})+(C_{r-1}+D_{r-1})E(ze^{i\delta_{r-1}}).$$

PROOF: In order to prove (11.22) we show that if (11.5) holds

$$(11.27)\quad \{h_1(s)+\sum_{r}^{\kappa} D_j e^{i\delta_j s}-\sum_{0}^{r-1} C_j e^{i\delta_j s}\} e^{-i\delta_r s} - \sum_{1}^{m} \sigma_j e^{-(s+2kb_j)\pi i}/\sin \pi(b_j-s)$$

is a bounded integral function which tends to zero for Im $s \to -\infty$. For then it follows that the function (11.27) is identically equal to zero and with (11.9) now (11.22) follows.

In view of (2.11), (11.1), (11.9) and (11.14) the singularities of the function (11.27) in the points $s = b_j+\nu$ ($j = 1, \ldots, m$; ν

integral) are removable and so that function can be considered as an integral function. Since the function has period 2 it is bounded for $|\text{Im}\, s| \leqq l$ (cf. (2.3) and (11.1)). Further from (4.8), (4.9) and an analogue of (2.54) (the latter applied to the last sum in (11.27)) we deduce that the function (11.27) is bounded for $\text{Im}\, s \geqq l$ while its modulus is at most

$$K\,|\exp i(\delta_{r-1}-\delta_r)s|+K_1|\exp -2\pi is|$$

for $\text{Im}\, s \leqq -l$. Here K and K_1 are constants independent of s. Hence the function (11.27) is bounded for $\text{Im}\, s \leqq -l$ and tends to zero for $\text{Im}\, s \to -\infty$. The assertions concerning (11.27) now follow.

Using the definitions IV and V in § 4.6 and § 11.2, (11.22) and (2.7) we deduce (11.23) if (11.5) is fulfilled. By taking limits we see that condition (11.5) is superfluous. If $p < m+n$ then $\delta_0 > 0$ and $\kappa = -1$ (cf. (11.7) and (4.7)). So if $p < m+n$ and $k = 0$ then $r = 0$ by (11.10) and the lefthand side of (11.23) reduces to $Q(z)$.

Further if $p < m+n$ then because of $\kappa = -1$, (4.7) and definition II in § 4.2: $C_0 = c_0$, $D_{-1} = -d_0$ and $C_{-1} = D_0 = 0$. From this, (4.3), (11.10) and (11.17) we deduce (11.24) in the case that $p<m+n$ and $k=0$ or $k=p-m-n$ (then $r=0$ or $r=-1$). With (11.9) this leads to (11.25) in the case that $p < m+n$.

Next suppose k is an integer which is arbitrary if $p \geqq m+n$ and which satisfies $k > 0$ or $k < p-m-n$ if $p < m+n$. The dependence of r on k will be denoted temporarily by $r(k)$. Then $r(k+1) = r(k)+1$ for the values of k we consider, in view of (11.10). Now we subtract the corresponding sides of (11.22) for $k||k$ and $k||k+1$. Then using (11.9), (4.7) and (11.18) we easily derive (11.24) for the numbers k we consider here. Hence (11.24) holds for the values of k mentioned in lemma 10.

If $k > 0$ or $k \leqq p-m-n$ then $r(k-1) = r(k)-1$ by (11.10). Using (11.24) and (11.9) we now infer (11.26) for $k \geqq 0$ or $k \leqq p-m-n$ except for $k = 0$, $p < m+n$.

§ 11.3. Before formulating the theorems on the G-function we make some preliminary remarks and we give a survey of these theorems.

In the theorems on the G-function we assume

1) (11.1), (11.2), $1 \leqq m \leqq q$, $0 \leqq n \leqq p$, $0 < \varepsilon < \pi/4$ with ε constant.

2) If $q > p$ then in definition III in § 4.4 we put

$$(11.28) \qquad \mu = q-p,\ \beta = 1,\ A_0 = (2\pi)^{(p-q+1)/2}(q-p)^{\alpha-\frac{1}{2}}$$

(cf. (1.8), (1.10), (3.24), (3.28)) and so

$$(11.29)\quad E(z) = \frac{1}{2\pi i} \sum_{0}^{\infty} A_j (q-p)^{-\alpha-j} z^{(1-\alpha-j)/(q-p)} \exp\{(q-p) z^{1/(q-p)}\}.$$

For the numbers $A_1, A_2, \ldots$ in (11.29) recurrence relations have been given by T. D. Riney, J. G. van der Corput and E. M. Wright (cf. the remark after lemma 2 in § 3).

3) $Q(z)$, λ_k, σ_h and $R(k; z)$ are given by the definitions IV (cf. § 4.6) and V (cf. § 11.2). So in view of (11.1), (1.3) and (4.26)

$$(11.30)\quad Q(z) = \sum \text{residues of } \frac{\prod_{1}^{n} \Gamma(1-a_j+s) \prod_{1}^{m} \Gamma(b_j-s)}{\prod_{m+1}^{q} \Gamma(1-b_j+s) \prod_{n+1}^{p} \Gamma(a_j-s)} z^s \text{ in the}$$

points $s = a_j-1-\nu$ $(j = 1, \ldots, n;\ \nu = 0, 1, 2, \ldots)$.

4) The lefthand side of (11.3) will be abbreviated by $G_{p,q}^{m,n}(z)$ if there is no reason for confusion.

The contents of the following theorems can be summed up as follows: If $q = p$ theorem 10 contains the analytic continuations and the behaviour near $z = \infty$ of the G-functions. If $q > p$ then in the majority of the cases the asymptotic expansion of the G-function is given by theorem 13. These expansions are exponentially infinite. Theorem 13 does not include the asymptotic expansions in the following cases (we assume $q > p$):

i) $p < q$, $p+q < 2(m+n)$ and (11.40) holds,
ii) $q = p+1$ and (11.42) holds where k is arbitrary if $p \geqq m+n$ and $k > 0$ or $k \leqq p-m-n$ if $p < m+n$ (k is always an integer),
iii) $\arg z$ belongs to a transitional region like (11.49), (11.56), (11.61), (11.63), (11.66) and (11.68).

In case i) theorem 11 and theorem 12 may be applied. They include the algebraic and exponentially small asymptotic expansions. In case ii) again theorem 11 may be applied giving algebraic expansions. Case iii) is covered by the theorems in § 11.6 and theorem 14.

The following theorems may be compared with the theorems A, B, C, D, E^* and 16—22 of C. S. Meijer [22]. These results include those of T. M. MacRobert [21]. In this paper MacRobert considers the asymptotic expansion of the E-function which is a special case of the G-function.

§ 11.4. In this section we deduce the analytic continuations and algebraic asymptotic expansions of the G-function.

THEOREM 10.

ASSERTION 1. *In the case* $1 \leqq p < m+n$ *the function* $G^{m,n}_{p,p}(z)$ *can be continued from* $0 < |z| < 1$ *into the sector*

$$|\arg z| < (m+n-p)\pi \tag{11.31}$$

by means of

$$G^{m,n}_{p,p}(z) = \frac{1}{2\pi i}\int_D \frac{\prod_1^m \Gamma(b_j-s)\prod_1^n \Gamma(1-a_j+s)}{\prod_{m+1}^p \Gamma(1-b_j+s)\prod_{n+1}^p \Gamma(a_j-s)} z^s\, ds \tag{11.32}$$

where D is a contour which runs from $s = \sigma - i\infty$ to $s = \sigma + i\infty$ (σ is an arbitrary real constant) so that the points $s = b_j + g$ ($j = 1, \ldots, m$; $g = 0, 1, \ldots$) resp. $s = a_j - 1 - h$ ($j = 1, \ldots, n$; $h = 0, 1, \ldots$) lie to the right resp. left of D. The function in (11.32) *can be continued analytically into the domain $|z| > 1$ by means of*

$$G^{m,n}_{p,p}(z) = Q(z) \tag{11.33}$$

if $1 \leqq p < m+n$. *$Q(z)$ now represents a convergent series for* $|z| > 1$ (*cf.* (11.30)).

ASSERTION 2. *Suppose* (11.5) *holds,* $p \geqq 1$, *k is an integer which is arbitrary if $m+n \leqq p$ while $k > 0$ or $k \leqq p-m-n$ if $m+n > p$. Then $G^{m,n}_{p,p}(z)$ can be continued analytically into the domain*

$$-\pi < \arg z + (m+n-p+2k-1)\pi < \pi \tag{11.34}$$

by means of

$$G^{m,n}_{p,p}(z) = -S(z) + \sum_{h=1}^m \sigma_h e^{-2k\pi i b_h} \cdot \frac{1}{2\pi i}\int_{D'} h_0(s)(ze^{(m+n-p+2k-1)\pi i})^s \frac{ds}{\sin \pi(b_h-s)} \tag{11.35}$$

where σ_h and $S(z)$ are given by definition V *in* § 11.2, *$h_0(s)$ is given by* (2.10) *with* (11.1) *and D' is a contour which runs from $\sigma - i\infty$ to $\sigma + i\infty$ (σ arbitrary real)* so *that the points* $s = a_\nu - 1 - \nu$

$(r = 1, \ldots, p;\ \nu = 0, 1, \ldots)$ *lie to the left of* D' *while those points* $s = b_j + g$ $(j = 1, \ldots, m;\ g = 0, 1, \ldots)$ *which differ from the points first mentioned lie to the right of* D'. *The function in* (11.35) *can be continued analytically into* $|z| > 1$ *by means of*

$$G_{p,p}^{m,n}(z) = R(k; z) \tag{11.36}$$

where now $R(k; z)$ *(cf.* (11.16)) *is a convergent series for* $|z| > 1$.

PROOF: Assertion 1 follows from remark 1 after theorem 2, especially (6.23), and $\mu = 0$, $\delta_0 > 0$ (cf. (1.8), (11.1), (11.7)). Assertion 2 follows from theorem 2, lemma 10, (11.9) with (11.10) and (11.11) with $j = r$.

REMARK. If $m = 1$, $n = p$ then (11.32) can be written in the form

$$G_{p,p}^{1,p}\left(z\Big|{}_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right) = \frac{1}{2i}\int_D h_0(s)z^s \frac{ds}{\sin \pi(b_1 - s)} \tag{11.37}$$

for $|\arg z| < \pi$. If moreover $b_1 = 0$ the G-function in (11.37) is equal to a constant times a function ${}_p\varphi_{p-1}(z)$ (cf. (11.6)).

From (11.35) and (11.37) it follows that if the assertions for (11.35) are satisfied then

$$G_{p,p}^{m,n}\left(z\Big|{}_{b_1,\ldots,b_p}^{a_1,\ldots,a_p}\right) = -S(z) + \sum_{h=1}^{m} \pi^{-1}\sigma_h e^{-2k\pi i b_h} \times G_{p,p}^{m,n}\left(ze^{(m+n-p+2k-1)\pi i}\Big|{}_{b_h, b_1,\ldots*\ldots,b_p}^{a_1,\ldots,a_p}\right) \tag{11.38}$$

where the asterisk denotes that b_h is omitted in the sequence $b_1, \ldots, b_p$. (11.38) can be deduced also from (11.4), (11.6) and (11.14). From (11.37) and (11.38) we can deduce (11.35).

THEOREM 11.

If $p < q$, $p+q < 2(m+n)$ *then the algebraic asymptotic expansion*

$$G_{p,q}^{m,n}(z) \sim Q(z) \tag{11.39}$$

holds for $|z| \to \infty$ *uniformly on*

$$|\arg z| \leqq (m+n-\tfrac{1}{2}(p+q))\pi - \varepsilon. \tag{11.40}$$

If $n = 0$, $p < q$ *and* $p+q < 2m$ *then better estimates are given in theorem* 12. *Further if* $p \geqq 1$ *then the algebraic asymptotic expansion*

$$G_{p,p+1}^{m,n}(z) \sim R(k; z) \tag{11.41}$$

holds for $|z| \to \infty$ *uniformly on*

(11.42) $\quad \frac{1}{2}\pi+\varepsilon \leqq \arg z+(m+n-p+2k)\pi \leqq \frac{3}{2}\pi-\varepsilon$

where k is an integer which is arbitrary if $p \geqq m+n$ while $k > 0$ or $k \leqq p-m-n$ if $p < m+n$.

Proof: Suppose first $p < q$, $p+q < 2(m+n)$. Then (2.13) is fulfilled in view of (11.7) and (11.28). From (11.7), (11.28) and the assertion in theorem 3 in § 6 concerning the case $\mu > 0$, $r = 0$ and (2.13) we infer (11.39) on (11.40). If moreover $n = 0$, then $Q(z)$ represents a formal series of zeros by (11.30) and then theorem 12 gives the exponentially small asymptotic expansions instead of (11.39).

Next suppose the conditions concerning (11.41) are fulfilled. Then (2.19) holds in view of (11.11) and (11.28). So we may apply theorem 3: (6.26), (6.27). Using (11.28), (11.9), (11.11) with $j = r$ and (11.23) we obtain (11.41) on (11.42).

§ 11.5. This section contains the exponentially small resp. infinite asymptotic expansions.

Theorem 12.

Suppose $0 \leqq p < q < 2m-p$. Then the following exponentially small asymptotic expansions hold: In the first place

(11.43) $$G^{m,0}_{p,q}(z) \sim \lambda_0 E(ze^{(m-p)\pi i})$$

for $|z| \to \infty$ uniformly on

(11.44) $$|\arg z| \leqq \tfrac{1}{2}(q-p)\pi-\varepsilon$$

if $m = q$, and uniformly on

(11.45) $$(\tfrac{1}{2}(p+q)-m)\pi+\varepsilon \leqq \arg z \leqq -\varepsilon$$

if $m < q$. In the second place

(11.46) $$G^{m,0}_{p,q}(z) \sim \lambda_{p-m} E(ze^{(p-m)\pi i})$$

for $|z| \to \infty$ uniformly on (11.44) if $m = q$, and on

(11.47) $$\varepsilon \leqq \arg z \leqq \big(m-\tfrac{1}{2}(p+q)\big)\pi-\varepsilon$$

if $m < q$. Finally if $m < q$ then

(11.48) $$G^{m,0}_{p,q}(z) \sim \lambda_0 E(ze^{(m-p)\pi i})+\lambda_{p-m} E(ze^{(p-m)\pi i})$$

for $|z| \to \infty$ uniformly on

(11.49) $$|\arg z| \leqq \varepsilon.$$

Proof: By (11.28) and (11.7) we have $\mu = q-p$, $\delta_0 = (m+n-p)\pi$. Since $n = 0$ and $p < q < 2m-p$ in theorem 12 the conditions $n = 0$, $\mu > 0$ and (2.13) in theorem 4 in § 7 are fulfilled. Using (4.3) and (11.17) we may translate the assertions of theorem 4 into those of theorem 12.

THEOREM 13.

Suppose k is an integer which is arbitrary if $p \geqq m+n-1$ while $k \geqq 0$ or $k \leqq p-m-n$ if $p < m+n-1$. Then

$$G_{p,q}^{m,n}(z) \sim \lambda_k E(ze^{(m+n-p+2k)\pi i}) \tag{11.50}$$

for $|z| \to \infty$ uniformly on

$$-\pi+\varepsilon \leqq \arg z+(m+n-p+2k)\pi \leqq \pi-\varepsilon \tag{11.51}$$

if $q \geqq p+2$ and uniformly on

$$-\tfrac{1}{2}\pi+\varepsilon \leqq \arg z+(m+n-p+2k)\pi \leqq \tfrac{1}{2}\pi-\varepsilon \tag{11.52}$$

if $q = p+1$. If $p < m+n-1$ and $p+3 \leqq q$ then (11.50) *with $k = 0$ also holds for $|z| \to \infty$ uniformly on*

$$\begin{aligned} -\pi+\varepsilon \leqq \arg z+(m+n-p)\pi \\ \leqq -\varepsilon+ \min(\tfrac{1}{2}(q-p),\, m+n-p)\pi \end{aligned} \tag{11.53}$$

and (11.50) *with $k = p-m-n$ also holds for $|z| \to \infty$ uniformly on*

$$\begin{aligned} \varepsilon-\min(\tfrac{1}{2}(q-p),\, m+n-p)\pi \\ \leqq \arg z+(p-m-n)\pi \leqq \pi-\varepsilon. \end{aligned} \tag{11.54}$$

The asymptotic expansions in this theorem are exponentially infinite.

PROOF: According to (11.11), (11.12) and (11.28) we have

$$\min\,(\mu\pi,\, \delta_j-\delta_{j-1}) \geqq 2\pi \text{ if } q \geqq p+2 \text{ resp. } = \pi \text{ if } q = p+1.$$

Hence theorem 5 in § 9, (11.24) and (11.9) imply (11.50) on (11.51) if $q \geqq p+2$ and (11.50) on (11.52) if $q = p+1$. If, however, $q \geqq p+3$, $p < m+n-1$ and $k = 0$ resp. $k = p-m-n$ then $r = 0$ resp. $r = -1$ by (11.10) and $\min(\delta_0-\delta_{-1},\, \mu\pi) > 2\pi$ by (11.12) and (11.28). So in these cases (11.50) with $k = 0$ resp. $k = p-m-n$ not only holds on (11.51) with $k = 0$ resp. $k = p-m-n$ but more generally on (11.53) resp. (11.54) (cf. (9.3)).

THEOREM 14.

Suppose k is an integer which is arbitrary if $m+n \leqq p+1$ while $k > 0$ or $k \leqq p-m-n$ if $m+n > p+1$. Then if $q \geqq p+3$

$$G_{p,q}^{m,n}(z) \sim \lambda_k E(ze^{(m+n-p+2k)\pi i})+\lambda_{k-1} E(ze^{(m+n-p+2k-2)\pi i}) \tag{11.55}$$

for $|z| \to \infty$ uniformly on

$$(11.56) \qquad \pi-\varepsilon \leqq \arg z+(m+n-p+2k)\pi \leqq \pi+\varepsilon.$$

Further if $0 \leqq p < q$, $p+1 < m+n < \frac{1}{2}(p+q)$ *then*

$$(11.57) \quad G^{m,n}_{p,q}(z) \sim \lambda_0 E(ze^{(m+n-p)\pi i})+\lambda_{p-m-n} E(ze^{(p-m-n)\pi i})$$

for $|z| \to \infty$ *uniformly on* (11.49). *The expansions in this theorem are exponentially infinite.*

Proof: If k satisfies the assumptions for (11.55) then by (11.10) and (11.11): $\delta_r-\delta_{r-1} = 2\pi$. So if $q \geqq p+3$ then $\delta_r-\delta_{r-1} < \mu\pi$ and theorem 6 in § 9 may be applied. Combining (9.17) and (9.18) with (11.26) and (11.25) (the latter in the case $k=0$, $p+1 = m+n$), (11.9) and $\delta_r-\delta_{r-1} = 2\pi$ we obtain (11.55) on (11.56).

If $p+1 < m+n < \frac{1}{2}(p+q)$ then using (11.12) and (11.28) we see that $\delta_0-\delta_{-1} < \mu\pi$. So the assertion in theorem 6 concerning the case $r = 0$ may be applied. In view of (11.7), (11.12) and (11.25) we obtain (11.57) on (11.49).

§ 11.6. In this section we consider the asymptotic expansions in the remaining transitional regions.

Theorem 15.

Suppose k *is an integer which is arbitrary if* $p \geqq m+n$ *while* $k > 0$ *or* $k \leqq p-m-n$ *if* $p < m+n$. *Then if* $p \geqq 1$

$$(11.58) \qquad G^{m,n}_{p,p+2}(z) \sim R(k; z)+\lambda_k E(ze^{(m+n-p+2k)\pi i}) +\lambda_{k-1}E(ze^{(m+n-p+2k-2)\pi i})$$

for $|z| \to \infty$ *uniformly on* (11.56). *If, however,* $p = n = 0$ *and* $q = 2$ *then* (11.55) *holds for* $|z| \to \infty$ *uniformly on* (11.56).

If $0 < n \leqq p < m+n = \frac{1}{2}(p+q)$ *then*

$$(11.59) \quad G^{m,n}_{p,q}(z) \sim Q(z)+\lambda_0 E(ze^{(m+n-p)\pi i})+\lambda_{p-m-n} E(ze^{(p-m-n)\pi i})$$

for $|z| \to \infty$ *uniformly on* (11.49). *If* $n = 0$, $m = \frac{1}{2}(p+q)$, $p <q$ *then* (11.57) *holds for* $|z| \to \infty$ *uniformly on* (11.49).

Proof: If the assumptions for (11.58) are satisfied then it follows from (11.11) and (11.28) that $\delta_r-\delta_{r-1} = \mu\pi$. So by theorem 7 in § 9, (9.24), (9.25), (11.28), (11.9), (11.23) and (11.26) we have (11.58) on (11.56). Next if $p = n = 0$ and $q = 2$ then again $\delta_r-\delta_{r-1} = \mu\pi$. Further now (9.31) represents a formal series of zeros and so by theorem 7 and remark 2 after theorem 7 (cf. § 9), (11.9), (11.28) and (11.26) we have (11.55) on (11.56).

If $0 \leqq n \leqq p < m+n = \frac{1}{2}(p+q)$ and $k = 0$ then $r = 0$ and $\delta_0-\delta_{-1} = \mu\pi$ by (11.10), (11.11), (11.12) and (11.28). From theorem 7, lemma 10, (11.7) and (11.28) we deduce (11.59) on (11.49). If moreover $n = 0$ we may apply remark 2 after theorem 7 and (11.25) to obtain (11.57) on (11.49).

THEOREM 16.

If $q > p$, $p+q < 2(m+n)$ *then*

(11.60) $$G^{m,n}_{p,q}(z) \sim Q(z)+\lambda_0 E(ze^{(m+n-p)\pi i})$$

for $|z| \to \infty$ *uniformly on*

(11.61) $$\tfrac{1}{2}(q-p)\pi-\varepsilon \leqq \arg z+(m+n-p)\pi \leqq \tfrac{1}{2}(q-p)\pi+\varepsilon.$$

If moreover $n = 0$ *we may delete* $Q(z)$ *in* (11.60).

If $p \geqq 1$ *and* k *is an integer which is arbitrary if* $p \geqq m+n$ *and which satisfies* $k > 0$ *or* $k \leqq p-m-n$ *if* $p < m+n$ *then*

(11.62) $$G^{m,n}_{p,p+1}(z) \sim R(k; z)+\lambda_k E(ze^{(m+n-p+2k)\pi i})$$

for $|z| \to \infty$ *uniformly on*

(11.63) $$\tfrac{1}{2}\pi-\varepsilon \leqq \arg z+(m+n-p+2k)\pi \leqq \tfrac{1}{2}\pi+\varepsilon.$$

Further

(11.64) $$G^{1,0}_{0,1}(z|b_1) = z^{b_1}e^{-z}.$$

PROOF: If $q > p$, $p+q < 2(m+n)$ then $p < m+n$ and so by (11.11) and (11.12): $\delta_0-\delta_{-1} = 2(m+n-p)\pi > (q-p)\pi = \mu\pi$. Now using theorem 8 in the case $r = 0$ (cf. § 9), (11.9) and (11.10) with $k = 0$ and lemma 10 in the case $k = 0$, $p < m+n$ we obtain (11.60) on (11.61) for $q > p$, $p+q < 2(m+n)$. If $n = 0$, $q > p$, $p+q < 2m$ then $Q(z)$ represents a formal series of zeros and we use the remark after theorem 8, (4.3) and (11.17). Then we get (11.60) with $Q(z)$ omitted on (11.61).

In the case that the assumptions of (11.62) are satisfied and $q = p+1$ then by (11.10), (11.11) and (11.28): $\delta_r-\delta_{r-1} > \mu\pi$. From theorem 8, (11.9) and lemma 10 we now derive (11.62) on (11.63). Finally (11.64) is a consequence of (11.4) and (11.6).

THEOREM 17.

If $p < q$, $p+q < 2(m+n)$ *then*

(11.65) $$G^{m,n}_{p,q}(z) \sim Q(z)+\lambda_{p-m-n} E(ze^{(p-m-n)\pi i})$$

for $|z| \to \infty$ *uniformly on*

(11.66) $$-\tfrac{1}{2}(q-p)\pi-\varepsilon \leqq \arg z+(p-m-n)\pi \leqq -\tfrac{1}{2}(q-p)\pi+\varepsilon.$$

If moreover $n = 0$ *then the term* $Q(z)$ *in* (11.65) *may be omitted.*

If $p \geqq 1$ and k is an integer which is arbitrary if $m+n \leqq p$ while $k > 0$ or $k \leqq p-m-n$ if $p < m+n$ then

$$(11.67) \qquad G^{m,n}_{p,p+1}(z) \sim R(k;z)+\lambda_{k-1}E(ze^{(m+n-p+2k-2)\pi i})$$

for $|z| \to \infty$ uniformly on

$$(11.68) \quad -\tfrac{1}{2}\pi-\varepsilon \leqq \arg z+(m+n-p+2k-2)\pi \leqq -\tfrac{1}{2}\pi+\varepsilon.$$

Proof: The proof is analogous to the proof of theorem 16; only instead of theorem 8 we apply theorem 9 and instead of (11.9) and (11.10) with $k = 0$ we use these formulae for $k = p-m-n$.

§ 12. Generalizations of the hypergeometric series

In this paragraph we consider some classes of series containing the hypergeometric series and the Bessel function as special cases. In § 12.1 and § 12.2 we treat the generalized hypergeometric function considered by E. M. Wright [32], [34]. In § 12.3 the exponentially small asymptotic expansions of another class of generalized hypergeometric series will be given. In § 12.4 we consider the generalized Bessel function introduced by E. M. Wright in [31] and [35].

§ 12.1. In this and the following section the function

$$(12.1) \qquad {}_p\psi_q(z) = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!} \prod_1^p \Gamma(\alpha_j\nu+a_j) \Big/ \prod_1^q \Gamma(\beta_j\nu+b_j)$$

will be investigated. Here we assume that p and q are non-negative integers, $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ are positive constants, $a_1, \ldots, a_p, b_1, \ldots, b_q$ are complex constants such that

$$(12.2) \quad \alpha_j\nu+a_j \neq 0, -1, -2, \ldots \quad (j = 1, \ldots, p;\ \nu = 0, 1, \ldots).$$

In § 12.1 and § 12.2 we use the number μ defined by

$$(12.3) \qquad \mu = 1+\sum_1^q \beta_j - \sum_1^p \alpha_j$$

instead of by (1.8), and the number β defined by (1.10) with the same p, q, α_j and β_j as in (12.1). Then if μ is positive the series in (12.1) is convergent for all values of z and it defines an integral function of z. If $\mu = 0$ then the series in (12.1) is convergent for $|z| < \beta^{-1}$ and it defines an analytic function of z for $|z| < \beta^{-1}$. This may be shown by means of theorem 1 (cf. § 6): Consider the special case of the function $H(z)$ in § 1 with the same p and $\alpha_j (j = 1, \ldots, p)$ as in (12.1) and with

$$(12.4) \quad \begin{cases} n = p, \ \ q \| q+1, \ \ m = 1, \ \ \beta_1 = 1, \ \ b_1 = 0, \ \ \beta_j \| \beta_{j-1} \ \text{and} \\ b_j \| 1-b_{j-1} \ \text{for} \ j = 2, \ldots, q+1, \ \ a_j \| 1-a_j \ (j = 1, \ldots, p). \end{cases}$$

The number μ defined by (1.8) assumes the value given by (12.3) if we make the substitution (12.4) in (1.8). Further the value of β defined by (1.10) does not alter if we substitute (12.4). Finally the conditions (1.4) and (1.5) are satisfied (cf. (12.2)).

From theorem 1 especially (6.1) and (6.5) we deduce that if we substitute (12.4) in the function $H(z)$ of § 1 then

$$(12.5) \qquad {}_p\psi_q(z) = H(-z)$$

for $\mu > 0$ and $z \neq 0$ and for $\mu = 0$ and $0 < |z| < \beta^{-1}$. The function $H(z)$ in (12.5) does not depend upon the value of arg z because ${}_p\psi_q(z)$ is one-valued (cf. (12.1)). Further it is easily seen that ${}_p\psi_q(z)$ is analytic in $z = 0$ and that if $\mu = 0$ and $|z| > \beta^{-1}$ then the series in (12.1) is divergent except in the case of a terminating series.

Special cases of the function ${}_p\psi_q(z)$ have been considered among others by G. Mittag-Leffler (cf. G. Sansone and J. C. H. Gerretsen [28] p. 345), E. W. Barnes [3], [4], D. Wrinch [38], [39], [40], C. Fox [17], C. V. Newsom [24], H. K. Hughes [20], C. S. Meijer [22], S. Bochner [5a], [6] and J. Boersma [7] while the general case of ${}_p\psi_q(z)$ with $\mu > 0$ has been considered by E. M. Wright [32] and [34].

The function of Mittag-Leffler is the special case ${}_1\psi_1(z)$ with $\alpha_1 = a_1 = b_1 = 1$. Barnes and Meijer have considered the ordinary generalized hypergeometric function which is the special case of ${}_p\psi_q(z)$ with $\alpha_1 = \ldots = \alpha_p = \beta_1 = \ldots = \beta_q = 1$.

Then ${}_p\psi_q(z)$ is a special case of the G-function and the asymptotic expansions and analytic continuations of this function (derived by Barnes and Meijer, cf. § 1.2) can be applied. Also a special case of the G-function and of ${}_p\psi_q(z)$ is the function considered by Boersma viz. the function in (12.1) with $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ positive rational. The properties of the G-function can be applied again in this case.

Fox has considered the more general special case of ${}_p\psi_q(z)$ that μ is positive rational. The method of Fox resembles in some aspects the method used by Barnes to obtain the exponential asymptotic expansions. However, in some of the expansions of Fox each coefficient is equal to zero (so-called "dummy" expansions).

Before Fox, Miss Wrinch considered the special case of ${}_p\psi_q(z)$ with $p = 0$ and $\beta_1 = \ldots = \beta_q = 1$ resp. $p = 1$, $q = 4$, $\alpha_1 = \beta_1 = \ldots = \beta_4 = 1$. She used results of Barnes and Kelvin's method of critical points. Newsom resp. Hughes has considered the special case of ${}_0\psi_q(z)$ with $\beta_1 = \ldots = \beta_q = 1$ resp. arbitrary positive $\beta_1, \ldots, \beta_q$. They use results derived in [23] and [19] for a more general class of functions (cf. below). In some cases Hughes does not obtain the exponentially small asymptotic expansions.

Bochner has considered the special case of ${}_p\psi_0(z)$ with $\mu = 0$. His method is related to the Laplace-Borel transformation.

More general series than the series in (12.1) have been considered by E. W. Barnes [2], G. N. Watson [29], W. B. Ford [16], C. V. Newsom [23], H. K. Hughes [19] and E. M. Wright [33] and [36]. However, in certain cases the asymptotic expansions obtained are "dummy" expansions.

Watson used his theory of asymptotic series and his results about the transformation of asymptotic series into convergent factorial series. Wright simplified and extended these methods of Watson (cf. the description of Wright's method in § 1.2).

Barnes used approximations by functions of which he previously derived a large number of properties by means of contour-integration.

Ford continued and extended the research of Barnes [2] in his book [16]. He used approximations by means of integrals which can be estimated by means of a method related to the method of steepest descents. Newson and Hughes again extended the research of Ford.

The results of the authors mentioned above concerning asymptotic expansions of the function ${}_p\psi_q(z)$ and its specializations in the case $\mu > 0$ are contained in the results of Wright in [32] and [34]. His results have the advantage that the asymptotic expansions hold uniformly on sectors a finite number of which cover the entire z-plane. Wright also deduces the exponentially small asymptotic expansions and moreover he gives relations for the coefficients in the asymptotic expansions.

In § 12.2 we deduce the analytic continuations and asymptotic expansions of the function ${}_p\psi_q(z)$ by specializing the theorems 2—9 with (12.4) and using (12.5).

The analytic continuations in the case $\mu = 0$ are contained in theorem 18. The asymptotic expansions of ${}_p\psi_q(z)$ in the case $0 < \mu < 2$ are given in the theorems 19, 21 and 22 except in the case that $p = 0$ for then the algebraic expansion in theorem 19 has to be replaced by the exponentially small expansion in theorem 20. In the case $\mu \geqq 2$ the theorems 21 and 22 yield the asymptotic expansions.

§ **12.2.** For the specialization of the theorems 2—9 to theorems about the function ${}_p\psi_q(z)$ defined by (12.1) we first transform some of the numbers and functions used in those theorems by means of (12.4). First by (3.24) and (12.4)

$$\alpha = \sum_1^q b_j - \sum_1^p a_j + \tfrac{1}{2}(p-q)+1. \tag{12.6}$$

Next according to (2.11) and (12.4) we have $h_1(s) = -\pi/\sin \pi s$, so

$$h_1(s) = 2\pi i \sum_0^\infty e^{(2\nu+1)\pi i s} \text{ resp. } -2\pi i \sum_0^\infty e^{-(2\nu+1)\pi i s}$$

for $\operatorname{Im} s > 0$ resp. $\operatorname{Im} s < 0$. Comparing this with lemma 4 and the definitions I and II in § 4 we see that

$$\begin{cases} \gamma_j = (2j+1)\pi \ (j=0, 1, \ldots),\ \delta_j = (2j+1)\pi \ (j=0, \pm 1, \ldots), \\ \kappa = -1,\ c_0 = C_0 = -d_0 = D_{-1} = 2\pi i,\ C_{-1} = D_0 = 0. \end{cases} \tag{12.7}$$

In definition III in § 4.4 the numbers $A_0, A_1, \ldots$ are determined by lemma 2 in § 3.2. So by (12.4) they are defined as follows in the case $\mu > 0$: $A_0, A_1, \ldots$ are the numbers independent of s so that if ε is a constant with $0 < \varepsilon < \pi$ and N is an arbitrary non-negative integer then

$$\begin{cases} (\beta\mu^\mu)^{-s} \prod_1^p \Gamma(\alpha_j s+a_j)/\{\Gamma(s+1) \prod_1^q \Gamma(\beta_j s+b_j)\} \\ \qquad = \sum_0^{N-1} A_j/\Gamma(\mu s+\alpha+j) \ +O(1)/\Gamma(\mu s+\alpha+N) \end{cases} \tag{12.8}$$

for $|s| \to \infty$ uniformly on (3.2). Especially

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)} \mu^{\alpha-\frac{1}{2}} \prod_1^p \alpha_j^{a_j-\frac{1}{2}} \prod_1^q \beta_j^{\frac{1}{2}-b_j}. \tag{12.9}$$

Here μ, β and α are given by (12.3), (1.10) and (12.6).

Finally

$$Q(z) = \sum \text{ residues of } z^s \Gamma(-s) \prod_1^p \Gamma(\alpha_j s+a_j)/\prod_1^q \Gamma(\beta_j s+b_j) \tag{12.10}$$

in the points

$$s = -(a_j+\nu)/\alpha_j \qquad (j = 1, \ldots, p;\ \nu = 0, 1, \ldots) \tag{12.11}$$

(cf. definition IV in § 4.6).

We now use the definitions (12.3), (1.10), (12.6), (12.10) and definition III in § 4.4 with $A_0, A_1, \ldots$ given by (12.8) and (12.9) in the following theorems.

THEOREM 18.

If $\mu = 0$ then ${}_p\psi_q(z)$ can be continued analytically into the sector $|\arg(-z)| < \pi$ by means of

$$(12.12) \qquad {}_p\psi_q(z) = \frac{1}{2\pi i}\int_D \frac{\prod_1^p \Gamma(\alpha_j s + a_j)}{\prod_1^q \Gamma(\beta_j s + b_j)}\, \Gamma(-s)(-z)^s\, ds$$

where D is a contour in the complex s-plane which runs from $s = \sigma - i\infty$ to $s = \sigma + i\infty$ (σ is an arbitrary real number) so that the points $s = 0, 1, 2, \ldots$ resp. (12.11) lie to the right resp. left of D.

From the sector $|\arg(-z)| < \pi$ the function ${}_p\psi_q(z)$ can be continued analytically into the domain $|z| > \beta^{-1}$ by means of

$$(12.13) \qquad {}_p\psi_q(z) = Q(-z).$$

Here $Q(-z)$ is a convergent series for $|z| > \beta^{-1}$.

PROOF: Use theorem 2 in § 6.2 with $r = 0$, (12.7) and (12.5).

THEOREM 19.

Suppose $0 < \mu < 2$, $p > 0$ and ε is a constant so that $0 < \varepsilon < \frac{1}{2}(2-\mu)\pi$. Then the algebraic asymptotic expansion

$$(12.14) \qquad {}_p\psi_q(z) \sim Q(-z)$$

holds for $|z| \to \infty$ uniformly on

$$(12.15) \qquad |\arg(-z)| \leqq (1-\tfrac{1}{2}\mu)\pi - \varepsilon.$$

PROOF: Use theorem 3 in § 6.3 with $r = 0$, (12.7) and (12.5).

THEOREM 20.

Suppose $p = 0$, $q > 0$, $\mu < 2$ and ε is a constant so that $0 < \varepsilon < \frac{1}{4}(2-\mu)\pi$. Then the following exponentially small asymptotic expansions hold:

$$(12.16) \qquad {}_0\psi_q(z) \sim 2\pi i E(z)$$

for $|z| \to \infty$ uniformly on

$$(12.17) \qquad \tfrac{1}{2}\mu\pi + \varepsilon \leqq |\arg z| \leqq \pi - \varepsilon.$$

Further

$$(12.18) \qquad {}_0\psi_q(z) \sim 2\pi i\,\{E(z)+E(ze^{\mp 2\pi i})\}$$

for $|z| \to \infty$ *uniformly on*

$$(12.19) \qquad \pi-\varepsilon \leqq \pm \arg z \leqq \pi+\varepsilon.$$

Here in (12.18) *and* (12.19) *the upper resp. lower signs belong together.*

PROOF: Since $p = 0$, $q > 0$ and $\mu < 2$ we have $\mu > 1$ and $\delta_0 > \frac{1}{2}\mu\pi$ (cf. (12.3) and (12.7)). So the assumptions of theorem 4 in § 7.3 are fulfilled. We apply this theorem with (12.7) and $z||ze^{\pm\pi i}$. Because $q > 0$ in theorem 20 we have to consider the case $q > m$ in theorem 4 (cf. (12.4)). Using (12.7) and (12.5) we now easily obtain (12.16) on (12.17) from (7.11) and (7.12) while (12.18) on (12.19) is a consequence of (7.15).

THEOREM 21.

Suppose μ *is positive and* ε *is a constant so that* $0 < \varepsilon < \frac{1}{4}\pi \min(\mu, 2)$. *Then the exponentially infinite asymptotic expansion*

$$(12.20) \qquad {}_p\psi_q(z) \sim 2\pi i E(z)$$

holds for $|z| \to \infty$ *uniformly on*

$$(12.21) \qquad |\arg z| \leqq \tfrac{1}{2}\pi \min(\mu, 2)-\varepsilon.$$

PROOF: Apply theorem 5 in § 9.1 with $r = 0$, $z||ze^{-\pi i}$, (12.7) and (12.5).

THEOREM 22.

In the transitional regions the following asymptotic expansions hold: If $\mu > 2$ *and* ε *is a constant so that* $0 < \varepsilon < \frac{1}{4}\min(2, \mu-2)$ *then*

$$(12.22) \qquad {}_p\psi_q(z) \sim 2\pi i\{E(z)+E(ze^{\mp 2\pi i})\}$$

for $|z| \to \infty$ *uniformly on* (12.19). *This expansion is exponentially infinite.*

If $\mu = 2$, $p > 0$ *and* ε *is a constant so that* $0 < \varepsilon < \pi/2$ *then*

$$(12.23) \qquad {}_p\psi_q(z) \sim Q(ze^{\mp\pi i})+2\pi i\,\{E(z)+E(ze^{\mp 2\pi i})\}$$

for $|z| \to \infty$ *uniformly on* (12.19). *If* $\mu = 2$, $p = 0$ *and* ε *is a constant so that* $0 < \varepsilon < \pi/2$ *then* (12.22) *holds for* $|z| \to \infty$ *uniformly on* (12.19).

If $0 < \mu < 2$, $p > 0$ *and* ε *is a constant so that* $0 < \varepsilon < \frac{1}{4}(2-\mu)\pi$ *then*

$$(12.24) \qquad {}_p\psi_q(z) \sim Q(ze^{\mp\pi i}) + 2\pi i E(z)$$

for $|z| \to \infty$ *uniformly on*

$$(12.25) \qquad \tfrac{1}{2}\mu\pi - \varepsilon \leqq \pm \arg z \leqq \tfrac{1}{2}\mu\pi + \varepsilon.$$

If $0 < \mu < 2$ *and* $p = 0$ *and* ε *is a constant so that* $0 < \varepsilon < \frac{1}{4}(2-\mu)\pi$ *then* (12.20) *holds for* $|z| \to \infty$ *uniformly on* (12.25).

In (12.22) *and* (12.19), (12.23) *and* (12.19), *and* (12.24) *resp.* (12.20) *and* (12.25) *the upper resp. lower signs belong together.*

Proof: Suppose first $\mu > 2$. Then in view of (12.7) we have $\delta_0 - \delta_{-1} < \mu\pi$ and so we apply theorem 6 in § 9.1 with $r = 0$ and $z||ze^{\mp\pi i}$. Using (12.5) and (12.7) we infer (12.22) on (12.19).

Next suppose $\mu = 2$. Then $\delta_0 - \delta_{-1} = \mu\pi$ by (12.7). Hence we may apply theorem 7 in § 9.2 with $z||ze^{\mp\pi i}$ and $r = 0$. Using (12.5) and (12.7) we obtain (12.23) on (12.19). If moreover $p = 0$ we use remark 2 after theorem 7. Then we see that (12.23) may be replaced by (12.22) on (12.19).

Finally suppose $0 < \mu < 2$. Then $\delta_0 - \delta_{-1} > \mu\pi$ by (12.7). In this case we use theorem 8 resp. 9 in § 9.2 with $z||ze^{-\pi i}$ resp. $ze^{\pi i}$ and $r = 0$. On account of (12.5) and (12.7) we may write the result as (12.24) on (12.25). If moreover $p = 0$ then (12.24) may be replaced by (12.20) in view of the remarks after the theorems 8 and 9.

§ 12.3. Here we consider another generalization of the hypergeometric function, which also contains the function ${}_p\psi_q(z)$ as a special case:

(12.26)
$$\chi(z) = \sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!} \prod_{1}^{n} \Gamma(\alpha_j\nu + a_j) \{\prod_{1}^{q} \Gamma(\beta_j\nu + b_j) \prod_{n+1}^{p} \Gamma(1 - a_j - \alpha_j\nu)\}^{-1}$$

where n, p and q are integers, $0 \leqq n \leqq p$, $q \geqq 0$ and $\alpha_1, \ldots, \alpha_p$, $a_1, \ldots, a_p, \beta_1, \ldots, \beta_q, b_1, \ldots, b_q$ satisfy the same assumptions as at (12.1) except that in (12.2) we replace p by n.

$\chi(-z)$ is the special case of $H(z)$ in § 1 with the same n, p and α_j as in (12.26) and with

$$(12.27) \quad \begin{cases} q||q+1,\ m = 1,\ \beta_1 = 1,\ b_1 = 0,\ \beta_j||\beta_{j-1} \text{ and} \\ b_j||1-b_{j-1}\ (j = 2, \ldots, q+1),\ a_j||1-a_j\ (j = 1, \ldots, p). \end{cases}$$

(cf. (6.5)). Then the numbers μ and β defined in (1.8) and (1.10) attain the values given by (12.3) and (1.10) (in (12.3) and (1.10) the parameters α_j etc. are those occurring in (12.26)). In the following μ denotes the number defined by (12.3).

From theorem 1 in § 6 it follows that $\chi(z)$ is defined, one-valued and analytic in z for $\mu = 0$, $|z| < \beta^{-1}$ resp. $\mu > 0$ and arbitrary z.

If $n = p$ we have $\chi(z) = {}_p\psi_q(z)$ (cf. (12.1)). If $p > n$ and (12.2) holds, then $\chi(z)$ can be expressed in functions ${}_p\psi_q(z)$: Using (2.6) and an analogue of (2.38) we see that

$$(12.28) \qquad \prod_{n+1}^{p} \Gamma(1-a_j-\alpha_j \nu)^{-1} = \prod_{n+1}^{p} \Gamma(\alpha_j \nu + a_j) \sum_{0}^{N} \rho_j e^{r_j \nu i}$$

and

$$(12.29) \qquad \chi(z) = \sum_{0}^{N} \rho_j \, {}_p\psi_q(ze^{ir_j}).$$

Here N is a positive integer, $r_0, \ldots, r_N$ resp. $\rho_0, \ldots, \rho_N$ are real resp. complex numbers independent of ν.

The analytic continuations resp. asymptotic expansions can be deduced from the theorems about $H(z)$. If $n < p$ and (12.2) holds, then in most cases these properties can also be deduced from (12.29) and the theorems in § 12.2 about ${}_p\psi_q(z)$. Only in the case that $n = 0$, $p > 0$ and

$$(12.30) \qquad \sum_{1}^{p} \alpha_j + \sum_{1}^{q} \beta_j < 1$$

$$(12.31) \qquad |\arg(-z)| \leqq \tfrac{1}{2}(1 - \sum_{1}^{p} \alpha_j - \sum_{1}^{q} \beta_j)\pi - \varepsilon$$

where ε is a constant so that

$$(12.32) \qquad 0 < \varepsilon < \tfrac{1}{4}(1 - \sum_{1}^{p} \alpha_j - \sum_{1}^{q} \beta_j)\pi,$$

theorem 19 and (12.29) lead to an algebraic asymptotic expansion of $\chi(z)$ in which each coefficient is zero, while an application of theorem 4 in § 7 shows that in this case $\chi(z)$ has an exponentially small asymptotic expansion. Therefore we give only the exponentially small expansions of $\chi(z)$ because the other expansions can be derived from the theorems 18—22 and (12.29).

Theorem 23.

Suppose $n = 0$, (12.30) *holds and* ε *is a constant satisfying* (12.32). $E(z)$ *will be defined by* (4.13) *where* μ, β, α *and* A_0 *are given by* (12.3), (1.10), (12.6) *and* (12.9), *and* $A_1, A_2, \ldots$ *are determined* *by* (12.8). *In the following formulae the upper resp. lower signs belong together.*

If $q = 0$, *then*

$$(12.33) \qquad \chi(z) \sim (2\pi)^{1-p} i \exp\{\pm(\tfrac{1}{2}p - \sum_1^p a_j)\pi i\} E(z \exp \mp \sum_1^p \alpha_j \pi i)$$

for $|z| \to \infty$ *uniformly on*

$$(12.34) \qquad \tfrac{1}{2}\pi(1 + \sum_1^p \alpha_j) + \varepsilon \leqq \pm \arg z \leqq \pi.$$

If $q > 0$ *then* (12.33) *holds for* $|z| \to \infty$ *uniformly on*

$$(12.35) \qquad \tfrac{1}{2}\pi(1 + \sum_1^p \alpha_j + \sum_1^q \beta_j) + \varepsilon \leqq \pm \arg z \leqq \pi - \varepsilon.$$

If $q > 0$ *then*

$$(12.36) \qquad \chi(z) \sim (2\pi)^{1-p} i [\exp\{\pm(\tfrac{1}{2}p - \sum_1^p a_j)\pi i\} E(z \exp \mp \sum_1^p \alpha_j \pi i) + \exp\{\mp(\tfrac{1}{2}p - \sum_1^p a_j)\pi i\} E(z \exp \mp (2 - \sum_1^p \alpha_j)\pi i)]$$

for $|z| \to \infty$ *uniformly on*

$$(12.37) \qquad \pi - \varepsilon \leqq \pm \arg z \leqq \pi + \varepsilon.$$

The asymptotic expansions (12.33) *and* (12.36) *are exponentially small.*

Proof: Substituting (12.27) in (2.12) and (4.3) we obtain

$$(12.38) \quad \begin{cases} \delta_0 = (1 - \sum_1^p \alpha_j)\pi, \ c_0 = (2\pi)^{1-p} i \exp(\tfrac{1}{2}p - \sum_1^p a_j)\pi i, \\ d_0 = -(2\pi)^{1-p} i \exp(\sum_1^p a_j - \tfrac{1}{2}p)\pi i. \end{cases}$$

In view of (12.3) and (12.30) the number μ is positive and (2.13) holds. From theorem 4: (7.11) with $z||ze^{-\pi i}$, (12.3), (12.38) and $\chi(z) = H(-z)$ with (12.27) we deduce that (12.33) with the upper sign holds for $|z| \to \infty$ uniformly on

$$-\tfrac{1}{2}\pi(1 - \sum_1^p \alpha_j) + \varepsilon \leqq \arg z - \pi \leqq \tfrac{1}{2}\pi(1 - \sum_1^p \alpha_j) - \varepsilon$$

if $q = 0$ and on

$$\tfrac{1}{2}\pi(-1 + \sum_1^p \alpha_j + \sum_1^q \beta_j) + \varepsilon \leqq \arg z - \pi \leqq -\varepsilon$$

if $q > 0$. This implies the assertion concerning (12.33) with the upper sign. In the same way the assertion concerning (12.33) with the lower sign follows from (7.12) with $z||ze^{\pi i}$. Finally using (7.15) with $z||ze^{\mp\pi i}$ in the case $\mu > 0$ we obtain (12.36) on (12.37).

§ 12.4. A special case of the function $\chi(z)$ in § 12.3 has been considered by E. M. Wright. In [35] he investigated the asymptotic behaviour for $|z| \to \infty$ of the generalized Bessel function $\varphi(z)$:

(12.39) $$\varphi(z) = \sum_0^\infty \frac{z^\nu}{\nu!\Gamma(b-\sigma\nu)}$$

where σ is a real number such that $0 < \sigma < 1$ and b is a complex number. This function is the special case of $\chi(z)$ (cf. (12.26)) with

(12.40) $$q = n = 0,\ p = 1,\ \alpha_1 = \sigma,\ a_1 = 1-b.$$

Previously in [31] Wright investigated the function $\varphi(z)$ with $\sigma < 0$ but this function is a special case of ${}_p\psi_q(z)$ (cf. (12.1)). Here we derive again the asymptotic expansion for $|z| \to \infty$ of $\varphi(z)$ in the case $0 < \sigma < 1$. These expansions will be derived from the theorems 3—9 and 23. We do not use (12.29) and the theorems of § 12.2 because then the case $1-b+\sigma\nu = 0, -1, -2, \ldots$ for $\nu = 0, 1, \ldots$ has to be excluded.

In view of (12.39) and (6.5) we have

(12.41) $$\varphi(z) = H(-z)$$

with

(12.42) $$q = m = 1, \beta_1 = 1, b_1 = 0, p = 1, n = 0, \alpha_1 = \sigma, a_1 = b.$$

So by (2.11) now $h_1(s) = -\sin \pi(b-\sigma s)/\sin \pi s$ and

(12.43) $$\begin{cases} h_1(s) = \left(e^{\pi i(b-\sigma s)} - e^{-\pi i(b-\sigma s)}\right) \sum_0^\infty e^{(2\nu+1)\pi i s} \text{ if } \operatorname{Im} s > 0 \\ h_1(s) = \left(e^{\pi i(\sigma s-b)} - e^{-\pi i(\sigma s-b)}\right) \sum_0^\infty e^{-(2\nu+1)\pi i s} \text{ if } \operatorname{Im} s < 0. \end{cases}$$

In view of (4.4) and because $0 < \sigma < 1$ this implies $\gamma_0 = \pi(1-\sigma)$, $\gamma_1 = \pi(1+\sigma)$, $\gamma_2 = \pi(3-\sigma)$. Combining this with (12.43) and the definitions I and II in § 4 we obtain

(12.44) $$\begin{cases} \delta_0 = \pi(1-\sigma),\ \ \delta_1 = \pi(1+\sigma),\ \ \delta_2 = \pi(3-\sigma),\ \ \delta_{-1} = \\ -\pi(1-\sigma),\ \ \delta_{-2} = -\pi(1+\sigma),\ \kappa = -1,\ C_j = 0 \text{ if } j < 0, \\ D_j = 0 \text{ if } j \geqq 0,\ \ C_0 = e^{b\pi i},\ \ C_1 = D_{-1} = -e^{-b\pi i}. \end{cases}$$

Further by (12.42), (1.8), (3.24) and the definitions III and IV in § 4:

$$(12.45)\quad \begin{cases} \mu = 1-\sigma,\ \alpha = b+\frac{1}{2},\ Q(z) = 0, \\ P(z) = \sum \text{residues of } z^s\Gamma(1-b+\sigma s)/\Gamma(1+s) \\ \qquad\qquad \text{in the points } s = (b-1-\nu)/\sigma\ (\nu = 0, 1, \ldots), \\ E(z) = \dfrac{1}{2\pi i(1-\sigma)} \exp\{(1-\sigma)^{1-\sigma}\sigma^\sigma z\}^{1/(1-\sigma)} \\ \qquad\qquad \cdot \sum_0^\infty A_j(\sigma^\sigma(1-\sigma)^{1-\sigma}z)^{(\frac{1}{2}-b-j)/(1-\sigma)}, \\ A_0 = \sqrt{2\pi}(1-\sigma)^b\sigma^{\frac{1}{2}-b} \end{cases}$$

where $A_1, A_2, \ldots$ are determined by

(12.46)

$$(\sigma^\sigma(1-\sigma)^{1-\sigma})^{-s}\Gamma(1-b+\sigma s)/\Gamma(1+s) \sim \sum_0^\infty A_j/\Gamma((1-\sigma)s+b+\tfrac{1}{2}+j)$$

for $|s| \to \infty$ on $|\arg s| \leqq \pi/2$.

Using the definitions (12.45) and (12.46) we prove

THEOREM 24.

Suppose $\frac{1}{3} < \sigma < 1$ *and* ε *is a constant so that*

$$(12.47)\qquad 0 < \varepsilon < \tfrac{1}{2}\pi(3\sigma-1).$$

Then the algebraic asymptotic expansion

$$(12.48)\qquad \varphi(z) \sim -e^{b\pi i}P(ze^{-\sigma\pi i})$$

holds for $|z| \to \infty$ *uniformly on*

$$(12.49)\qquad |\arg z| \leqq \tfrac{1}{2}(3\sigma-1)\pi-\varepsilon.$$

PROOF: Because $\frac{1}{3} < \sigma < 1$ and because of (12.44) and (12.45) we have $\delta_1-\delta_0 > \mu\pi$. Hence we may apply theorem 3 in § 6.3 with $r = 1$ and $z||ze^{-\pi i}$. Using further (12.41), (12.44) and (12.45) we obtain the assertions of theorem 24.

THEOREM 25.

Suppose $0 < \sigma < 1$ *and* ε *is a constant so that*

$$0 < \varepsilon < \tfrac{1}{4}\pi \min(1-\sigma,\ 2\sigma).$$

Then $\varphi(z)$ *possesses the exponentially small asymptotic expansion*

$$(12.50)\qquad \varphi(z) \sim \pm e^{\pm b\pi i}E(ze^{\mp\sigma\pi i})$$

for $|z| \to \infty$ *uniformly on*

(12.51) $$\tfrac{1}{2}\pi(1+\sigma)+\varepsilon \leqq \pm \arg z \leqq \pi.$$

(12.50) also holds for $|z| \to \infty$ *uniformly on*

(12.52) $$\tfrac{1}{2}\pi(3\sigma-1)+\varepsilon \leqq \pm \arg z \leqq \pi \textit{ resp. } \varepsilon \leqq \pm \arg z \leqq \pi$$

if $\frac{1}{3} < \sigma < 1$ *resp.* $0 < \sigma \leqq \frac{1}{3}$. *This expansion is exponentially infinite on the sector*

(12.53) $$\tfrac{1}{2}\pi(3\sigma-1)+\varepsilon \leqq \pm \arg z \leqq \tfrac{1}{2}\pi(1+\sigma)-\varepsilon$$
$$\textit{resp. } \varepsilon \leqq \pm \arg z \leqq \tfrac{1}{2}\pi(1+\sigma)-\varepsilon,$$

if $\frac{1}{3} < \sigma < 1$ *resp.* $0 < \sigma \leqq \frac{1}{3}$. *In the formulae above the upper resp. lower signs belong together.*

Proof: The exponentially small expansion (12.50) on (12.51) is an immediate consequence of theorem 23 and (12.40).

Next we apply theorem 5 in § 9 with $r = 0$ and $z||ze^{-\pi i}$. Using also (12.41), (12.44) and (12.45) we obtain the exponentially infinite expansion (12.50) with the upper sign for $|z| \to \infty$ uniformly on

(12.54) $$\varepsilon-\tfrac{1}{2}\pi \min(1-\sigma, 2\sigma) \leqq \arg z -\pi\sigma \leqq -\varepsilon+\tfrac{1}{2}\pi(1-\sigma).$$

In the same way using $\delta_0-\delta_{-1} > \mu\pi$ we derive from the remark after theorem 8 in § 9.2 that (12.50) with the upper sign holds for $|z| \to \infty$ uniformly on

(12.55) $$\tfrac{1}{2}\pi(1-\sigma)-\varepsilon \leqq \arg z-\pi\sigma \leqq \tfrac{1}{2}\pi(1-\sigma)+\varepsilon.$$

Combining (12.51), (12.54) and (12.55) we obtain the assertions concerning (12.50) with the upper sign.

Using theorem 5 in § 9 with $r = -1$, $z||ze^{\pi i}$ and the remark after theorem 9 in § 9.2 with $z||ze^{\pi i}$ we obtain the assertions concerning (12.50) with the lower sign.

Theorem 26.

If $0 < \sigma < \frac{1}{3}$ *and* ε *is a constant so that*

$$0 < \varepsilon < \tfrac{1}{4}\pi \min(1-3\sigma, 2\sigma)$$

then

(12.56) $$\varphi(z) \sim e^{b\pi i}E(ze^{-\sigma\pi i})-e^{-b\pi i}E(ze^{\sigma\pi i})$$

for $|z| \to \infty$ *uniformly on*

(12.57) $$|\arg z| \leqq \varepsilon.$$

If $\sigma = \frac{1}{3}$ *and* ε *is a constant so that* $0 < \varepsilon < \frac{1}{3}\pi$ *then*

(12.58) $$\varphi(z) \sim e^{b\pi i}E(ze^{-\frac{1}{3}\pi i})-e^{-b\pi i}E(ze^{\frac{1}{3}\pi i})-e^{b\pi i}P(ze^{-\frac{1}{3}\pi i})$$

for $|z| \to \infty$ *uniformly on* (12.57).

If $\frac{1}{3} < \sigma < 1$ *and* ε *is a constant satisfying*

$$0 < \varepsilon < \tfrac{1}{4}\pi \min(2-2\sigma, 3\sigma-1),$$

then

$$\varphi(z) \sim \pm e^{\pm b\pi i} E(ze^{\mp\sigma\pi i}) - e^{b\pi i} P(ze^{-\sigma\pi i}) \tag{12.59}$$

for $|z| \to \infty$ *uniformly on*

$$\tfrac{1}{2}\pi(3\sigma-1)-\varepsilon \leqq \pm \arg z \leqq \tfrac{1}{2}\pi(3\sigma-1)+\varepsilon \tag{12.60}$$

where the upper resp. lower signs belong together.

PROOF: If $0 < \sigma < \frac{1}{3}$ then $\delta_1 - \delta_0 < \mu\pi$ by (12.44) and (12.45). Now apply theorem 6 in § 9 with $r = 1$, $z||ze^{-\pi i}$. Using also (12.41), (12.44) and (12.45) we deduce (12.56) on (12.57).

If $\sigma = \frac{1}{3}$ then $\delta_1 - \delta_0 = \mu\pi$ by (12.44) and (12.45) and so theorem 7 in § 9 with $r = 1$ can be applied. This leads to (12.58) on (12.57).

If $\frac{1}{3} < \sigma < 1$ then $\delta_1 - \delta_0 > \mu\pi$ according to (12.44) and (12.45) and consequently theorem 8 resp. 9 with $r = 1$ can be applied. Replacing z by $ze^{-\pi i}$ in these theorems and using (12.41), (12.44) and (12.45) we arrive at (12.59) on (12.60).

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