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Higher-dimensional field theory. III. Normalization

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Higher-Dimensional field Theory

III. Normalization

by

E. Snapper

Conventions.

We continue all conventions of [1] and [2], referred to respectively as *FI* and *FII*. Hence, E/F is a finitely generated field extension whose degree of transcendency is $r \geqq 1$; F^* is the algebraic closure in E of F and the two fields F and F^* may very well be distinct. "Module" always means "finitely generated module of E" and for any module M, $F\langle M\rangle$ denotes the integral closure in E of the ring $F[M]$ and A the algebraic closure in E of the field $F(M)$. When we speak of a valuation V_B of E, it is always understood that $B \neq E$ and that $F \subset B$; by the place $\mathfrak{P}$ of V_B or of B, we mean the unique maximal ideal of B. An ideal of a ring is called nontrivial if it is neither the 0-ideal nor the whole ring. Again, the "remarks" do not form a part of the logical development of the theory, but they have been added only to show those readers who know the theory of algebraic varieties, how our field theory ties in with geometry. The introduction of *FI* contains a short introduction to the present paper.

1. *The local rings of a projective class of modules.* Let C be a projective class of modules; we make no assumption about the dimension of C. The rings $F[M]$, where $M \in C$, are called *the coordinate rings of* C. The rings of quotients $(F[M])_{\mathfrak{p}}$, where $M \in C$ and $\mathfrak{p}$ is a nontrivial prime ideal of $F[M]$, are called *the local rings of* C; of course, $(F[M])_{\mathfrak{p}}$ consists of the quotients a/b, where $a, b \in F[M]$ and $b \notin \mathfrak{p}$. When $\dim(C) = 0$, we know that all the coordinate rings of C are equal to the field $F(C)$ of C and hence C has then no local rings. In general, one and the same local ring of C may arise from many different coordinate rings of C and, in this connection, we need the following statement.

STATEMENT 1.1. *Let $M_j \in C$ and let $\mathfrak{p}_j$ be a nontrivial prime ideal of $F[M_j]$, for $j = 1, 2$. Then, $(F[M_1])_{\mathfrak{p}_1} = (F[M_2])_{\mathfrak{p}_2}$ if and only* *if there exists a valuation V_B of E, with place $\mathfrak{P}$, which has the following two properties*: (1) $M_1 \subset B$ *and* $M_2 \subset B$; (2) $F[M_1] \cap \mathfrak{P} = \mathfrak{p}_1$ *and* $F[M_2) \cap \mathfrak{P} = \mathfrak{p}_2$.

PROOF. Let $(F[M_1])_{\mathfrak{p}_1} = (F[M_2])_{\mathfrak{p}_2}$. We designate this ring by R and its unique maximal ideal by $\mathfrak{p}$; then, $F[M_1] \cap \mathfrak{p} = \mathfrak{p}_1$ and $F[M_2] \cap \mathfrak{p} = \mathfrak{p}_2$. Since $\mathfrak{p}$ is a nontrivial prime ideal of R, there exists a valuation V_B of E, with place $\mathfrak{P}$, which is such that $R \subset B$ and $R \cap \mathfrak{P} = \mathfrak{p}$; clearly, V_B has the two properties of statement 1.1. Conversely, let the valuation V_B of E have the two required properties. For reasons of symmetry, all we have to show is that then $(F[M_1])_{\mathfrak{p}_1} \subset (F(M_2])_{\mathfrak{p}_2}$. Hereto, let $c \in (F[M_1])_{\mathfrak{p}_1}$, i.e. $c = a/b$ where a, $b \in F[M_1]$ and $b \notin \mathfrak{p}_1$; this implies that b is a unit of B. Since a, $b \in F[M_1]$ and $1 \in M_1$, there exists an $h \geqq 0$ such that $a, b \in M_1^h$. Since $M_1, M_2 \in C$, there exists an $e \in M_1'$ such that $(1/e)M_1 = M_2$, and hence such that $(1/e^h)M_1^h = M_2^h$. We now first conclude that $c = (a/e^h)/(b/e^h)$, where a/e^h, $b/e^h \in M_2^h \subset F[M_2]$. Furthermore, since $1 \in M_1$, $1/e \in M_2 \subset B$, and since also $e \in M_1 \subset B$, we see that e is a unit of B. This shows that b/e^h is a unit of B and consequently that $b/e^h \notin \mathfrak{p}_2$. We have now proved that $c \in (F[M_2])_{\mathfrak{p}_2}$, and hence we are done.

We have defined in *FII*, definition 1.3, when a subset C_0 of C covers C. Statement 1.1 enables us to give the following characterization of a covering C_0 of C in terms of local rings.

STATEMENT 1.2. *A subset C_0 of C covers C if and only if the local rings, which arise from the modules of C_0, constitute the complete set of all local rings of C.*

PROOF. *Let C_0 cover C.* We have to show that, when $M \in C$ and $\mathfrak{p}$ is a nontrivial prime ideal of $F[M]$, we can find an $N \in C_0$ and a nontrivial prime ideal $\mathfrak{q} \in F[N]$, which are such that $(F[M])_{\mathfrak{p}} = (F[N])_{\mathfrak{q}}$. Let V_B be a valuation of E with place $\mathfrak{P}$, where $M \subset B$ and $F[M] \cap \mathfrak{P} = \mathfrak{p}$. Since C_0 covers C, there exists a module $N \in C_0$, such that $N \subset B$. Clearly, $F[N] \cap \mathfrak{P} \neq F[N]$; furthermore, $F[N] \cap \mathfrak{P} \neq 0$, since otherwise the field $F(N) = F(M)$ of C would be contained in B, which contradicts the fact that $F[M] \cap \mathfrak{P} \neq 0$. We designate the ideal $F[N] \cap \mathfrak{P}$ by $\mathfrak{q}$ and conclude from statement 1.1 that $(F[M])_{\mathfrak{p}} = (F[N])_{\mathfrak{q}}$. Conversely, let the subset C_0 of C be such that the local rings, which arise from the modules of C_0, constitute the complete set of local rings of C. We have to show that, when V_B is a valuation of E with place $\mathfrak{P}$, there exists an $N \in C_0$ such that $N \subset B$. Hereto choose an $M \in C$ which is such that $M \subset B$. Designating $F[M] \cap \mathfrak{P}$ by $\mathfrak{p}$, there exists an $N \in C_0$ and a nontrivial prime ideal $\mathfrak{q} \subset F[N]$, such that $(F[M])_{\mathfrak{p}} = (F[N])_{\mathfrak{q}}$. Clearly, $N \subset (F[N])_{\mathfrak{q}} \subset B$, and we are done.

REMARK 1.1. Let S be an irreducible algebraic variety, defined over F as groundfield, and with E as field of rational functions. Let C be the projective class of modules which arises from the affine models of S (See *FII*, remark 1.1). All terms, introduced in this section, agree exactly with the terms used in the study of the geometry of S. Our coordinate rings are the affine coordinate rings of S.

2. *Locally normal projective classes of modules.* We say that *the projective class C is locally normal, if all its coordinate rings are integrally closed in E*, i.e. if $F[M] = F\langle M\rangle$ for all $M \in C$.

STATEMENT 2.1. *If the subset C_0 of C covers C and if $F[M] = F\langle M\rangle$ for all $M \in C_0$, C is already locally normal.*

PROOF. If $\dim(C) = 0$, every nonempty subset C_0 of C covers C and the only coordinate ring of C is then its field $F(C)$; consequently, statement 2.1 is then obviously correct. Let us now assume that $\dim(C) \geqq 1$, that C_0 is as in statement 2.1 and that $M \in C$; we have to show that then $F[M] = F\langle M\rangle$. If $\mathfrak{p}$ is a nontrivial prime ideal of $F[M]$, there exists an $N \in C_0$ and a nontrivial prime ideal $\mathfrak{q} \subset F[N]$, such that $(F[N])_{\mathfrak{q}} = (F[M])_{\mathfrak{p}}$; consequently, since $(F[N]_{\mathfrak{q}}$ is integrally closed in E, so is $(F[M])_{\mathfrak{p}}$. Furthermore, $F[M]$ is a Noetherian ring which is not a field, and hence $F[M] = \bigcap_{\mathfrak{p}} (F[M])_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through the nontrivial prime ideals of $F[M]$. This shows that $F[M]$ is integrally closed in E and we are done.

For any projective class C and rational integer $h \geqq 0$, the class $|C^h|_i$ is well defined (see *FII*, sections 1 and 3).

THEOREM 2.1. *To every projective class of modules C is associated a rational integer $h_0 \geqq 0$, which is such that when $h \geqq h_0$, the class $|C^h|_i$ is locally normal.*

PROOF. Let $M_1, \ldots, M_n$ be a finite set of modules of C which covers C. To each module M_j is associated a rational integer $s_0^{(j)} \geqq 0$, which has the property that, if $s \geqq s_0^{(j)}$, $F[|M_j^s|_i] = F\langle M_j\rangle$ (see *FI*, statement 5.1 and remark 5.1); since $F\langle M_j\rangle = F\langle |M_j^s|_i\rangle$, when $s \geqq 1$, this means that $F[|M_j^s|_i]$ is integrally closed in E, when $s \geqq s_0^{(j)}$. Clearly then, any rational integer $h_0 \geqq \text{Max}\,(s_0^{(1)}, \ldots, s_0^{(n)})$ has the required property. Namely, when $h \geqq h_0$, we conclude from *FII*, statements 1.5 and 3.3, that the modules $|M_1^h|_i, \ldots, |M_n^h|_i$ cover $|C^h|_i$. Furthermore, each of the rings $F[|M_j^h|_i]$, for $j = 1, \ldots, n$, is integrally closed in E, and hence statement 2.1

implies that the class $|C^h|_i$ is locally normal. The theorem is now proved.

If $\dim(C) \geqq 1$, we know from *FII*, statements 1.5 and 3.3, that the sets of divisors of the first kind of the projective classes C and $|C^h|_i$, for $h \geqq 1$, coincide. This fact, together with the just proven theorem, enables us in any investigation which involves only the divisors of the first kind of a projective class, to restrict ourselves to locally normal classes. (We can even restrict ourselves to arithmetically normal classes, according to theorem 3.1).

In order to investigate locally normal classes further, we have to make a few remarks about the module $Q(M; h)$, discussed in *FI*, section 5.

STATEMENT 2.2. *Let M be any nonzero module. For every rational integer $h \geqq 0$, there exists a rational integer s_h depending on h, which is such that when $s \geqq s_h$, $M^s Q(M; h) = M^{s+h}$.*

PROOF. Let M and h be fixed. Since $M^h \subset Q(M; h)$, $M^{s+h} \subset M^s Q(M; h)$ for all $s \geqq 0$. Denote, for $s \geqq 0$, by N_s the module which consists of all elements $e \in E$, which have the property that $eM^s \subset M^{s+h}$; then clearly, $Q(M; h) = \bigcup_{s=0}^{\infty} N_s$ and $N_s \subset N_{s+1}$. The finite dimensionality of $Q(M; h)$ implies that there exists an s_h such that, if $s \geqq s_h$, $N_s = N_{s_h}$ and hence $N_s = Q(M; h)$. We see immediately that $M^s N_s \subset M^{s+h}$, for all $s \geqq 0$, and hence statement 2.2 is proved.

The equivalence of (1) and (2) of the next theorem implies that, when C is locally normal and $M \in C$, the integer s_h of statement 2.2 can be chosen so large that it is independent of h. The author does not know whether the same is true for all nonzero modules M.

THEOREM 2.2. *If C is a projective class of modules, the following three statements are equivalent*: (1) *C is locally normal*; (2) *there exists a rational integer $u_0 \geqq 0$, which is such that when $u \geqq u_0$ and $M \in C$, $M^{u+h} = M^u |M^h|_i$ for all $h \geqq 0$*; (3) *there exists a rational integer $t_0 \geqq 0$, which is such that when $t \geqq t_0$ and $M \in C$, $M^t = |M^t|_i$.*

PROOF. (1) *implies* (2). Let C be locally normal. If $M \in C$ and $a \in M'$, $(1/a)M \in C$ and hence $F[(1/a)M] = F\langle(1/a)M\rangle$; consequently, we conclude from *FI*, statement 5.2, that then $Q(M; h) = |M^h|_i$ for all $h \geqq 0$. Let the rational integer k_0 be as in *FI*, theorem 3.1, and let $u_0 \geqq \mathrm{Max}(s_0, \ldots, s_{k_0})$, where s_h has the same meaning as in statement 2.2, above; we show that u_0 has the required property. If $u \geqq u_0$, statement 2.2 tells us that $M^{u+h} = M^u |M^h|_i$ for $h = 0, 1, \ldots, k_0$. If $h = k_0 + n$ where

$n \geqq 0$, $M^{u+k_0+n} = M^{u+n}|M^{k_0}|_i$ because of statement 2.2, and $M^n|M^{k_0}|_i = |M^{n+k_0}|_i$ because of *FI*, theorem 3.1; hence, $M^{u+h} = M^u M^n|M^{k_0}|_i = M^u|M^h|_i$. This proves that $M^{u+h} = M^u|M^h|_i$ for all $h \geqq 0$. Of course, if the module N is proportional to M, i.e. if $N = eM$ for some $e \in E'$, also $N^{u+h} = N^u|N^h|_i$ for all $h \geqq 0$ and $u \geqq u_0$; this then holds in particular for all the modules of C and we are done.

(2) *implies* (3). Let u_0 be as in (2), let again k_0 be as in theorem 3.1 of *FI*, and let $m = \text{Max}(u_0, k_0)$; we show that $t_0 = 2m$ has the required property. If $M \in C$, $M^{2m} = M^m|M^m|_i$ because of (2), and $|M^{2m}|_i = |M^m|_i M^m$ because of theorem 3.1 of *FI*; hence certainly, $M^{t_0} = |M^{t_0}|_i$. Now let $t = t_0 + n$, where $n \geqq 0$. Then, $|M^{t_0+n}|_i = |M^{t_0}|_i M_n$ because of theorem 3.1 of *FI*, and hence $|M^t|_i = M^{t_0} M^n = M^t$. It follows again that $|N^t|_i = N^t$, when N is proportional to M and $t \geqq t_0$, and hence we are done.

(3) *implies* (1). Let t_0 be as in (3) and let $M \in C$. We know that $F\langle M\rangle = F[|M^t|_i]$, when t is large enough (see *FI*, statement 5.1 and remark 5.1). Consequently, when $t \geqq t_0$, $F\langle M\rangle = F[M^t]$, and we have often observed that $F[M^t] = F[M]$ when $t \geqq 1$. Theorem 2.2 is now completely proved.

We have observed that, when the class C is locally normal and $M \in C$, $Q(M; h) = |M^h|_i$ for all $h \geqq 0$; hence we conclude from theorem 2.2 that then $Q(M; h) = M^h$, when h is large enough. The author does not know whether always, for any nonzero module M whatsoever, $Q(M; h) = M^h$ when h is large enough.

In order to interpret theorem 2.2 in terms of linear systems, we assume that C is an r-dimensional projective class. We consider the linear systems relative to the set $\mathfrak{B}$ of divisors of the first kind of C. Let $M \in C$ and let $G(M)$ be the cycle of M. The linear system without fixed cycle $g(M; -G(M))$ is evidently independent of the choice of $M \in C$; we call this system, *the linear system of the hyperplane sections of* C and call its elements *the hyperplane sections of* C. When $h \geqq 0$, the multiple $hg(M; -G(M)) = g(M^h; -hG(M))$ of the linear system of hyperplane sections of C clearly is well-behaved, since $\mathfrak{B}(M^h) = \mathfrak{B}(M) = \mathfrak{B}$ when $h \geqq 1$. Consequently, the complete linear system which contains $hg(M; -G(M))$ is the system $|-hG(M)| = g(|M^h|_i; -hG(M))$ (see *FII*, theorem 6.2). When we furthermore assume that $F = F^*$, $g(M^h; -hG(M)) = g(|M^h|_i; -hG(M))$ if and only if $M^h = |M^h|_i$. We have now arrived at the following formulation of the equivalence of (1) and (3) of theorem 2.2. *Let* C *denote an* r*-dimensional projective class, let* $F = F^*$ *and let* g *denote the linear system of* *the hyperplane sections of C. Then, C is locally normal if and only if there exists a rational integer $t_0 \geqq 0$, which is such that when $t \geqq t_0$, the linear system tg is complete.* After the above explanations it is also clear that the interpretation in terms of linear systems of the equivalence of (1) and (2) of theorem 2.2 is the following. *When C, F and g are as above, C is locally normal if and only if there exists a rational integer u_0, such that when $u \geqq u_0$, $ug+hg = ug+|hg|$.* Here, $|hg|$ denotes the complete linear system to which hg belongs; the sum of linear systems was discussed in *FII*, section 6.

Remark 2.1. Let S and C be as in remark 1.1. All terms, used in this paper, then agree exactly with those used by Zariski in the study of the geometry of S. The notion of local normality, theorems 2.1 and 2.2 interpreted in terms of S and its linear systems, can all be found in [3] or in the manuscript Z, mentioned in *FI*. We do not require for our theorems that the field $F(C)$ of C is E, since we have no opportunity to use this restriction; we even allow the dimension of C to be less than r. Whenever $F(C) \neq E$, our theorems refer to normalizations with respect to S of rational images of S. The same remarks are valid for the next section.

3. *Arithmetically normal projective classes of modules.* Our theorem on arithmetic normalization depends on the following corollary of theorem 3.1 of *FI*.

Statement 3.1. *Let M be a module and let k_0 be as in theorem 3.1 of FI. Then, for any $k \geqq k_0$ and $t \geqq 1$, $(|M^k|_i)^t = |M^{kt}|_i$.*

Proof. Let M, k_0, k and t be as in statement 3.1. When $t = 1$, the statement is trivial, and hence we make the induction hypothesis that the statement has been proved for all rational integers $t' < t$. We know that the product of the integral closures of a set of modules is always contained in the integral closure of the product of these modules, and hence $(|M^k|_i)^t \subset |M^{tk}|_i$. Since $t \geqq 1$, the meaning of k_0 implies that $|M^{tk}|_i = |M^k|_i M^{k(t-1)}$ from which we conclude that $|M^{tk}|_i \subset |M^k|_i\, |M^{k(t-1)}|_i$. According to our induction hypothesis, $|M^{k(t-1)}|_i = (|M^k|_i)^{t-1}$, which implies that $(|M^k|_i)^t \subset |M^{tk}|_i \subset (|M^k|_i)^t$, and we are done.

C always designates a projective class of modules.

Definition 3.1. *Let $M \in C$. Then, C is called arithmetically normal if $M^t = |M^t|_i$ for all $t \geqq 1$.*

It is clear that, since the modules of C are proportional, the notion of arithmetic normality does not depend on the choice of $M \in C$, but only on C itself. Furthermore, it follows from the equivalence of (1) and (3) of theorem 2.2, that an arithmetically normal class is locally normal.

THEOREM 3.1. *Let $M \in C$ and let k_0 be as in theorem 3.1 of FI. Then, when $k \geqq k_0$, the class $|C^k|_i$ is arithmetically normal.*

PROOF. Since $M \in C$, $|M^k|_i \in |C^k|_i$ and hence we have to prove that $(|M^k|_i)^t = |(|M^k|_i)^t|_i$ for $k \geqq k_0$ and $t \geqq 1$. According to statement 3.1, the left hand side of this equality is equal to $|M^{kt}|_i$ and the right hand side to $||M^{kt}|_i|_i$; for any module N, it is always true that $|N|_i = ||N|_i|_i$, and hence the theorem is proved.

Observe that the rational integer k_0 of theorem 3.1 does not depend on the choice of $M \in C$.

Let us again assume that $\dim(C) = r$, that $F = F^*$ and that g is the linear system of the hyperplane sections of C. It is clear from the remarks of the previous section, that definition 3.1 then states that *C is arithmetically normal if and only if tg is complete for all $t \geqq 1$.*

REMARK 3.1. Let S and C be as in remark 1.1. Theorem 3.1, expressed for S and its linear systems, occurs in Zariski's manuscript Z, mentioned in *FI*.

4. *Extension to several classes.* We mention here briefly that many of the theorems can be extended to the product of several projective classes. We have given enough material in section 6 of *FI*, so that the reader can carry out these extensions. As an example, we state the extension of theorem 2.2.

Let $C_1, \ldots, C_n$ be projective classes of modules. The product class $C_1 \cdot \ldots \cdot C_n$ was discussed in *FII*, section 1.

THEOREM 4.1. *If $C_1, \ldots, C_n$ are projective classes of modules, the following three statements are equivalent.* (1) *The product class $C_1 \cdot \ldots \cdot C_n$ is locally normal*; (2) *there exist nonnegative rational integers $u_0^{(1)}, \ldots, u_0^{(n)}$, which are such that when $u_j \geqq u_0^{(j)}$ and $M_j \in C_j$* for $j=1, \ldots, n$, $M_1^{u_1+h_1} \cdot \ldots \cdot M_n^{u_n+h_n} = M_1^{u_1} \cdot \ldots \cdot M_n^{u_n}|M_1^{h_1} \cdot \ldots \cdot M_n^{h_n}|_i$ for all $h_j \geqq 0$; (3) there *exist nonnegative rational integers $t_0^{(1)}, \ldots, t_0^{(n)}$, which are such that when $t_j \geqq t_0^{(j)}$ and $M_j \in C_j$ for $j = 1, \ldots, n$, $M_1^{t_1} \cdot \ldots \cdot M_n^{t_n} = |M_1^{t_1} \cdot \ldots \cdot M_n^{t_n}|_i$.*

REMARK 4.1. We have already pointed out in *FII*, remark 1.3, the connection between the notions of product class C_1C_2 and of graph of algebraic correspondences. Zariski extends several of his theorems to the graph of correspondences in his manuscript Z.

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