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On the Rotation Number of a Normal Curve

by

John S. Griffin, Jr.

In the mid-thirties Hopf [1] gave a proof that the rotation number of a simple closed curve is either 2π or -2π; Whitney [2] then showed how to calculate the rotation number of any normal curve from properties of its crossing points.

From these theorems and the consideration of a few simple examples, it would seem that the differential-geometric notion of rotation number should be related to the topological notion of index, at any rate for normal curves. The object of the present note is to exhibit such a relation.

The author takes pleasure in recording his indebtedness to the remarks of Professors Samelson and H. Hopf.

1. *The Rotation Number of a Curve.* Let R be the field of real numbers. Let $\mathscr{P}$ be the Euclidean plane with a chosen rectangular coordinate system, that is, suppose that $\mathscr{P} = R \times R$. In particular, then, $\mathscr{P}$ is a two-dimensional vector space with basis vectors $e_1 = (1, 0)$ and $e_2 = (0,1)$ and with norm defined by $\| (x, y) \| = \sqrt{x^2 + y^2}$.

Let S be the unit circle, that is the family of all vectors which have norm 1, and define $p : R \to S$ by

$$p(t) = (\cos t, \ \sin t)$$

for all points t of R. Now R is a covering space for S under the projection p, and hence there is the following well-known lemma: If A is any simply connected arcwise connected topological space, and $f : A \to S$ is continuous, then for any a in A and any t in R such that $p(t) = f(a)$ there is a unique continuous function $f^* : A \to R$ such that $pf^* = f$ and $f^*(a) = t$; such a function f^* is called a covering function for f. This is essentially the proposition of Hopf (1b of [1]) and thus may be taken as a basis for the measure of angles. Indeed, if $A = [a, b]$ is any closed interval and $u : A \to \mathscr{P}$ is any continuous function which never vanishes, then the angle turned by u is defined to be $v^*(b) - v^*(a)$, where

v^* is any covering function for the function $v : A \to S$ defined by

$$v(t) = \frac{u(t)}{\| u(t) \|}$$

for all real numbers t in A. Again, if c and d are any two linearly independent vectors then $\sphericalangle (c, d)$ the *angle from c to d* is defined to be the angle turned by u, where

$$u(t) = c + t(d - c)$$

for $0 \leqq t \leqq 1$.

Let f be a continuous function on a closed interval $[a, b]$ to $\mathscr{P}$. Then f is a *closed curve* if $f(a) = f(b)$, and a *simple closed curve* if furthermore $f(s) \neq f(t)$ for s different from a, b, and t. If f is a closed curve and c is a point of $\mathscr{P}$ which is not on f, then the *index* of c with respect to f is $\frac{\theta}{2\pi}$, where θ is the angle turned by u, the function u being defined to satisfy

$$u(t) = f(t) - c$$

for all points t of the interval $[a, b]$. If f is a simple closed curve then by the Jordan theorem the index of each exterior point is 0, and the indices of all interior points are equal, and have either the value $+ 1$ or else the value $- 1$. The *orientation number* ω_f of a simple closed curve f may thus be defined to be the index of any interior point of f.

A *smooth curve* is a continuous function f on a closed interval A to $\mathscr{P}$ which has a continuous non-vanishing derivative throughout A; this derivative f' is the *tangent curve* of f. A *smooth closed curve* is a closed curve which is smooth and whose tangent curve is also closed; the *rotation number* of a smooth closed curve f is the angle turned by its tangent curve f'.

If $f : [a, b] \to \mathscr{P}$ is a closed curve, the *multiplicity* of a point c of $\mathscr{P}$ is the number (possibly infinite) of points s such that $a \leqq s < b$ and $f(s) = c$. A *crossing point* of f is a point c of multiplicity 2 such that if $f(s) = c$ then f has a derivative at s, and furthermore if $f(s) = c$ and $f(t) = c$ but $s < t$ then $f'(s)$ and $f'(t)$ are linearly independent unless $s = a$ and $t = b$, in which case $f'(s) = f'(t)$. A smooth curve is *normal* if there are but a finite number of points whose multiplicity is greater than 1, and each of these is a crossing point. As shown by Whitney [2], any smooth closed curve f may be deformed into a normal curve by a homotopy which modifies both f and f' only a very little, and which in particular leaves the rotation number of f unchanged.

A class of curves which arise in the study of normal curves are the broken curves: Let $f : [a, b] \to \mathscr{P}$ be a closed curve; f is *broken* if there are points c_i, $a = c_0 < c_1 < \ldots < c_n = b$, such that for $i = 1, 2, \ldots, n$ $f|_{[c_{i-1}, c_i]}$ is a smooth curve, say with tangent curve τ_i; furthermore for $i = 1, 2, \ldots, n-1$, the vectors $\tau_i(c_i)$ and $\tau_{i+1}(c_i)$ are linearly independent; and finally either the vectors $\tau_1(a)$ and $\tau_n(b)$ are equal, or else they are linearly independent. The *corner points* of f are the points $c_1, \ldots, c_{n-1}$ together with a if $\tau_1(a) \neq \tau_n(b)$.

The concept of rotation number may be extended to the class of broken curves by making proper allowance for corner points: if α_i is the angle turned by τ_i then the rotation number of f is

$$\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n-1} \sphericalangle\, [\tau_i(c_i), \tau_{i+1}(c_i)] + \beta$$

where $\beta = \sphericalangle\, [\tau_n(b), \tau_1(a)]$ if a is a corner point, and $\beta = 0$ otherwise.

In the paper mentioned above Hopf showed that the rotation number of a simple closed curve is always 2π of else -2π, and he gave a rule to determine the sign. Since it is implicit in his proof that this sign is the sign of the orientation number, his theorem may be rephrased as follows:

If f is a simple closed curve, either smooth or broken, then the rotation number of f is $2\pi\, \omega_f$.

2. *Circuits in a Closed Curve.* Let $[a, b]$ be a closed interval, and let $f : [a, b] \to \mathscr{P}$ be a closed curve. It may happen that there is a proper sub-interval $[c, d]$ of $[a, b]$ such that $f|_{[c, d]}$ is also a closed curve; if so, then $f(c) = f(d)$, and therefore the function g defined by

$$g(t) = f(t + a) \qquad \text{for } 0 \leqq t \leqq c - a$$

and

$$g(t) = f(t + d) \qquad \text{for } c - a \leqq t \leqq (c - a) + (b - d)$$

is continuous, and hence is also a closed curve. Now $f|_{[c, d]}$ or g may be easier to deal with than f, as for example if $f|_{[c, d]}$ were a simple closed curve; hence the study of f might be facilitated by this sort of decomposition.

These considerations may easily be formalized. First let us agree that if A is a union of a finite number of closed intervals, say

$$A = \cup_{i=1}^{n} [a_i, b_i],$$

and f is a continuous function on A to $\mathscr{P}$ such that $f(b_i) = f(a_{i+1})$ for $i = 1, 2, \ldots n-1$ then

$$f^{\sim} : [0, \sum_{i=1}^{n} (b_i - a_i)] \rightarrow \mathscr{P}$$

is defined by

$$f^{\sim}(t) = f(t + a_i)$$

for $0 \leqq t \leqq b_1 - a_1$ if $i = 1$, and for $\sum_{j=1}^{i-1} b_j - a_i \leqq t \leqq \sum_{j=1}^{i} b_j - a_i$ for $1 < i < n$. Thus in the above example $g = (f\,|_{[a,\,c] \cup [d,\,b]})^{\sim}$.

A *circuit* in a closed curve $f : A \rightarrow \mathscr{P}$ is a restiction of f to a subset B of A which is a union of a finite number of closed intervals, say

$$B = \cup_{i=1}^{n} [a_i, b_i],$$

and such that $f(b_i) = f(a_{i+1})$ for each i and $f\,|_B^{\sim}$ is a simple closed curve. A *decomposition* of f is a family of circuits $\phi_1, \phi_2, \ldots, \phi_n$, such that if $\phi_i = f\,|_{B_i}$ for each i then $A = \cup_{i=1}^{n} B_i$ and $B_i \cap B_j$ is discrete unless $i = j$.

Lemma. *If $f : A \rightarrow \mathscr{P}$ is a normal curve then there is a decomposition of f into circuits; if $\phi_1, \ldots, \phi_n$ is any such decomposition then the rotation number of f is the sum of the rotation numbers of the curves $\phi_i^{\sim}$.*

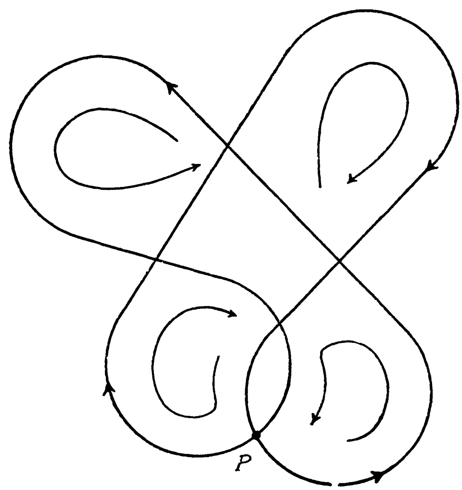
Proof. Let C be the family of all curves satisfying all the hypotheses of the lemma except possibly smoothness: members of C may be broken but they must be closed and have only a finite number of points of multiplicity greater than 1, all these being crossing points. For $g : A \rightarrow \mathscr{P}$ a member of C, let S_g be the family of all points s of A such that $g(s)$ is the image of at least one point other than s. Clearly S_g is always finite; the end-points of A are members of S_g, and S_g has no other members if and only if g is simple.

An induction on the number of members of S_g now yields the desired decomposition. Trivially for any member g of C for which S_g has exactly two points there is a decomposition. Suppose therefore that there is a decomposition for any $g \in C$ such that S_g has at most n points, and let $f : [a, b] \rightarrow P$ be a member of C such that S_f has exactly $n + 1$ points, say $s_0, s_1, \ldots, s_n$. Let $\mathscr{I}$ be the family of all intervals $[s_i, s_j]$ such that $f(s_i) = f(s_j)$, and partially order $\mathscr{I}$ by inclusion. Since $\mathscr{I}$ is finite, there is a minimal member, say $[s_p, s_q]$. Clearly $f\,|_{[s_p,\, s_q]}$ is a circuit, and if

$$g = (f\,|_{[a,\, s_p] \cup [s_q,\, b]})^{\sim}$$

then $g \in C$ and S_g has at most n points; it follows that g, and hence f, has a decomposition into circuits.

Observe that neither the decomposition nor even the number of circuits is unique for a given curve. For example, the algorithm implicit in the above argument yields a decomposition of the curve in the accompanying figure into two circuits, each with one corner



at P; but this curve can also be decomposed into four circuits as indicated by the arrows.

To return to the proof, it remains to show that if $\phi_1, \ldots, \phi_n$ is a decomposition of the normal curve $f : A \to P$ into circuits, and r_i is the rotation number of $\phi_i^{\sim}$, then $\sum_i r_i$ is the rotation number of f. Suppose that for each i ϕ_i is defined on the intervals $I_0^i, I_1^i, \ldots, I_{p(i)}^i$, where for each j

$$I_j^i = [a_j^i, b_j^i];$$

thus we are assuming that

$$A = \cup_{i,j} I_j^i,$$

that for each j

$$b_j^i \neq a_{j+1}^i \quad \text{but} \quad f(b_j^i) = f(a_{j+1}^i),$$

that each $\phi_i^{\sim}$ is a simple closed curve, and that I_j^i and I_k^h have at most one common point unless $i = h$ and $j = k$. Note that it may happen that I_j^i is to the right of I_k^i even though $j < k$, as for example in two of the four circuits indicated in the figure.

Let α_j^i be the angle turned by f' on the interval I_j^i, and let β_j^i, $j = 0, 1, 2, \ldots p(i)$, represent the angles at the corners of $\phi_i^{\sim}$: if $j < p(i)$

$$\beta_j^i = \sphericalangle\, (f'(b_j^i),\ f'(a^i{}_{+1}))$$

and

$$\beta^i_{p(i)} = \measuredangle\ (f'(b^i_{p(i)}),\ f'(a^i_0))$$

unless a^i_0 and $b^i_{p(i)}$ are the left and right end points of A, in which case leave $\beta^i_{p(i)}$ undefined. One then has

$$r_i = \sum_j \alpha^i_j + \sum_j \beta^i_j,$$

and hence

$$\sum_i r_i = \sum_i \sum_j \alpha^i_j + \sum_i \sum_j \beta^i_j.$$

Since f is smooth, the rotation number of f is $\sum_i \sum_j \alpha^i_j$, and it remains to see that the second sum vanishes. Here the crucial fact is that the corners of members of the decomposition occur in pairs: corners arise from crossing points and therefore must have mates. Let then $c_1, c_2, \ldots, c_n$ be the images of the points b^i_j under f (excluding the right end-point b^i_j of A if the left end point of A is a^i_0), and let the pair (i, j) belong to σ_k if and only if $f(b^i_j) = c_k$; it follows that

$$\sum_i \sum_j \beta^i_j = \sum_{k=1}^{n} \sum_{(i,j)\in\sigma_k} \beta^i_j.$$

And now finally for each k

$$\sum_{(i,j)\in\sigma_k} \beta^i_j = 0:$$

if $(i, j) \in \sigma_k$ there is exactly one member (h, m) of σ_k distinct from (i, j) and

$$f(b^i_j) = f(b^h_m) = c_k,$$

and furthermore

$$f(a^i_{j+1}) = f(a^h_{m+1}) = c_k$$

(unless $j + 1 = p(i)$ respectively $m + 1 = p(h)$, in which case $j + 1$ respectively $m + 1$ must be replaced with 0). Since c_k is of multiplicity two,

$$b^i_j = a^h_{m+1} \quad \text{and} \quad b^h_m = a^i_{j+1};$$

since

$$\beta^i_j = \measuredangle\ (f'(b^i_j),\ f'(a^i_{j+1}))$$

and

$$\beta^h_m = \measuredangle\ (f'(b^h_m),\ f'(a^h_{m+1}))$$

it follows that $\beta^i_j = -\beta^h_m$.

3. *The Subdivision of a Normal Curve.* The proposed relationship now follows.

THEOREM. *A normal curve f may be decomposed into a finite* *number of circuits; if* $\phi_1, \ldots, \phi_n$ *is such a decomposition, and* ω_i *is the orientation number of* $\phi_i^{\sim}$*, then the rotation number of* f *is* $2\pi \sum_i \omega_i$*, and the index with respect to* f *of a point* P *not* f *is* $\sum_i a_i \omega_i$*, where* a_i *is* 1 *or* 0 *according as* P *is interior or exterior to* $\phi_i^{\sim}$.

PROOF. In virtue of the lemma and the proposition of Hopf, it remains only to consider the index of a point P not on f.

Using the notation of § 2, let s_i, $i = 0, 1, \ldots m$ be the points of s_f, and suppose that

$$s_0 < s_1 < \ldots < s_m;$$

and let $\phi_i = f\,|_{B_i}$, so that each interval $[s_j, s_{j+1}]$ is a subset of exactly one of the sets B_i. Now the index of any point c of $\mathscr{P}$ which is not on f is the angle turned by the curve α divided by 2π, where

$$\alpha(t) = f(t) - c;$$

so if

$$\beta j = \alpha\,|_{[s_j, s_{j+1}]}$$

then the index of c with respect to f is the sum of the angles turned by the curves β_j, again divided by 2π. On the other hand, the index of c with respect to $\phi_i^{\sim}$ is the angle turned by γ_i divided by 2π, where

$$\gamma_i(t) = \phi_i^{\sim}(t) - c.$$

The angle turned by γ_i is the sum of the angles turned by those β_j for which $[s_j, s_{j+1}] \subset B_i$. Therefore the index of c with respect to f is the sum of the indices of c with respect to the curves $\phi_i^{\sim}$; each of these, however, is ω_i or 0 according as c is interior or exterior to $\phi_i^{\sim}$. Thus the latter part of the theorem is proved.

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