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Proximate Orders and Distribution of a -points of Meromorphic Functions

by

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§ 1. Let $f(z)$ be a meromorphic function of order ρ ($0 < \rho < \infty$) and lower order λ ($0 \leq \lambda < \infty$). Let $M(r, f)$, $T(r, f)$, $n(r, a)$, $N(r, a)$ have their usual meanings.

We define $\rho(r)$ to be proximate order D of $f(z)$ for $T(r, f)$, having the following properties;

- 1.1 $\rho(r)$ is real, continuous and piecewise differentiable;
- 1.2 $\rho(r) \rightarrow \rho$ as $r \rightarrow \infty$,
- 1.3 $r\rho'(r) \log r \rightarrow 0$ as $r \rightarrow \infty$,
- 1.4 $T(r, f) \leq r^{\rho(r)}$ for $r \geq r_0$
 $\quad = r^{\rho(r)}$ for a sequence of values of $r \rightarrow \infty$.

For the existence of this proximate order see [7] where $\rho(r)$ is constructed with $\log M(r, f)$ and $f(z)$ is an entire function. The same reasoning may be applied to construct $\rho(r)$ with the above properties. From the properties 1.1 to 1.4 we can deduce the following,

- 1.5 $r^{\rho(r)}$ is an increasing function of $r \geq r_0$.
- 1.6 $(ur)^{\rho(ur)} \sim u^{\rho} r^{\rho(r)}$ for $r \geq r_0$.
- 1.7 $n(r, a) < K r^{\rho(r)}$ for all $r \geq r_0$. [13]

§ 2. We define $\lambda(r)$ to be proximate order L for $f(z)$ for $T(r, f)$ having the following properties.

- 2.1 $\lambda(r)$ is non-negative, continuous function of r for $r \geq r_0$.
- 2.2 $\lambda(r)$ is differentiable except at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist.
- 2.3 $\lambda(r) \rightarrow \lambda$ as $r \rightarrow \infty$.
- 2.4 $r\lambda'(r) \log r \rightarrow 0$ as $r \rightarrow \infty$.
- 2.5 $T(r, f) \geq r^{\lambda(r)}$ for $r \geq r_0$.
 $\quad = r^{\lambda(r)}$ for a sequence of values of $r \rightarrow \infty$.

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For the existence of this proximate order see [8] where $\lambda(r)$ is constructed with $\log M(r, f)$ and $f(z)$ is an entire function. The same argument may be applied to construct $\lambda(r)$ with the above properties.

From properties 2.1—2.5 we can easily deduce the following

2.6 $r^{\lambda(r)}$ is an increasing function of $r \geq r_0$.

2.7 $(ur)^{\lambda(ur)} \sim u^{\lambda} r^{\lambda(r)}$ for $r \geq r_0$. [4]

§ 3. Applying the properties of $\rho(r)$ and $\lambda(r)$ we prove a number of results. For convenience we set

3.1 $n(r) = n(r, a) + n(r, b)$

3.2 $N(r) = N(r, a) + N(r, b)$

where $a \neq b$, $0 \leq a \leq \infty$, $0 \leq b \leq \infty$
and prove the following theorems

THEOREM 1. If

3.3 $\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = \alpha < \infty$

and

3.4 $\frac{N(r)}{r^{\lambda(r)}} \rightarrow 0$ as $r \rightarrow \infty$.

Then for $x \neq a, b$

$$1 = \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \leq \alpha < \infty.$$

By putting $b = \infty$, we can easily deduce from this theorem the analogous result for entire functions. Also consider the following function

$$f(z) = \prod_1^{\infty} \left(1 + \frac{2}{A_n} \right)^{ku_n}$$

where

$$\begin{aligned} k &= [\rho] + 1 \\ U_n &= A_n^{\rho+n} \\ A_n &= n^{n^n} \end{aligned}$$

then

$$\limsup_{r \rightarrow \infty} \frac{n(r, 0)}{\log m(r, f)} = \infty. \quad [6]$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{n(r, 0)}{r^{\lambda(r)}} = \infty$$

so that

$$\limsup_{r \rightarrow \infty} \frac{N(r, 0)}{r^{\lambda(r)}} = \infty. \quad [3]$$

Hence the condition 3.3 is essential.

THEOREM 2. If

$$3.6 \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} = \beta > 0$$

and

$$3.7 \quad \frac{N(r)}{r^{\rho(r)}} \rightarrow 0 \text{ as } r \rightarrow \infty$$

Then for $x \neq a, b$,

$$3.8 \quad 0 < \beta \leq \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} \leq 1.$$

And since [3]

$$3.9 \quad 0 < \limsup_{r \rightarrow \infty} \frac{n(r, a)}{r^{\rho(r)}} < \infty$$

if and only if $0 < \limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\rho(r)}} < \infty$

we can easily deduce analogous results for entire functions by putting $b = \infty$ and replacing $N(r, a)$ by $n(r, a)$. See [13].

§ 4. To see whether the converse of theorem 1 and 2 is true or not we note that if $N(r, x)/r^{\lambda(r)} \rightarrow \infty$, then $T(r, f)/r^{\lambda(r)} \rightarrow \infty$ as $r \rightarrow \infty$ also. Hence without any restrictions on $N(r, x)/r^{\lambda(r)}$ we cannot prove anything, in general. We prove the following

THEOREM 3.

If

$$4.1 \quad \limsup_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} < \infty \quad \text{for } x = a, b, c.$$

Then

$$4.2 \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} < \infty.$$

Imposing more restrictions on $f(z)$ we prove the following

THEOREM 4.

If $f(z)$ is a meromorphic function of non-integral order where $p(p \geq 1)$ is the genus and

$$4.3 \quad \limsup_{r \rightarrow \infty} \frac{N(r)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

$$4.4 \quad \frac{\alpha}{2} \leq \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq 3e(p+1)^2 \alpha (2 + \log p) \pi \operatorname{cosec} \pi(\lambda - p).$$

THEOREM 5.

If $f(x)$ is a meromorphic function of non-integral order and genus $p \geq 1$, then

$$4.5 \quad \limsup_{r \rightarrow \infty} \frac{N(r)}{T(r, f)} \geq \frac{\sin \pi(\rho - p)}{3e\rho(2 + \log p)(1 + p)\pi}$$

$$4.6 \quad \geq \frac{\sin \pi(\rho - p)}{3e(2 + \log p)(1 + p)^2\pi}.$$

§ 5. S. K. Singh [10] has proved

If $f(z)$ be an entire function of non-integral order, then

$$5.1 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} > 0 \text{ for all } a, (0 \leq |a| < \infty).$$

S. M. Shah [8] has proved that for functions of order less than one

$$5.2 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} \geq 1 - \rho$$

We here prove

THEOREM 6.

If $f(z)$ be an entire function of non-integral finite order and genus p , and

$$5.3 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

$$5.4 \quad \frac{\alpha}{\lambda} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leq \pi \alpha 3e(p+1)^2 (2 + \log p) \operatorname{cosec} \pi(\lambda - p).$$

THEOREM 7.

If $f(z)$ is an entire function of genus zero and $0 < \lambda < 1$ and

$$5.5 \quad \limsup_{r \rightarrow \infty} \frac{n(r, a)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

$$5.6 \quad \frac{\alpha}{\lambda} \leq \limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leq \pi \alpha \operatorname{cosec}(\pi \lambda).$$

THEOREM 8.

If $f(z)$ is an entire function of non-integral order ρ and genus p , then

$$5.7 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} \geq \frac{\sin \pi(\rho - p)}{3e(p + 1)^2(2 + \log p)\pi}.$$

THEOREM 9.

If $f(z)$ is an entire function of order ρ , $0 < \rho < 1$ and genus zero, then

$$5.8 \quad \limsup_{r \rightarrow \infty} \frac{N(r, a)}{\log M(r, f)} \geq \frac{\sin \pi \rho}{\pi \rho}.$$

This theorem has been proved by Valirom [12], but we give a different proof by using proximate orders.

§ 6. PROOF OF THEOREM 1.

By 2.5 we have

$$6.1 \quad \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} = 1.$$

Also for $x \neq a, b$

$$T(r, f) < N(r) + N(r, x) + o(\log r).$$

Hence

$$\begin{aligned} 1 &= \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq \liminf_{r \rightarrow \infty} \frac{N(r)}{r^{\lambda(r)}} + \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \\ &\leq \liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\lambda(r)}} \\ &\leq \liminf_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \\ &= 1 \end{aligned}$$

and the left hand equality follows.

The right hand inequality follows from the fact that $N(r, x) \leq T(r, f)$ for all x and the theorem is proved.

PROOF of THEOREM 2.

By 1.4 we have

$$6.2 \quad \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} = 1$$

and so the right hand inequality is obvious.

To prove the left hand inequality, suppose if possible

$$\liminf_{r \rightarrow \infty} \frac{N(r, x)}{r^{\rho(r)}} = 0 \quad \text{for } x \neq a, b.$$

Hence

$$\left[\frac{N(r)}{r^{\rho(r)}} + \frac{N(r, x)}{r^{\rho(r)}} \right] \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

and so

$$\frac{T(r, f)}{r^{\rho(r)}} \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

and this contradicts 3.6 and the theorem follows.

PROOF of THEOREM 3.

Let

$$\limsup_{r \rightarrow \infty} \frac{N(r, x_i)}{r^{\lambda(r)}} = \alpha_i \quad (i = 1, 2, 3).$$

Then

$$N(r, x_i) < (\alpha_i + \varepsilon_i) r^{\lambda(r)} \quad (i = 1, 2, 3).$$

We have

$$\begin{aligned} T(r, f) &\leq \sum_{i=1}^3 N(r, x_i) + o(\log r) \\ &\leq \sum_{i=1}^3 (\alpha_i + \varepsilon_i) r^{\lambda(r)} + o(\log r) \\ &= \beta r^{\lambda(r)} + o(\log r) \quad (\beta < \infty). \end{aligned}$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq \beta < \infty$$

and the Theorem follows.

PROOF of THEOREM 4.

Since

$$T\left(r, \frac{\alpha f + \beta}{r f + \delta}\right) = T(r, f) \quad 0(1)$$

we may suppose $a = 0$, and $b = \infty$, without any loss of generality and so we have

$$6.3 \quad n(r) = n(r, 0) + n(r, \infty)$$

$$6.4 \quad N(r) = N(r, 0) + N(r, \infty).$$

Also we know [5] that

$$6.5 \quad T(r, f) \leq 0(r^p) + 3e(2 + \log p)(1 + p) \int_0^\infty \frac{n(t)r^{p+1}dt}{t^{p+1}(t+r)}$$

By lemma 1 [2] we have

$$6.6 \quad \int_0^\infty \frac{n(t)r^{p+1}dt}{t^{p+1}(t+r)} \leq (p+1) \int_0^\infty \frac{N(t)r^{p+1}dt}{t^{p+1}(t+r)}.$$

Setting $S = 3e(2 + \log p)(1 + p)^2$ and since from 4.3

$$N(r) \leq (\alpha + \varepsilon)r^{\lambda(r)} = \beta r^{\lambda(r)} \quad (\beta < \infty)$$

we get

$$T(r, f) \leq S\beta \int_0^\infty \frac{t^{\lambda(t)}r^{p+1}dt}{t^{p+1}(t+r)} + 0(r^p).$$

Put $t = ur$

$$T(r, f) \leq S\beta \int_0^\infty \frac{(ur)^{\lambda(ur)}r^{p+1}r du}{(ur)^{p+1}(ur+r)} + 0(r^p)$$

$$\sim S\beta \int_0^\infty r^{\lambda(r)} \frac{u^{\lambda-p-1}}{u+1} du + 0(r^p), \quad \text{by 2.7}$$

$$\sim S\beta r^{\lambda(r)} \pi \operatorname{cosec} \pi(\lambda - p) + 0(r^p), \quad \text{since } 0 < \lambda - p < 1.$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\lambda(r)}} \leq S\alpha\pi \operatorname{cosec} \pi(\lambda - p)$$

and the right hand inequality is proved.

The left hand inequality is obvious since $N(r) \leq 2T(r, f)$ and the theorem follows.

PROOF OF THEOREM 5.

From 1.7 we have

$$\limsup_{r \rightarrow \infty} \frac{n(r)}{r^{\rho(r)}} = H_1 < \infty.$$

Also since

$$6.7 \quad \int_{r_0}^r t^{\rho(t)-1} dt \sim \frac{r^{\rho(r)}}{\rho} \quad [1]$$

$$6.8 \quad N(r) \leq \frac{H}{\rho} r^{\rho(r)}.$$

From [5] we have

$$6.9 \quad T(r, f) \leq O(r^p) + 3e(2 + \log p)(1 + p) \int_0^\infty \frac{n(t)r^{p+1}dt}{t^{p+1}(t+r)}.$$

Applying lemma 1 [2] we get

$$6.10 \quad T(r, f) \leq O(r^p) + 3e(2 + \log p)(1 + p)^2 \int_0^\infty \frac{N(t)r^{p+1}}{t^{p+1}(t+r)} dt.$$

In 6.10, set $S = 3e(2 + \log p)(1 + p)^2$.

Using 6.8 we have

$$\begin{aligned} T(r, f) &\leq O(r^p) + S \int_0^\infty \frac{H}{\rho} \frac{t^{\rho(t)} r^{p+1}}{t^{p+1}(t+r)} dt \\ &\leq O(r^p) + \frac{S.H.}{\rho} \int_0^\infty \frac{(ur)^{\rho(ur)} r^{p+1} r}{(ur)^{p+1}(ur+r)} du \\ &\sim O(r^p) + \frac{S.H.}{\rho} r^{\rho(r)} \int_0^\infty \frac{u^{\rho-1}}{u+1} du. \end{aligned}$$

Hence

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}} &\leq S.\pi. \operatorname{cosec} \pi (\rho - p) \frac{H}{\rho} \\ &\leq S.\pi. \operatorname{cosec} \pi (\rho - p) \cdot \limsup_{r \rightarrow \infty} \frac{N(r)}{r^{\rho(r)}}. \end{aligned}$$

So

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{N(r)} \leq \frac{\limsup_{r \rightarrow \infty} \frac{T(r, f)}{r^{\rho(r)}}}{\limsup_{r \rightarrow \infty} \frac{N(r)}{r^{\rho(r)}}} \leq S.\pi. \operatorname{cosec} \pi (\rho - p)$$

and 4.6 follows.

Starting with 6.9 and proceeding similarly we have 4.5 and we note that 4.6 is a better inequality than 4.5, since $\rho < p + 1$. Proofs of Theorems 6 and 8 are omitted since they are similar to the proofs of Theorems 4 and 5.

PROOF OF THEOREM 7

$$\log f(z) \leq r \int_0^\infty \frac{n(t, a)}{t(t+r)} dt. \quad [11]$$

From 5.6,

$$n(r, a) \leq (\alpha + \varepsilon) r^{\lambda(r)} = \beta r^{\lambda(r)}, \quad \beta < \infty.$$

Hence

$$\begin{aligned} \log M(r, f) &\leq r\beta \int_0^\infty \frac{t^{\lambda(r)}}{t(t+r)} dt \\ &\sim \beta r^{\lambda(r)} \int_0^\infty \frac{u^\lambda}{u(u+1)} dt && \text{by 2.7.} \\ &= \beta r^{\lambda(r)} \frac{\pi}{\sin \pi\lambda} \end{aligned}$$

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leq \frac{\alpha\pi}{\sin \pi\lambda}.$$

Left hand inequality is obvious.

PROOF OF THEOREM 9.

From 1.4 we have

$$N(r, a) \leq T(r, f) \leq r^{\rho(r)}.$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\rho(r)}} = \alpha \leq 1$$

we have [11]

$$\begin{aligned} \log M(r, f) &\leq \int_0^\infty \frac{n(t)r}{t(t+r)} dt \\ &\leq \int_0^\infty \frac{N(t)r}{(t+r)^2} dt \\ &\leq \int_0^\infty \alpha \frac{t^{\rho(t)} r}{(t+r)^2} dt \\ &\sim \alpha \int_0^\infty \frac{r^{\rho(r)} u^\rho}{(u+1)^2} du. \end{aligned}$$

Hence

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho(r)}} \leq \frac{\alpha \pi \rho}{\sin \pi \rho}$$

$$\liminf_{r \rightarrow \infty} \frac{\log M(r, f)}{N(r, a)} \leq \frac{\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\rho(r)}}}{\limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\rho(r)}}} \leq \frac{\pi \rho}{\sin \pi \rho}.$$

Lastly we note that if we use the properties of lower proximate order and assume

$$\limsup_{r \rightarrow \infty} \frac{N(r, a)}{r^{\lambda(r)}} < \infty.$$

Then we have

$$\limsup_{r \rightarrow \infty} \frac{\log M(r, f)}{N(r, a)} \leq \frac{\pi \lambda}{\sin \pi \lambda}$$

and since

$$\frac{\pi \lambda}{\sin \pi \lambda} \leq \frac{\pi \rho}{\sin \pi \rho}$$

and so in one way we have a better inequality.

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