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EUGENE LUKACS

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On Certain Periodic Characteristic Functions

by

Eugene Lukacs

Office of Naval Research, Washington, D.C.

1. Introduction.

The purpose of this note is to derive certain properties of periodic characteristic functions and to determine those distributions whose characteristic functions are entire periodic functions.

Let $F(x)$ be a probability distribution, that is, a never-decreasing, right continuous function such that $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$. The Fourier transform of $F(x)$, that is the function

$$f(z) = \int_{-\infty}^{\infty} e^{izx} dF(x) \tag{1}$$

is called the characteristic function of $F(x)$.

A distribution is called a *lattice distribution* if it is purely discontinuous and if its discontinuity points form a (proper or improper) subset of a sequence of equidistant points.

A characteristic function is said to be an *analytic characteristic function* if it coincides with an analytic function in some neighborhood of the origin.

We prove the following theorems:

THEOREM 1. *An analytic characteristic function which is single valued and periodic has either a real or a purely imaginary period. The period is real if, and only if, the characteristic function belongs to a lattice distribution.*

THEOREM 2. *A characteristic function is an entire periodic function (not $\equiv 1$) if, and only if, it is the characteristic function of a lattice distribution.*

2. The lemmas.

For the proof of these theorems we need two lemmas which we derive in this section.

LEMMA 1. *A characteristic function $f(t)$ assumes the value 1 for some real $t_0 \neq 0$ if, and only if, it is the characteristic function of a lattice distribution.*

PROOF: Let us first assume that for some real $t_0 \neq 0$ the value $f(t_0) = f(0) = 1$. We see then from (1) that

$$\int_{-\infty}^{\infty} (1-e^{it_0x})dF(x) = 0$$

and therefore also

$$\int_{-\infty}^{\infty} (1-\cos t_0x)dF(x) = 0$$

Since the function $1-\cos t_0x$ is continuous and nonnegative this relation can hold only if $F(x)$ is a purely discontinuous distribution such that its discontinuity points are contained in the point set $2\pi s/t_0$ ($s = 0, \pm 1, \pm 2, \ldots$) where $1-\cos t_0x$ vanishes. The distribution function $F(x)$ has therefore necessarily the form

$$F(x) = \sum_{s=-\infty}^{\infty} p_s \varepsilon(x-2\pi s/t_0) \tag{2}$$

where

$$p_s \geqq 0, \quad \sum_{s=-\infty}^{\infty} p_s = 1$$

and where

$$\varepsilon(x) = \begin{cases} 0 \text{ if } x < 0 \\ 1 \text{ if } x \geqq 0 \end{cases} \tag{3}$$

is the degenerate distribution. On the other hand, if $F(x)$ has the form (2) then

$$f(t) = \sum_{s=-\infty}^{\infty} p_s \exp(2\pi its/t_0) \tag{4}$$

so that $f(t_0) = 1$.

Lemma 1 is a particular case of a somewhat more general result of A. Wintner ([2], p. 48). Wintner proves that $|f(\xi)| = 1$ for some real $\xi \neq 0$ if, and only if, $F(x)$ is a lattice distribution.

LEMMA 2. *Let $f(z)$ be an analytic characteristic function with the strip of convergence*[1] $-\alpha < Im(z) < \beta$ *and assume that there are three real numbers n_1, n_2, n_3 such that*

(i) $-\alpha < n_1 < n_2 < n_3 < \beta$,
(ii) $n_1 + n_3 = 2n_2$,
(iii) $f(in_1) = f(in_2) = f(in_3)$,

Then $f(z) \equiv 1$.

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[1]) For the properties of analytic characteristic functions see [1].

PROOF: We rewrite (iii) as

$$\begin{cases} f(in_2)-f(in_1) = 0 \\ f(in_3)-f(in_2) = 0 \end{cases} \tag{5}$$

We conclude from (i) that the three points in_j ($j = 1, 2, 3$) are in the interior of the strip of convergence of $f(z)$ so that the integral representation (1) is valid at these points. Then

$$f(in_j) = \int_{-\infty}^{\infty} e^{-xn_j} dF(x) \text{ for } j = 1, 2, 3$$

and

$$f(in_{j+1})-f(in_j) = \int_{-\infty}^{\infty} [e^{-xn_{j+1}}-e^{-xn_j}]\, dF(x) \text{ for } j = 1, 2.$$

If we put

$$\begin{cases} g_j(x) = e^{-xn_{j+1}}-e^{-xn_j} & (j = 1, 2) \\ I_j = \int_{-\infty}^{\infty} g_j(x)\, dF(x) & (j = 1, 2) \end{cases} \tag{6}$$

We can write (5) as

$$I_1 = I_2 = 0 \tag{7}$$

We carry the proof of lemma 2 indirectly and assume therefore tentatively that $f(z) \not\equiv 1$ and show that this leads to a contradiction with (7). If $f(z) \not\equiv 1$, then also $f(t) \not\equiv 1$ for real t and the corresponding distribution function is nondegenerate and therefore there exists a real $h > 0$ such that at least one of the two relations

$$\int_{+h}^{+\infty} dF(x) = 1-F(h) > 0 \tag{8a}$$

$$\int_{-\infty}^{-h} dF(x) = F(-h) > 0 \tag{8b}$$

is satisfied.

From (ii) we see that $n_3-n_2 = n_2-n_1$ and obtain therefore from (6)

$$g_1(x)-g_2(x) = -e^{-xn_1}[e^{-x(n_2-n_1)}-1]^2.$$

Therefore

$$\begin{cases} g_1(x) < g_2(x) \text{ if } x \neq 0 \\ g_1(0) = g_2(0) = 0 \end{cases} \tag{9}$$

Let us first assume that (8a) is satisfied and choose h accordingly. We see then from (9) that

$$\int_0^h g_1(x)\, dF(x) \leqq \int_0^h g_2(x)\, dF(x) \leqq 0 \tag{10a}$$

and

(11) $$0 \leqq \int_{-\infty}^{0} g_1(x)\,dF(x) \leqq \int_{-\infty}^{0} g_2(x)\,dF(x)$$

From (8a) we conclude that there exists a finite $A>h$ such that $\int_h^A dF(x)>0$. We see from (9) that $g_2(x)-g_1(x)>0$ for $0<h\leqq x\leqq A$ and there exists a real number $\mu>0$ such that $g_2(x)-g_1(x)\geqq\mu>0$ for $h\leqq x\leqq A$. Since $g_2(x)-g_1(x)$ is positive for $x\geqq h$, we see that

$$\int_h^\infty [g_2(x)-g_1(x)]dF(x) \geqq \int_h^A [g_2(x)-g_1(x)]\,dF(x) \geqq \mu\int_h^A dF(x) > 0$$

and therefore

(10b) $$\int_h^\infty g_1(x)\,dF(x) < \int_h^\infty g_2(x)\,dF(x)$$

Adding (10a), (10b) and (11) we obtain

$$\int_{-\infty}^{+\infty} g_1(x)\,dF(x) < \int_{-\infty}^{+\infty} g_2(x)\,dF(x)$$

or

(12) $$I_1<I_2.$$

In case (8b) is valid we obtain by a similar reasoning again (12). Thus if at least one of the conditions (8a) of (8b) is satisfied, relation (12) must be valid in contradiction to (7). Therefore no h exists such that at least one of the relations (8a) or (8b) is satisfied. But this means that $1-F(h) = F(-h) = 0$ for any $h>0$, in other words, $\int_{-h}^{+h} dF(x) = 1$ for any $h>0$, i.e. the distribution $F(x)$ is necessarily equal to the degenerate distribution $\varepsilon(x)$.

In this proof condition (iii) is apparently not used fully. In the argument only the relation $f(in_2)-f(in_1) = f(in_3)-f(in_2) = M$ was used. Nevertheless (iii) can not be weakened since for $M \neq 0$ the function $f(t)$ could not be a characteristic function.

3. Proof of the theorems.

We next prove theorem 1. Let us therefore assume that $f(z)$ is a single valued analytic characteristic function which has the period

(13) $$\omega = \xi+i\eta \qquad (\xi,\ \eta \text{ real}).$$

It is known (see for instance [1]) that the relation

(14) $$f(-\bar{z}) = \overline{f(z)}$$

is satisfied for every analytic characteristic function in its domain of regularity. On account of the periodicity of $f(z)$ we see that

$$f(-\xi-i\eta) = f(0) = 1,$$

we deduce then from (14) that

$$f(\xi-i\eta) = 1.$$

Adding to the argument the period $\omega = \xi+i\eta$, we obtain

$$f(2\xi) = 1. \tag{15}$$

We consider first the case where $\xi \neq 0$. In this case we see from (15) and from lemma 1 that $f(z)$ is the characteristic function of a lattice distribution. We see from (2) and (4) that its characteristic function is given by

$$f(t) = \sum_{s=-\infty}^{\infty} p_s \exp(it\pi s/\xi) \qquad (p_s \geqq 0,\ \sum_{-\infty}^{+\infty} p_s = 1) \tag{16}$$

Therefore $f(t)$ is a simply periodic function with the real period 2ξ while $\eta = 0$. If on the other hand $\xi = 0$, then (15) is satisfied and we see from (13) that $\omega = i\eta$, i.e. that the period is purely imaginary. This establishes theorem 1.

The case of a characteristic function which has a purely imaginary period can actually occur. As an example we mention the well known characteristic function $f(t) = \dfrac{1}{\cosh t}$.

From the proof of theorem 1 we obtain the following corollary:

Corollary to Theorem 1. *A characteristic function which does not reduce to a constant can not be doubly periodic.*

We proceed to prove theorem 2. Let therefore $f(t)$ be an entire characteristic function which does not reduce to a constant and assume that it is periodic. From theorem 1 we see that $f(t)$ must be simply periodic with either a real or a purely imaginary period. From lemma 2 we see that $f(t)$ can not have a purely imaginary period. The theorem follows then from lemma 1.

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