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Proximate Orders and Distribution of a-points of Meromorphic Functions

by

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§ 1. Let $f(z)$ be a meromorphic function of order $\rho(0 < \rho < \infty)$ and lower order $\lambda(0 \leqq \lambda < \infty)$. Let $M(r, f)$, $T(r, f)$, $n(r, a)$, $N(r, a)$ have their usual meanings.

We define $\rho(r)$ to be proximate order D of $f(z)$ for $T(r, f)$, having the following properties;

1.1 $\rho(r)$ is real, continuous and piecewise differentiable;

1.2 $\rho(r) \to \rho$ as $r \to \infty$,

1.3 $r\rho'(r) \log r \to 0$ as $r \to \infty$,

1.4 $T(r, f) \leqq r^{\rho(r)}$ for $r \geqq r_0$
$= r^{\rho(r)}$ for a sequence of values of $r \to \infty$.

For the existence of this proximate order see [7] where $\rho(r)$ is constructed with $\log M(r, f)$ and $f(z)$ is an entire function. The same reasoning may be applied to construct $\rho(r)$ with the above properties. From the properties 1.1 to 1.4 we can deduce the following,

1.5 $r^{\rho(r)}$ is an increasing function of $r \geqq r_0$.

1.6 $(ur)^{\rho(ur)} \sim u^{\rho} r^{\rho(r)}$ for $r \geqq r_0$.

1.7 $n(r, a) < K\, r^{\rho(r)}$ for all $r \geqq r_0$. [13]

§ 2. We define $\lambda(r)$ to be proximate order L for $f(z)$ for $T(r, f)$ having the following properties.

2.1 $\lambda(r)$ is non-negative, continuous function of r for $r \geqq r_0$.

2.2 $\lambda(r)$ is differentiable except at isolated points at which $\lambda'(r - 0)$ and $\lambda'(r + 0)$ exist.

2.3 $\lambda(r) \to \lambda$ as $r \to \infty$.

2.4 $r\lambda'(r) \log r \to 0$ as $r \to \infty$.

2.5 $T(r, f) \geqq r^{\lambda(r)}$ for $r \geqq r_0$.
$= r^{\lambda(r)}$ for a sequence of values of $r \to \infty$.

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For the existence of this proximate order see [8] where $\lambda(r)$ is constructed with $\log M(r, f)$ and $f(z)$ is an entire function. The same argument may be applied to construct $\lambda(r)$ with the above properties.

From properties 2.1—2.5 we can easily deduce the following

2.6 $r^{\lambda(r)}$ is an increasing function of $r \geqq r_0$.

2.7 $(ur)^{\lambda(ur)} \sim u^{\lambda} r^{\lambda(r)}$ for $r \geqq r_0$. [4]

§ 3. Applying the properties of $\rho(r)$ and $\lambda(r)$ we prove a number of results. For convenience we set

3.1 $$n(r) = n(r, a) + n(r, b)$$

3.2 $$N(r) = N(r, a) + N(r, b)$$

where $a \neq b$, $0 \leqq a \leqq \infty$, $0 \leqq b \leqq \infty$
and prove the following theorems

THEOREM 1. If

3.3 $$\operatorname*{Lim\,sup}_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} = \alpha < \infty$$

and

3.4 $$\frac{N(r)}{r^{\lambda(r)}} \to 0 \text{ as } r \to \infty.$$

Then for $x \neq a, b$

$$1 = \liminf_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \leqq \alpha < \infty.$$

By putting $b = \infty$, we can easily deduce from this theorem the analogous result for entire functions. Also consider the following function

$$f(z) = \prod_1^\infty \left(1 + \frac{2}{A_n}\right)^{ku_n}$$

where

$$k = [\rho] + 1$$
$$U_n = A_n^{\rho+n}$$
$$A_n = n^{n^n}$$

then

$$\limsup_{r\to\infty} \frac{n(r, 0)}{\log m(r, f)} = \infty. \qquad [6]$$

Hence

$$\limsup_{r\to\infty} \frac{n(r, 0)}{r^{\lambda(r)}} = \infty$$

so that

$$\limsup_{r\to\infty} \frac{N(r, 0)}{r^{\lambda(r)}} = \infty. \qquad [3]$$

Hence the condition 3.3 is essential.

THEOREM 2. If

3.6 $$\liminf_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}} = \beta > 0$$

and

3.7 $$\frac{N(r)}{r^{\rho(r)}} \to 0 \text{ as } r \to \infty$$

Then for $x \neq a, b$,

3.8 $$0 < \beta \leqq \liminf_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \leqq \limsup_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} \leqq 1.$$

And since [3]

3.9 $$0 < \limsup_{r\to\infty} \frac{n(r, a)}{r^{\rho(r)}} < \infty$$
$$\text{if and only if } 0 < \limsup_{r\to\infty} \frac{N(r, a)}{r^{\rho(r)}} < \infty$$

we can easily deduce analogous results for entire functions by putting $b = \infty$ and replacing $N(r, a)$ by $n(r, a)$. See [13].

§ 4. To see whether the converse of theorem 1 and 2 is true or not we note that if $N(r, x)/r^{\lambda(r)} \to \infty$, then $T(r, f)/r^{\lambda(r)} \to \infty$ as $r \to \infty$ also. Hence without any restrictions on $N(r, x)/r^{\lambda(r)}$ we cannot prove anything, in general. We prove the following

THEOREM 3.
If

4.1 $$\limsup_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} < \infty \qquad \text{for } x = a, b, c.$$

Then

4.2 $$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} < \infty.$$

Imposing more restrictions on $f(z)$ we prove the following

THEOREM 4.

If $f(z)$ is a meromorphic function of non-integral order where $p(p \geqq 1)$ is the genus and

4.3
$$\limsup_{r\to\infty} \frac{N(r)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

4.4
$$\frac{\alpha}{2} \leqq \limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} \leqq 3e(p+1)^2\alpha(2+\log p)\pi \operatorname{cosec}\pi(\lambda-p).$$

THEOREM 5.

If $f(x)$ is a meromorphic function of non-integral order and genus $p \geqq 1$, then

4.5
$$\limsup_{r\to\infty} \frac{N(r)}{T(r, f)} \geqq \frac{\sin \pi(\rho - p)}{3e\rho(2+\log p)(1+p)\pi}$$

4.6
$$\geqq \frac{\sin \pi(\rho - p)}{3e\ (2 + \log p)(1 + p)^2\pi}.$$

§ 5. S. K. Singh [10] has proved

If $f(z)$ be an entire function of non-integral order, then

5.1
$$\limsup_{r\to\infty} \frac{N(r, a)}{\log M(r, f)} > 0 \text{ for all } a, \ (0 \leqq |a| < \infty).$$

S. M. Shah [8] has proved that for functions of order less than one

5.2
$$\limsup_{r\to\infty} \frac{N(r, a)}{\log M(r, f)} \geqq 1 - \rho$$

We here prove

THEOREM 6.

If $f(z)$ be an entire function of non-integral finite order and genus p, and

5.3
$$\limsup_{r\to\infty} \frac{N(r, a)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

5.4
$$\frac{\alpha}{\lambda} \leqq \limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leqq \pi\alpha 3e(p+1)^2(2+\log p)\operatorname{cosec}\pi(\lambda-p).$$

THEOREM 7.

If $f(z)$ is an entire function of genus zero and $0 < \lambda < 1$ and

5.5 $$\limsup_{r\to\infty} \frac{n(r, a)}{r^{\lambda(r)}} = \alpha < \infty.$$

Then

5.6 $$\frac{\alpha}{\lambda} \leqq \limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leqq \pi\alpha \operatorname{cosec} (\pi\lambda).$$

THEOREM 8.

If $f(z)$ is an entire function of non-integral order ρ and genus p, then

5.7 $$\limsup_{r\to\infty} \frac{N(r, a)}{\log M(r, f)} \geqq \frac{\sin \pi(\rho - p)}{3e(p+1)^2(2+\log p)\pi}.$$

THEOREM 9.

If $f(z)$ is an entire function of order ρ, $0 < \rho < 1$ and genus zero, then

5.8 $$\limsup_{r\to\infty} \frac{N(r, a)}{\log M(r, f)} \geqq \frac{\sin \pi\rho}{\pi\rho}.$$

This theorem has been proved by Valirom [12], but we give a different proof by using proximate orders.

§ 6. PROOF OF THEOREM 1.

By 2.5 we have

6.1 $$\liminf_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} = 1.$$

Also for $x \neq a, b$

$$T(r, f) < N(r) + N(r, x) + 0 (\log r).$$

Hence

$$\begin{aligned} 1 = \liminf_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} &\leqq \liminf_{r\to\infty} \frac{N(r)}{r^{\lambda(r)}} + \liminf_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \\ &\leqq \liminf_{r\to\infty} \frac{N(r, x)}{r^{\lambda(r)}} \\ &\leqq \liminf_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} \\ &= 1 \end{aligned}$$

and the left hand equality follows.

The right hand inequality follows from the fact that $N(r, x) \leqq T(r, f)$ for all x and the theorem is proved.

PROOF of THEOREM 2.
By 1.4 we have

6.2 $$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}} = 1$$

and so the right hand inequality is obvious.

To prove the left hand inequality, suppose if possible

$$\liminf_{r\to\infty} \frac{N(r, x)}{r^{\rho(r)}} = 0 \quad \text{for} \quad x \neq a, b.$$

Hence

$$\left[\frac{N(r)}{r^{\rho(r)}} + \frac{N(r, x)}{r^{\rho(r)}}\right] \to 0 \text{ as } r \to \infty$$

and so

$$\frac{T(r, f)}{r^{\rho(r)}} \to 0 \text{ as } r \to \infty$$

and this contradicts 3.6 and the theorem follows.

PROOF of THEOREM 3.
Let

$$\limsup_{r\to\infty} \frac{N(r, x_i)}{r^{\lambda(r)}} = \alpha_i \qquad (i = 1, 2, 3).$$

Then

$$N(r, x_i) < (\alpha_i + \varepsilon_i)r^{\lambda(r)} \qquad (i = 1, 2, 3).$$

We have

$$\begin{aligned} T(r, f) &\leqq \sum_{i=1}^{3} N(r, x_i) + 0\,(\log r) \\ &\leqq \sum_{i=1}^{3} (\alpha_i + \varepsilon_i)r^{\lambda r} + 0\,(\log r) \\ &= \beta r^{\lambda(r)} + 0\,(\log r) \qquad (\beta < \infty). \end{aligned}$$

Hence

$$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} \leqq \beta < \infty$$

and the Theorem follows.

Proof of Theorem 4.
Since

$$T\left(r, \frac{\alpha f+\beta}{rf+\delta}\right) = T(r, f)\, 0(1)$$

we may suppose $a = 0$, and $b = \infty$, without any loss of generality and so we have

6.3 $$n(r) = n(r, 0) + n(r, \infty)$$

6.4 $$N(r) = N(r, 0) + N(r, \infty).$$

Also we know [5] that

6.5 $$T(r, f) \leqq 0(r^p) + 3e(2 + \log p)(1 + p)\int_0^\infty \frac{n(t)r^{p+1}dt}{t^{p+1}(t+r)}$$

By lemma 1 [2] we have

6.6 $$\int_0^\infty \frac{n(t)r^{p+1}dt}{t^{p+1}(t+r)} \leqq (p+1)\int_0^\infty \frac{N(t)r^{p+1}dt}{t^{p+1}(t+r)}.$$

Setting $S = 3e(2 + \log p)(1 + p)^2$ and since from 4.3

$$N(r) \leqq (\alpha+\varepsilon)r^{\lambda(r)} = \beta r^{\lambda(r)} \quad (\beta < \infty)$$

we get

$$T(r, f) \leqq S\beta\int_0^\infty \frac{t^{\lambda(t)}\, r^{p+1}\, dt}{t^{p+1}(t+r)} + 0(r^p).$$

Put $t = ur$

$$T(r, f) \leqq S\beta\int_0^\infty \frac{(ur)^{\lambda(ur)}\, r^{p+1}\, r\, du}{(ur)^{p+1}(ur+r)} + 0(r^p)$$

$$\sim S\beta\int_0^\infty r^{\lambda(r)}\, \frac{u^{\lambda-p-1}}{u+1}du + 0(r^p), \quad \text{by 2.7}$$

$$\sim S\beta r^{\lambda(r)}\, \pi \operatorname{cosec} \pi(\lambda-p) + 0(r^p), \text{ since } 0 < \lambda - p < 1.$$

Hence

$$\limsup_{r\to\infty} \frac{T(r, f)}{r^{\lambda(r)}} \leqq S\alpha\pi \operatorname{cosee} \pi(\lambda - p)$$

and the right hand inequality is proved.

The left hand inequality is obvious since $N(r) \leqq 2T(r, f)$ and the theorem follows.

Proof of Theorem 5.
From 1.7 we have

$$\limsup_{r\to\infty} \frac{n(r)}{r^{p(r)}} = H_1 < \infty.$$

Also since

6.7 $$\int_{r_0}^{r} t^{\rho(t)-1}\,dt \sim \frac{r^{\rho(r)}}{\rho}$$ [1]

6.8 $$N(r) \leqq \frac{H}{\rho} r^{\rho(r)}.$$

From [5] we have

6.9 $$T(r, f) \leqq 0(r^p) + 3e(2 + \log p)(1 + p)\int_0^\infty \frac{n(t) r^{p+1} dt}{t^{p+1}(t + r)}.$$

Applying lemma 1 [2] we get

6.10 $$T(r, f) \leqq 0(r^p) + 3e(2 + \log p)(1 + p)^2\int_0^\infty \frac{N(t) r^{p+1}}{t^{p+1}(t + r)}\,dt.$$

In 6.10, set $S = 3e(2 + \log p)(1 + p)^2$.
Using 6.8 we have

$$\begin{aligned} T(r, f) &\leqq 0(r^p) + S\int_0^\infty \frac{H}{\rho} \frac{t^{\rho(t)} r^{p+1}}{t^{p+1}(t + r)}\,dt \\ &\leqq 0(r^p) + \frac{S.H.}{\rho}\int_0^\infty \frac{(ur)^{\rho(ur)} r^{p+1} r}{(ur)^{p+1}(ur + r)}\,du \\ &\sim 0(r^p) + \frac{S.H.}{\rho} r^{\rho(r)}\int_0^\infty \frac{u^{\rho-p-1}}{u + 1}\,du. \end{aligned}$$

Hence

$$\begin{aligned} \limsup_{r\to\infty} \frac{T(r, f)}{r^{\rho(r)}} &\leqq S.\pi.\ \mathrm{cosec}\ \pi\ (\rho - p)\frac{H}{\rho} \\ &\leqq S.\pi.\ \mathrm{cosec}\ \pi(\rho - p) \cdot \limsup_{r\to\infty} \frac{N(r)}{r^{\rho(r)}}. \end{aligned}$$

So

$$\liminf_{r\to\infty} \frac{T(r, f)}{N(r)} \leqq \frac{\limsup_{r\to\infty} \frac{T(r, t)}{r^{\rho(r)}}}{\limsup_{r\to\infty} \frac{N(r)}{r^{\rho(r)}}} \leqq S.\pi.\mathrm{cosec}\ \pi(\rho - p)$$

and 4.6 follows.

Starting with 6.9 and proceeding similarly we have 4.5 and we note that 4.6 is a better inequality than 4.5, since $\rho < p + 1$. Proofs of Theorems 6 and 8 are omitted since they are similar to the proofs of Theorems 4 and 5.

PROOF of THEOREM 7

$$\log f(z) \leqq r \int_0^\infty \frac{n(t, a)}{t(t + r)} dt. \qquad [11]$$

From 5.6,

$$n(r, a) \leqq (\alpha + \varepsilon) r^{\lambda(r)} = \beta r^{\lambda(r)}, \qquad \beta < \infty.$$

Hence

$$\begin{aligned} \log M(r, f) &\leqq r\beta \int_0^\infty \frac{t^{\lambda(r)}}{t(t + r)} dt \\ &\sim \beta r^{\lambda(r)} \int_0^\infty \frac{u^\lambda}{u(u + 1)} dt \qquad \text{by 2.7.} \\ &= \beta r^{\lambda(r)} \frac{\pi}{\sin \pi\lambda} \end{aligned}$$

$$\limsup_{r \to \infty} \frac{\log M(r, f)}{r^{\lambda(r)}} \leqq \frac{\alpha\pi}{\sin \pi\lambda}.$$

Left hand inequality is obvious.

PROOF of THEOREM 9.
From 1.4 we have

$$N(r, a) \leqq T(r, f) \leqq r^{\rho(r)}.$$

Hence

$$\limsup_{r \to \infty} \frac{N(r, a)}{r^{\rho(r)}} = \alpha \leqq 1$$

we have [11]

$$\begin{aligned} \log M(r, f) &\leqq \int_0^\infty \frac{n(t) r}{t(t + r)} dt \\ &\leqq \int_0^\infty \frac{N(t) r}{(t + r)^2} dt \\ &\leqq \int_0^\infty \alpha \frac{t^{\rho(t)} r}{(t + r)^2} dt \\ &\sim \alpha \int_0^\infty \frac{r^{\rho(r)} u^\rho}{(u + 1)^2} du. \end{aligned}$$

Hence

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{r^{\rho(r)}} \leqq \frac{\alpha\pi\rho}{\sin \pi\rho}$$

$$\liminf_{r\to\infty} \frac{\log M(r, f)}{N(r, a)} \leqq \frac{\limsup\limits_{r\to\infty} \dfrac{\log M(r, f)}{r^{\rho(r)}}}{\limsup\limits_{r\to\infty} \dfrac{N(r, a)}{r^{\rho(r)}}} \leqq \frac{\pi\rho}{\sin \pi\rho}.$$

Lastly we note that if we use the properties of lower proximate order and assume

$$\limsup_{r\to\infty} \frac{N(r, a)}{r^{\lambda(r)}} < \infty.$$

Then we have

$$\limsup_{r\to\infty} \frac{\log M(r, f)}{N(r, a)} \leqq \frac{\pi\lambda}{\sin \pi\lambda}$$

and since

$$\frac{\pi\lambda}{\sin \pi\lambda} \leqq \frac{\pi\rho}{\sin \pi\rho}$$

and so in one way we have a better inequality.

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