COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 14 (1959-1960), p. 269-273

<http://www.numdam.org/item?id=CM_1959-1960__14__269_0>

A Counterexample to Discrete Spectral Synthesis

by

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Malliavin [2] has proved that spectral synthesis holds for the convolution algebra $L^1(G)$ of a locally compact abelian group G precisely when G is compact, i.e., precisely when $L^1(G)$ has discrete character space. Question: Does spectral synthesis hold for an arbitrary commutative semi-simple Banach algebra A with discrete maximal ideal space M? Answer: No. Not even when primary ideals are absent at infinity. We exhibit a counterexample that arises, in fact, as a convolution algebra on the circle.

§ 1. Definitions.

Let us say that **spectral synthesis** holds for A (with M discrete or not) if each closed ideal is the intersection of regular maximal ideals. (Historically and logically the above label should be applied rather to an equivalent property of the dual space A^*: that each weak-star closed invariant subspace of A^* is generated by multiplicative functionals. But here we shall be more concerned with the algebra A itself.) In studying spectral synthesis it is usual to require in advance that A be **Šilov-regular**: for each point $m \in M$, and for each neighborhood V of m, there is some $x \in A$ with $x(m) = 1$ and $x \equiv 0$ outside V. See [1] p. 83. (An algebra A with discrete M is always Šilov-regular, and in fact contains every **finitely-non-zero sequence**, i.e., every complex function x on M such that $x(m) \neq 0$ for finitely many m only.) When A is Šilov-regular, spectral synthesis can be reformulated as follows. For each (not necessarily closed) ideal $I \leqq A$, define the set $Z(I) \leqq M$ consisting of the common zeroes of the $x \in I$. For each closed subset $N \leqq M$, define the ideal $K(N) \leqq A$ consisting of all $x \in A$ that vanish identically on N, and also the ideal $J(N) \leqq A$ consisting of all compact-supported $x \in A$ that vanish identically on open sets containing N. (When M is discrete, $J(N)$ is simply all the finitely-non-zero sequences that vanish identically on N.) Then for every ideal I with $Z(I) = N$ one can prove that $J(N) \leqq I \leqq K(N)$. (The proof is especially easy with discrete M.) Hence a necessary and sufficient condition for spectral synthesis in a Šilov-regular A is that $K(N)$ be the closure of $J(N)$ for each closed $N \leqq M$. In other words, every ideal is determined by its zeroes.

§ 2. The Tauberian property.

In particular, a trivial necessary condition for spectral synthesis in a Šilov-regular A is that the ideal $J(\infty)$ consisting of all compact-supported $x \in A$ (hence, when M is discrete, consisting of all finitely-non-zero sequences) be dense in A. Such an A is called **Tauberian.** (Notice, however, that when M is discrete, $x \in A$ need not be approximable in the most straightforward way by its own "partial sums." For instance, take $A =$ the Fourier transform of the convolution algebra of continuous functions on the circle. Then one needs Fejer sums.) An ideal I with empty $Z(I)$ is sometimes called a **primary ideal at infinity.** To say that A is Tauberian is to say it contains no proper closed primary ideals at infinity. (Primary ideals at finite distance can occur in general, but not when M is discrete.)

Non-Tauberian algebras are easy to come by. Here is one with discrete M. Let A consist of all complex sequences $x = \{x(m)\}$ for which $\lim_m mx(m)$ exists and is finite. Put $||x|| = \sup_m m|x(m)|$. Since the most general continuous functional ξ is of the form $\langle x, \xi \rangle = \sum_m x(m)\xi(m)$, with $\sum_m \frac{|\xi(m)|}{m}$ finite, then the most general multiplicative functional μ is of the form $\langle x, \mu \rangle = x(m)$ for some fixed m. Hence the maximal ideal space M is the positive integers. But the closure of $J(\infty)$ in A consists of all sequences y for which $\lim_m m\, y(m) = 0$. Hence A is non-Tauberian, and spectral synthesis fails for A in a trivial and uninteresting way. We need only restrict our attention to the closed subalgebra $J(\infty)$ in order to have spectral synthesis work beautifully. Moreover, a certain natural extension of spectral synthesis holds even for the algebra A itself, viz. every closed ideal is the intersection of regular maximal ideals and closed primary ideals.

§ 3. Construction of the counterexample.

Now for a Tauberian algebra in which spectral synthesis genuinely fails. Consider the vector space $\mathscr{E}$ of 2π-periodic complex functions square-integrable on $(-\pi, \pi)$ and continuous on the closed subinterval $[-\pi/2, \pi/2]$.

(i) The norm

$$\|f\| = \left(\int_{-\pi}^{\pi} |f(s)|^2 ds\right)^{1/2} + \sup\{|f(s)| : -\pi/2 \leqq s \leqq \pi/2\}$$

makes $\mathscr{E}$ a Banach space.

(ii) With ordinary convolution as multiplication, $\mathscr{E}$ is a Banach algebra. For when $f, g \in L^2(-\pi, \pi)$, then $f*g$ is periodic-continuous (in fact, has absolutely convergent Fourier series). To see that the multiplicative inequality $\|f*g\| \leqq \|f\| \, \|g\|$ holds, write $\sqrt{2\pi} \, \|f\|_2$ for the first term in the definition of $\|f\|$. Then

$$|(f*g)(s)| = \left|\frac{1}{2\pi} \int_{-\pi}^{\pi} f(s-t)g(t)dt\right| \leqq \|f\|_2\|g\|_2.$$

Hence

$$\sup\{|(f*g)(s)| : -\pi/2 \leqq s \leqq \pi/2\} \leqq \|f\|_2\|g\|_2,$$

and also

$$\|f*g\|_2 \leqq \|f\|_2\|g\|_2.$$

Hence

$$\|f*g\| \leqq (\sqrt{2\pi} + 1)\|f\|_2\|g\|_2 \leqq 2\pi \, \|f\|_2\|g\|_2 \leqq \|f\| \, \|g\|.$$

(iii) The trigonometric polynomials form a dense subalgebra of $\mathscr{E}$. For they are uniformly dense in the space of 2π-periodic continuous functions. And the continuous functions are L^2-dense in $L^2(-\pi, \pi)$.

(iv) The most general non-zero multiplicative functional μ on $\mathscr{E}$ is of the form

$$\mu(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) e^{-ims} ds$$

for some fixed m. For, by (iii) it is enough that μ have this form for each trigonometric monomial p_n (definition: $p_n(s) = e^{ins}$). And because the p_n are orthogonal idempotents, then $\mu(p_m) = 1$ for some one p_m while $\mu(p_n) = 0$ for $n \neq m$.

Finally, we define the algebra E as the Fourier transform of $\mathscr{E}$. The elements x of E are thus the double-ended sequences $\{x(m)\}$ that arise as Fourier coefficients of the $f \in \mathscr{E}$. Under pointwise multiplication, E is an algebra isomorphic to the convolution algebra $\mathscr{E}$. And by transporting the $\mathscr{E}$ norm to E we make it also a Banach algebra. By (iv) E has the integers as maximal ideal space. And by (iii) E is Tauberian.

§ 4. Approximate units.

Returning momentarily to a general commutative semi-simple Banach algebra A, let us say that an $x \in A$ has **ap-**

proximate units if x belongs to the closure of the principal ideal Ax. (Among the various possible notions of approximate unit, this is perhaps the very weakest. In an L^1 algebra, for instance, there always exists a single bounded net $\{y_\nu\}$ such that $y_\nu * x$ converges to x for all x.) Our present interest in approximate units stems from the following fact (essentially due to Ditkin). *In order that spectral synthesis hold for a Tauberian A with discrete M it is necessary and sufficient that every $x \in A$ have approximate units.* Necessity: Let $N \leqq M$ be the zeroes of $x \in A$. Then also $N = Z(Ax)$. It follows that $K(N)$ coincides with the closure of Ax, since these two ideals have the same zeroes. In particular, since $x \in K(N)$, then x belongs to the closure of Ax. Sufficiency: Define N as above. We must approximate x by something in $J(N)$. Choose $y \in A$ such that $||x - yx|| < \varepsilon$. And choose $w \in J(\infty)$ such that $||y - w|| < \varepsilon$. Then $wx \in J(N)$ and

$$||x - wx|| \leqq ||x - yx|| + ||yx - wx|| \leqq \varepsilon + \varepsilon ||x||.$$

§ 5. An element without units.

We have to find some unitless element x in our algebra E. That spectral synthesis must then fail for E depends, of course, only on the first half of Ditkin's theorem. It will be more convenient to work directly with the convolution algebra $\mathscr{E}$. We claim that the characteristic function f of the closed interval $[-\pi/2, \pi/2]$ does not have approximate units in $\mathscr{E}$. More explicitly, we claim that if $f(s) = 1$ for $|s| \leqq \pi/2$ and $f(s) = 0$ for $\pi/2 < |s| \leqq \pi$, then $||g * f - f|| > 1/5$ for all $g \in \mathscr{E}$.

For suppose, $||g*f - f|| \leqq 1/5$. In particular, $|(g*f)(-\pi/2) - 1| \leqq 1/5$ and $|(g*f)(\pi/2) - 1| \leqq 1/5$. In terms of the indefinite integral $G = \dfrac{1}{2\pi}\displaystyle\int_{-\pi} g(s)ds$, we can write

$$(g*f)(\pi/2) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} g(\pi/2 - s)ds = \frac{1}{2\pi}\int_0^{\pi} g(t)dt = G(\pi) - G(0).$$

Similarly, $(g*f)(-\pi/2) = G(0) - G(-\pi/2) = G(0)$. Hence, $|G(\pi) - 2| = |G(\pi) - G(0) - 1 + G(0) - 1| \leqq |G(\pi) - G(0) - 1\,| + |G(0) - 1| = |(g*f)(\pi/2) - 1)| + |(g*f)(-\pi/2) - 1\,| \leqq 1/5 + 1/5$.

To sum up,

$$|G(\pi) - 2| \leqq 2/5.$$

On the other hand, let us write x and y respectively for the elements of E corresponding to the above elements f and g of $\mathscr{E}$.

We are supposing $||yx - x|| \leqq 1/5$, hence $|y(0)x(0) - x(0)| \leqq 1/5$. But $x(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(s)ds = 1/2$, and $y(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} g(s)ds = G(\pi)$; hence $|(1/2)G(\pi) - 1/2\,| \leqq 1/5$, or

$$|G(\pi) - 1| \leqq 2/5.$$

This estimate contradicts the estimate obtained in the last paragraph. Hence $x \in E$ does not have approximate units and spectral synthesis fails for E.

§ 6. Character multipliability.

In conclusion, let us point out that the pathology of E has its source in the fact that the multiplier algebra for E does not contain most functions z of the form $z(m) = \zeta^m$ for fixed complex ζ of modulus 1. Or what is the same, $\mathscr{E}$ is not translation-invariant. For *suppose, that the commutative semi-simple Tauberian Banach algebra A has a discrete abelian group G as maximal ideal space. And suppose that for every $x \in A$ and every character* z on G *we have $zx \in A$. Then spectral synthesis must hold for A.* Proof: By the closed graph theorem, each mapping $x \to zx$ is continuous $A \to A$. And because A is Tauberian, then by Šilov [3] p. 9 and p. 13, each mapping $z \to zx$ is continuous $\hat{G} \to A$. With all this continuity it is legitimate to use A-valued integration, and then the approximate units y_ν in $L^1(\hat{G})$ will work also for A. Hence spectral synthesis must hold for A.

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[Oblatum 26 september 1959].