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Autohomeomorphism Groups of 0-dimensional Spaces

by

J. de Groot and R. H. McDowell 1)

If $T$ is a topological space, we denote by $A(T)$ the group of all homeomorphisms of $T$ onto itself. In [2], it was shown that given an arbitrary group $G$, one can find a topological space $T$ such that $G$ and $A(T)$ are isomorphic; in fact, such a $T$ can be found among the compact connected Hausdorff spaces. In general, no such $T$ can be found among the spaces with a base of open and closed sets, i.e., the spaces $T$ such that $\dim T = 0$. The present paper investigates the following question. What can be said, in general, about $A(T)$ if $T$ is a completely regular Hausdorff space and $\dim T = 0$?

If $\alpha$ is any cardinal $\geq 1$, we shall denote by $S_\alpha$ the restricted permutation group on $\alpha$ objects; that is, the group of all those permutations which involve only finitely many objects. We will find it convenient to let $S_0$ denote the group of one element. $\Sigma C_2$ will denote the direct sum of $\aleph_1$ groups of order two. Throughout this paper, “space” will be used to mean “completely regular Hausdorff space”. For any 0-dimensional space $T$, we shall show that $A(T)$ must

1) consist of a single element (in which case we say $T$ is “rigid”),
2) contain a subgroup $S_\alpha$ for some $\alpha$,
3) contain a subgroup of the form $S_\alpha + \Sigma C_2$. This result is best possible, in the sense that for any cardinal $\alpha$, we can construct spaces whose autohomeomorphism group is precisely $S_\alpha$ or $S_\alpha + \Sigma C_2$. We produce examples of arbitrarily high weight, 2) but we leave open the problem of constructing compact rigid 0-dimensional spaces of arbitrarily high weight.

In particular, if $T$ is dense in itself, $A(T)$ equals the unit

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2) The weight of a space is $m$ if there exists an open base of $m$ and not less than $m$ sets.
element or contains a subgroup \( \sum C_2 \). On the other hand, one can construct compact 0-dimensional Hausdorff spaces \( H \), dense in itself, for which \( A(H) = 1 \) or \( A(H) \) equals the direct sum of continuously many groups of order two (in the last case one takes the Čech-Stone compactification of \([2; \S \, 5, \text{example I}]\)).

Some of the results of this paper were announced in \([3]\).

I. \( A(T) \) for 0-dimensional Spaces

1.1. Lemma. Let \( \{x_i\} \) and \( \{y_i\}, i \in N \) (the natural numbers) be sets of distinct isolated points in the space \( T \) such that, for every \( J \subseteq N \), \( \{x_j\}, j \in J \), have identical boundaries in \( T \); then \( T \) admits of uncountably many distinct autohomeomorphisms of order two.

**Proof.** It is easy to see that the map interchanging \( x_i \) and \( y_i \) for each \( i \) in \( N \), and leaving all other points of \( T \) fixed, is an autohomeomorphism; the same is clearly true for every subset \( J \) of \( N \), and there are uncountably many such subsets.

In what follows, we shall need the following well known (and easily proved) result from group theory.

1.2. Proposition. If \( G \) is a group in which all elements distinct from the identity have order two, then \( G \) can be represented as the direct sum of cyclic groups of order two.

1.3. Theorem. Let \( T \) be a 0-dimensional completely regular Hausdorff space, containing \( \alpha \) isolated points (\( \alpha \) may be 0). Then either \( A(T) = S_\alpha \), or \( A(T) \) contains a subgroup of the form \( S_\alpha + \Sigma C_2 \).

**Proof.** \( A(T) \) clearly contains a subgroup isomorphic to \( S_\alpha \), since every one – one onto map moving a finite number of isolated points, and leaving all other points fixed, is a homeomorphism. Thus we need only show that if \( T \) admits any autohomeomorphism which does more than this, then \( T \) contains a subgroup isomorphic to \( S_\alpha + \Sigma C_2 \).

Note first of all that if \( \alpha > \aleph_0 \), there is no problem, since \( S_\alpha \) itself contains such a subgroup. So we assume \( \alpha \leq \aleph_0 \), and we distinguish two cases.

(1) There is an autohomeomorphism \( \varphi \) on \( T \) which moves a non-isolated point \( p \). Then we can find an open – and – closed set \( U \) containing \( p \) such that \( U \cap \varphi(U) = \emptyset \). If \( U \) has no countable base, we can find more than \( \aleph_0 \) distinct open- and -closed subsets \( K \subseteq U \), and interchanging \( K \) and \( \varphi(K) \) gives us an autohomeo-
morphism of order two. If $U$ has a countable base, let $D = \{x_i\}$ be the set of all isolated points in $U$. If $D$ is finite, then $M = (U \setminus D) \cup \varphi(U \setminus D)$ is open-and-closed, dense in itself, separable, metrizable and 0-dimensional, and is therefore homeomorphic to a dense-in-itself subset of the Cantor set. Since $M$ is not rigid, $A(M)$ (and hence $A(T)$) contains a subgroup of the form $\Sigma C_2$, by [2; p. 90, (i)]. If $D$ is infinite and closed, let $\{x_i\}$ be any enumeration of $D$; then $\{x_i\}$ and $\{\varphi(x_i)\}$ satisfy the hypotheses of Lemma 1.1; if $D$ is not closed, it has a limit point $q$ and a subsequence $\{y_i\}$ converging to $q$. In that case, $\{y_{2i-1}\}$ and $\{y_{2i}\}$ satisfy the hypotheses of 1.1.

(2) No autohomeomorphism moves a non-isolated point. Let $\varphi$ be a homeomorphism moving an infinite set of isolated points $\{y_i\}$. If we can find a set of isolated points $\{x_i\}$ such that $\{x_i\} \cap \{\varphi(x_i)\} = \emptyset$, then $\{x_i\}$ and $\{\varphi(x_i)\}$ clearly satisfy the hypotheses of 1.1. But such a set $\{x_i\}$ is easily found, for if there is a $y \in \{y_i\}$ with infinite orbit, let $x_i = \varphi^{2i}y$; if each $y_i$ has finite orbit, form $\{x_i\}$ by choosing one point from each of the orbits determined by $y_i$.

It follows that $A(T)$ contains a group isomorphic to $\Sigma C_2$; from the construction, it is easily seen that by dividing the isolated points into two disjoint infinite sets if necessary, one can find a subgroup isomorphic to $S_\omega + \Sigma C_2$.

It should be pointed out that in only one case in the proof of 1.3 do we fail to find continuously many distinct autohomeomorphisms of order two. We could replace $\Sigma C_2$ in the statement of the theorem by the direct sum of continuously many groups of order two if we could prove the following: if $U$ and $V$ are 0-dimensional, disjoint homeomorphic spaces having no countable base, and $X = U \cup V$, then $A(X)$ contains $c$ elements of order two.

II. Rigid Spaces

In this section, we extend the methods of [2] to produce rigid 0-dimensional spaces of arbitrary (infinite) weight. We shall require some ideas in the theory of uniform spaces; the reader is referred to [1] and [4] for a development of these ideas.

First, we extend a metric space theorem to uniform spaces in a routine manner.

2.1. DEFINITION. An intersection of $m$ open sets will be called a $G_{m^*}$-set; a $G_{m^*}$-set will be called, as usual, a $G_\delta$-set.

2.2. THEOREM. Let $X$ be a completely regular Hausdorff space
of weight $m$, complete in a uniformity $\mathcal{D}$ generated by a set $D$ of $m$ pseudometrics. Then every continuous map $f$ from a subset $H$ of $X$ into $X$ can be extended continuously to a map $\bar{f}$ from a $G_{m^\delta}$-set $G \supset H$ into $X$.

**Proof.** For each $d \in \mathcal{D}$, and each $x \in \bar{H}$, let $\omega_d(x)$ be the oscillation of $f$ at $x$ with respect to $d$. Let

$$G_d = \{x \in \bar{H} : \omega_d(x) = 0\}.$$

$G_d$ is evidently a $G_\delta$-set. Let

$$G = \bigcap_{d \in \mathcal{D}} G_d;$$

then $G \supset H$ is a $G_{m^\delta}$-set.

Now $f$ can be extended continuously over $G$. For let $\{h_\alpha\}$ be any net in $H$ converging to a point $x \in G$. Then, in the uniformity generated by $\mathcal{D}$, $\{f(h_\alpha)\}$ is a Cauchy net, by the definition of $G$. Hence $\{f(x_\alpha)\}$ converges to some point $p \in X$; set $f(x) = p$. $f$ is evidently continuous at $x$.

Now, using 2.2, we extend some of the results in [2].

2.3. **Definition.** If $X$ is a topological space, and $f$ a map from a subset of $X$ into $X$, then $f$ is called a continuous displacement of order $M$ if $f$ is continuous, and is a displacement of order $M$. A continuous displacement of order $c$ will be called, as usual, a continuous displacement [2; § 2].

2.4. **Theorem.** Let $X$ be a completely regular Hausdorff space of weight $m$, complete in a uniformity $\mathcal{D}$ generated by $m$ pseudometrics, and let $|X| = 2^m = M$. Further, let $\{K_\beta\}$ be any family of $M$ subsets of $X$, each of cardinal $M$. Then there is a family $\{F_\gamma\}$ of $2^M$ subsets of $X$ such that

1. For $\gamma \neq \gamma'$, $|F_\gamma \setminus F_{\gamma'}| = M$.
2. No $F_\gamma$ admits of any continuous displacement of order $M$ onto itself or any other $F_{\gamma'}$.
3. For every $\beta, \gamma$, $|F_\gamma \cap K_\beta| = M$, and $|(X \setminus F_\gamma) \cap K_\beta| = M$.

**Proof.** There exist only $M$ $G_{m^\delta}$-sets in $X$, and a fixed subset of $X$ admits at most $M$ continuous maps into $X$, and therefore at most $M$ continuous displacements of order $M$. Let $f_\beta$ be a continuous displacement of order $M$ whose domain is a $G_{m^\delta}$-set. The family $\{f_\beta\}$ of all such mappings has cardinal at most $M$. This family is non-empty (otherwise the theorem is trivial), so by counting a given displacement $M$ times if necessary, we may assume that $|\{f_\beta\}| = M$. 


Now we apply \([2; \text{Lemma 1}]\), with \(X = N, M = m\), and \(\{f_\beta\}\). We obtain a family \(\{F_\gamma\}\) of \(2^M\) subsets of \(X\) satisfying (1) and (3). Suppose (2) is false, and there is a continuous displacement of order \(M\), \(\varphi\), from \(F_\gamma\) onto \(F_\gamma\). This \(\varphi\) can be extended (Theorem 2.4) to a continuous map \(\hat{\varphi}\) of a \(G_m\)-set \(G_\gamma \supset F\) into \(X\), so \(\hat{\varphi} = f_\beta\) for some \(\beta\). Hence, by \([2; \text{Lemma 1}, (2.3)]\), for every pair \(\gamma, \gamma\)', \(f_\beta F_\gamma \backslash F_\gamma' \neq \phi\), and so, since \(\varphi = f_\beta\) on \(F_\gamma\), \(\varphi F_\gamma \backslash F_\gamma' \neq \phi\), i.e., \(\varphi\) maps \(F_\gamma\) onto no member of \(\{F_\gamma\}\).

2.5. \textbf{Lemma.} Let \(P\) be a space in which every open set has cardinal at least \(M\). If \(\varphi : P \to P\) is non-trivial, and is either locally topologically into \(P\) or continuous onto \(P\), then \(\varphi\) is a displacement of order \(M\).

\textbf{Proof.} The proof is word for word the proof of \([2; \text{Lemma 2}]\), with "\(\aleph\)" replaced by "\(M\)”, and “continuous displacement” replaced by “continuous displacement of order \(M\)”.

2.6. \textbf{Theorem.} Let \(X\) be a locally compact Hausdorff space of weight \(m\), complete in a uniformity generated by \(m\) pseudometrics, such that every open set in \(X\) has \(2^m\) points. Let \(K\) be the set of all compact subsets of \(X\) whose cardinal is \(2^m\). Then the sets \(\{F_\gamma\}\) constructed in \text{Theorem 2.4} are such that no \(\{F_\gamma\}\) can be mapped topologically into or continuously onto itself or any other \(F_\gamma\).

\textbf{Proof.} Each open set in each \(F_\gamma\) will have \(2^m\) points. By \text{Lemma 2.5} and (1), \text{Theorem 2.4}, any non-trivial \(\varphi\) satisfying either condition of the theorem is a continuous displacement of order \(M\). But this contradicts (2), \text{Theorem 2.4}.

2.7. \textbf{Example.} \text{Theorem 2.6} enables us to construct many examples of rigid 0-dimensional spaces of arbitrary weight. For instance, let

\[ X = \Pi_{a \in A} X_a, \]

where \(|A| = m\), and, for each \(a\), \(X_a\) is a discrete space of cardinal two. Then \(X\) has weight \(m\), \(X\) is compact, and hence complete in any uniformity, so \(X\) is complete in a uniformity generated by \(m\) pseudometrics. Further, every open set in \(X\) contains \(2^m\) points. Now, applying \text{Theorem 2.6}, we get a collection of \(2^{2m}\) sets \(\{F_a\}\), each of weight \(m\) and dimension 0, such that \(F_\gamma\) is rigid for each \(\gamma\), and the \(F_\gamma\) are topologically distinct.

2.8. \textbf{Problem.} The rigid spaces constructed in the preceding example are proper dense subsets of a compact space, hence they
are not themselves compact. We have not been able to construct examples of compact, rigid 0-dimensional spaces of arbitrarily high weight; such spaces would be of interest in the study of Boolean rings (see, for example, [2; § 8.1]).

III. Spaces whose Autohomeomorphism Groups are \( S_\alpha \) or \( S_\alpha + \sum C_2 \).

If \( \alpha \) is finite, the discrete space of cardinal \( \alpha \) has \( S_\alpha \) as its autohomeomorphism group. This is not the case for \( \alpha \) infinite, of course. In Example 3.1, however, we produce for each infinite \( \alpha \) a space having \( \alpha \) isolated points whose autohomeomorphism group is precisely \( S_\alpha \). In Example 3.2, we find spaces whose autohomeomorphism group is the direct sum of \( S_\alpha \) and the sum of continuously many groups of order two; this group is then isomorphic to \( S + \sum C_2 \) if we assume the continuum hypothesis.

In this connection one should recall the remark following the proof of Theorem 1.3; it is conceivable that \( \aleph_1 \) can be replaced by \( \aleph \) throughout this paper.

In both 3.1 and 3.2, the spaces \( S_\alpha \) which play a part in the construction can evidently be chosen to have arbitrarily high weight, hence the same is true for our examples.

3.1. Example. Let \( P \) be a discrete space of cardinal \( \alpha \), and let \( \beta P \) be its (0-dimensional) Čech-Stone compactification. With each \( p \in P \), we associate a 0-dimensional space \( S_p \) such that

1. for each \( p \in P \), \( S_p \) is rigid and dense-in-itself,
2. if \( p \) and \( q \) are distinct elements of \( P \), then no non-empty open subset of \( S_p \) is homeomorphic to an open subset of \( S_q \).

Such a collection \( \{S_p\} \) can be constructed by using Example 2.7, as follows: with each \( p \in P \), we associate a cardinal \( \alpha_p \) such that if \( p \neq q \), \( 2^{\alpha_p} \neq 2^{\alpha_q} \). Taking \( \alpha_p = \aleph \) in 2.1, we obtain a rigid space which we can denote by \( S_p \) such that each open subset of \( S_p \) contains \( 2^{\alpha_p} \) points. The collection \( \{S_p\}, p \in P \) evidently satisfies (1) and (2).

Now let

\[ X = \bigcup_{p \in P} S_p \cup \beta P. \]

We topologize \( X \) by prescribing a base for the open sets, consisting of

1. the sets \( \{p\}, p \in P \),
2. the open-and-closed sets in \( S_p \) for each \( p \in P \),
3. the sets
where $U$ is open-and-closed in $P$.

The space $X$ so defined is evidently a 0-dimensional completely regular Hausdorff space. The topology on each $S_p$ as a subspace of $X$ is the same as its original topology.

Every mapping of $X$ onto $X$ which permutes a finite number of the (isolated) points of $P$ and leaves all other points of $X$ fixed, is clearly a homeomorphism. These are the only autohomeomorphisms of $X$. For if an autohomeomorphism $\varphi$ leaves each $p \in P$ pointwise fixed, then the points of $\beta P$ are fixed, so

$$\bigcup_{p \in P} S_p$$

must be mapped topologically on itself. But from (1) and (2), this space is rigid, so $\varphi$ is the identity map. On the other hand, if $\varphi$ displaces an infinite subset $D$ of $P$, then $\varphi$ must move some point of $\beta P \setminus P$ (since the closures of $D$ and $\varphi(D)$ in $\beta P$ are non-empty and disjoint), hence there is a $p \in P$ such that $S_p \cap \varphi S_p = \emptyset$. But $\varphi S_p \cap \beta P = \emptyset$, since no open set in $S_p$ contains an isolated point. It follows that $\varphi S_p \cap S_{p'} \neq \emptyset$ for some $p \neq p'$, contradicting (2).

3.2. Example. For each $\alpha$, we construct a space $T_\alpha$ such that $A(T_\alpha)$ is precisely $S_\alpha + \sum C_2$ (assuming the continuum hypothesis). Let $M$ be a 0-dimensional subset of the real numbers such that $A(M)$ is the direct sum of continuously many groups of order two [2; § 5, Example 1], and let $X$ be the space constructed in Example 3.1, so that $A(X) = S_\alpha$. Let $T_\alpha = X \cup M$. If $\varphi$ is any autohomeomorphism of $T$, then $x \in M$ if and only if $\varphi(x) \in M$, since $x \in M$ if and only if the least cardinal of a base at $x$ is $\aleph_0$. It follows that $A(T_\alpha) = A(X) + A(M) = S_\alpha + \sum C_2$.

REFERENCES

L. Gillman and M. Jerison
J. de Groot
J. de Groot
J. L. Kelley
(Oblatum 10-9-62).