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Higher-Dimensional field Theory

II. Linear Systems

by

E. Snapper

Conventions.

We continue the conventions of [1] (referred to as *FI*), whose introduction contains a short introduction to the present paper. (Square brackets refer to the references.) In particular, E/F is a finitely generated field extension and "module" always means "finitely generated module." We further introduce the following conventions. Since all fields which occur in this paper contain F, "degree of transcendency" always means "degree of transcendency with respect to F as groundfield." The, necessarily finite, degree of transcendency of E is denoted by r. We indicate by E' the set of nonzero elements of E and use this same notation for any subset of any field; for example, if M is a module of E, we indicate by M' the set of nonzero elements of this module. We denote by V_B the nontrivial valuation of E whose valuation ring is B. Hence, $B \neq E$ and if $e \in E'$, $V_B(e)$ belongs to the value group of V_B; of course, $V_B(0) = \infty$ where the symbol ∞ is handled in the usual way, and $V_B(e) \geqq 0$ if and only if $e \in B$. It is always assumed that $F \subset B$, i.e. that when $c \in F'$, $V_B(c) = 0$. Observe that such a valuation V_B can not exist when $r = 0$ and *we consequently always assume that* $r \geqq 1$. The unique maximal ideal of B is denoted by $\mathfrak{P}$ and is called the place of V_B or B; of course, any one of the three concepts V_B, B, $\mathfrak{P}$ determines the other two. The degree of transcendency of the residueclass field $B/\mathfrak{P}$ is, as customary, called the dimension of V_B and we know that it can not exceed $r-1$; on the other hand, we should be careful to note that $B/\mathfrak{P}$ may not be finitely generated over F.

Several remarks have been added to show briefly how the terminology and conventions tie in with the underlying geometry. These remarks do not form a part of the logical development of our field theory and are meant only for those readers who know the theory of algebraic varieties.

1. *The divisors of the first kind of a projective class of modules.* If N is a nonzero module of E, the modules $(1/a)N$ where $a \in N'$, already played a role in sections 4 to 6 of *FI*. The nonempty class C of proportional modules which arises in this way from N, clearly satisfies the following two conditions:
(1) When $M \in C$, $1 \in M$ and (2) When $M \in C$ and $a \in M'$, $(1/a)M \in C$. Conversely, any nonempty class C of proportional modules of E, satisfying these two conditions, can be considered as to consist of all the modules $(1/a)M$, where $a \in M'$ for any fixed module M of C. Although other arrangements are possible, the author feels that it is best to associate the notion of "divisor of the first kind" with such a class of modules. This makes the following definition convenient.

DEFINITION 1.1. *A nonempty class of proportional modules of E which satisfies the above two conditions is called a projective class of modules.*

REMARK 1.1. Let $y_0, \ldots, y_n$ be the homogeneous coordinates of a generic point of an irreducible algebraic variety S, defined over F as groundfield and with E as field of rational functions. By choosing planes, not containing S, as planes at infinity, we obtain the different affine models of S. Each such model has a generic point with coordinates $y_0/a, \ldots, y_{j-1}/a, y_{j+1}/a, \ldots, y_n/a$, where a is any nonzero element $c_0y_0 + \ldots + c_ny_n$ of the module N which is generated by $y_0, \ldots, y_n$; this module belongs to the field $E(y_0, \ldots, y_n)$, while $c_0, \ldots, c_n$ belong to F and $c_j \neq 0$. Since furthermore $y_k/a \in E$, for $k = 0, \ldots, n$, the module $(1/a)N$ belongs to E and we see immediately that $(1/a)N = (y_0/a, \ldots, y_{j-1}/a, y_{j+1}/a, \ldots, y_n/a, 1)$. This shows that the modules $(y_0/a, \ldots, y_{j-1}/a, y_{j+1}/a, \ldots, y_n/a, 1)$ of E which are generated by 1 together with the coordinates $y_0/a, \ldots, y_{j-1}/a, y_{j+1}/a, \ldots, y_n/a$ of the generic points of the different affine models of S, form a projective class of modules. This projective class of modules gives us the same information about the underlying algebraic variety S as a projective model of S, which is the reason why we can study linear systems in terms of projective classes of modules. Observe that we did not require in definition 1.1 that $F(M) = E$, when M belongs to the class, and hence we are actually dealing with the rational images of S.

Let C be a projective class of modules of E. It is evident that the field $F(M)$, where $M \in C$, does not depend on the choice of $M \in C$. We call this field *the field of* C and denote it by $F(C)$; actually, this notation needs no explanation, since the ordinary

field adjunction to F of all the elements which occur in the different modules of C results exactly in the field of C. The, necessarily finite, degree of transcendency of $F(C)$ is called *the dimension of* C.

Let V_B be a valuation of E with place $\mathfrak{P}$ and let H_B denote the natural homomorphism from B onto $B/\mathfrak{P} = H_B(B)$. If N is any module of E, the minimum value $V_B(a)$, where $a \in N$, is well defined, since N is finitely generated, and we denote it by $V_B(N)$. Clearly, for some $a \in N'$, $(1/a)N \subset B$ if and only if $V_B(a) = V_B(N)$. This shows in particular that C contains a *nonempty* subset C_B which consists of those modules of C which happen to lie in B. Every $M \in C_B$ is mapped by H_B on a module $H_B(M)$ of the field $H_B(B)$ and we denote this set of modules by $H_B(C_B)$.

STATEMENT 1.1. *The set of modules $H_B(C_B)$ is a projective class of modules of the field $H_B(B)$.*

PROOF. It is clear that $1 \in H_B(M)$ for every $M \in C_B$. Hence all we have to show is that if $M \in C_B$ and a is a nonzero element of $H_B(M)$, then $(1/a)H_B(M) \in H_B(C_B)$. The fact that a is a nonzero element of $H_B(M)$ means that there exists an element $e \in M$, which is a unit of B, and which is such that $H_B(e) = a$. It follows that $(1/e)M$ belongs to C_B, and not just to C, and hence that $(1/a)H_B(M) \in H_B(C_B)$; done.

We call the projective class of modules $H_B(C_B)$ *the center of* V_B *on* C. In order to investigate this center further, let d be the dimension of C, let $M \in C_B$ and let A be the algebraic closure of the field $F(M)$ in E; then $F[M]$ belongs to the valuation ring $F(M) \cap B$ of $F(M)$. The ordinary integral closure $F\langle M\rangle$ of the ring $F[M]$ in E is the integral closure in A of that same ring and hence both $F[M]$ and $F\langle M\rangle$ belong to the valuation ring $A \cap B$ of A. Consequently, $H_B(F[M]) \subset H_B(F(M) \cap B) \subset H_B(A \cap B)$ and $H_B(F[M]) \subset H_B(F\langle M\rangle) \subset H_B(A \cap B)$; both $H_B(F(M) \cap B)$ and $H_B(A \cap B)$ are fields, while the first field already contains the modules of the center $H_B(C_B)$. If $A \subset B$, the valuation ring $A \cap B$ is the trivial one and the contraction of H_B on A is then an isomorphism; of course, $A \subset B$ if and only if $F(M) \subset B$. If $A \not\subset B$, the degree of transcendency of the fields $H_B(F(M) \cap B)$ and $H_B(A \cap B)$ is at most $d-1$.

STATEMENT 1.2. *If $A \subset B$, the center $H_B(C_B)$ of V_B on C is C itself; if $A \not\subset B$, the dimension of this center is at most $d-1$. In both cases, the field of quotients of the ring $H_B(F\langle M\rangle)$ is a finite algebraic extension of the field of this center.*

PROOF. The modules of C all belong to $F(M)$ and hence, if

$A \subset B$, $C_B = C$; since furthermore the contraction of H_B on A is then an isomorphism, we can consider $H_B(C_B)$ as identical with C. If $A \not\subset B$, we conclude from the fact that the field of $H_B(C_B)$ is contained in $H_B(F(M) \cap B)$, that the dimension of $H_B(C_B)$ is at most $d-1$. Concerning the last sentence of statement 1.2, we have to be careful, since the field $H_B(A \cap B)$ may not be finitely generated over F. However, since $A/F(M)$ is a finite algebraic field extension, $F\langle M\rangle$ is a finitely generated module over $F[M]$ (See *FI*, section 1), and consequently $H_B(F\langle M\rangle)$ is a finitely generated module over $H_B(F[M])$. We conclude that $H_B(F\langle M\rangle) = F[a_1, \ldots, a_m]$, where $a_1, \ldots, a_m$ are integral over the ring $H_B(F[M])$. Consequently, the field of quotients of $H_B(F\langle M\rangle)$ is $F(a_1, \ldots, a_m)$, where $a_1, \ldots, a_m$ are algebraic over the field of quotients of $H_B(F[M])$; since this last field of quotients is exactly the field of the center $H_B(C_B)$, we are done.

REMARK 1.2. Let the irreducible algebraic variety S be as in remark 1.1 and let C be the projective class of modules $(y_0/a, \ldots, y_{j-1}/a, y_{j+1}/a, \ldots, y_n/a, 1)$, which arises from the affine models of S. When V_B is a valuation of E, our definition of the center of V_B on C is the exact field-theoretic equivalent of the customary definition of the center of V_B on S. In the theory of algebraic varieties, the center of a valuation of E on a rational image of S, which is not a birational image of S, is usually not explicitly defined, but there is of course no reason not to use the same definition of center for the rational images of S as for its birational images. We found it expedient to do this in our field-theoretic approach because, as a result, we won't have to restrict ourselves in theorem 6.2 to linear systems whose associated rational transformation of S leaves the dimension of S unchanged. Exactly the same remarks are valid with respect to our definition of "divisors of the first kind."

In the sequel, the projective classes C of dimension $d = 0$ will often be somewhat exceptional. The reason is that *the following three statements are equivalent*: (1) *the elements of the module M of E are all algebraic over F*; (2) *$F[M]$ is a field*; (3) *$F\langle M\rangle$ is a field.* Namely, we know from *FI*, section 1, that not only $F[M]$ but als $F\langle M\rangle$ has a finite number of ring generators over F (i.e., $F\langle M\rangle = F[a_1, \ldots, a_n]$, where $a_1, \ldots, a_n \in F\langle M\rangle$). The equivalence of these three statements now follows immediately from the fact that a ring with a finite number of ring generators over a field F is itself a field if and only if all these generators are algebraic over F (see theorem 2 of [2]). When all the elements

of M are algebraic over F, we say that M is algebraic over F; in that case, $F[M] = F(M)$ and $F\langle M\rangle$ is the algebraic closure F^* of F in E. Now, if $F[M]$ and hence $F\langle M\rangle$ are not fields, the minimal prime ideals of these rings are defined as usual; in particular, a prime ideal of one of these rings is then minimal, if and only if residuation with respect to that ideal diminishes the degree of transcendency of that ring by 1. The notion of minimal prime ideal is meaningless for fields and this is why it is often best to treat the 0-dimensional projective classes, i.e. the classes whose modules are algebraic over F, separately from the others.

A valuation V_B of E is, as customary, called a divisor when its dimension is $r-1$. We know that in that case the field $B/\mathfrak{P}$ is finitely generated over F.

DEFINITION 1.2. *Let C be a projective class of dimension $d \geqq 1$, consisting of modules of E, and let V_B be a divisor of E. Then, V_B is called a divisor of the first kind of C, if the dimension of its center on C is $d-1$.*

If C is a 0-dimensional projective class, no definition of "divisor of the first kind" is made; in that case, the dimension of the center of any valuation V_B on C is 0, since this center is then C itself.

Let V_B be a divisor of the first kind of C. Let d, M, $F\langle M\rangle$, A, H_B, C_B all have the same meaning as in statement 1.2, where $d \geqq 1$. We conclude from this statement that $A \not\subset B$ and see further that the degree of transcendency of both fields $H_B(F(M) \cap B)$ and $H_B(A \cap B)$ is necessarily $d-1$. In other words, *the valuations induced by V_B in A and in $F(M)$ are divisors of these fields*; of course, these induced valuations may be divisors without V_B being of the first kind for C. Precisely, the requirement that V_B be of the first kind for C requests that the degree of transcendency of the ring $H_B(F[M])$ be $d-1$. We see from statement 1.2 that the integral domains $H_B(F[M])$ and $H_B(F\langle M\rangle)$ have the same degree of transcendency and hence the following statement is clearly correct.

STATEMENT 1.3. *Let C be a projective class of dimension $d \geqq 1$, consisting of modules of E, and let V_B be a divisor of E with place $\mathfrak{P}$. We choose any $M \in C_B$ and denote the prime ideals $F[M] \cap \mathfrak{P}$ and $F\langle M\rangle \cap \mathfrak{P}$ by respectively $\mathfrak{q}$ and $\mathfrak{p}$. Then, the following three statements are equivalent*: (1) *V_B is a divisor of the first kind for C*; (2) *$\mathfrak{q}$ is a minimal prime ideal of $F[M]$*; (3) *$\mathfrak{p}$ is a minimal prime ideal of $F\langle M\rangle$.*

If C is a 0-dimensional projective class and V_B any valuation of E, the ideals $\mathfrak{q}$ and $\mathfrak{p}$ of statement 1.3 are always the 0-ideals

of respectively the fields $F[M] = F(M)$ and $F\langle M\rangle = F^*$.

We now return to a projective class C of dimension $d \geqq 1$ and a valuation V_B of E with place $\mathfrak{P}$; we choose an $M \in C_B$ and study the ideal $\mathfrak{p} = F\langle M\rangle \cap \mathfrak{P}$. Clearly, $(F\langle M\rangle)_\mathfrak{p} \subset A \cap B$, where $(F\langle M\rangle)_\mathfrak{p}$ denotes the ring of quotients of $F\langle M\rangle$ which consists of the quotients a/b, where $a, b \in F\langle M\rangle$ and $b \notin \mathfrak{p}$. Since $F\langle M\rangle$ is the integral closure in A of the ring $F[M]$, A is the field of quotients of $F\langle M\rangle$, and $F\langle M\rangle$ is integrally closed and Noetherian. Consequently, if $\mathfrak{p}$ is a minimal prime ideal of $F\langle M\rangle$, the ring $(F\langle M\rangle)_\mathfrak{p}$ is a valuation ring of A, whose associated valuation is a divisor of A, and this ring is not contained in any other proper subring of A. We conclude that, if V_B is a divisor of the first kind of C, $(F\langle M\rangle)_\mathfrak{p} = A \cap B$ and hence that $H_B(A \cap B)$ is then the field of quotients of the ring $H_B(F\langle M\rangle)$. It is now clear that the following statement is correct.

STATEMENT 1.4. *If C is a projective class of dimension $d \geqq 1$, consisting of modules of E, we find all its divisors of the first kind as follows. For each $M \in C$, construct all rings of quotients $(F\langle M\rangle)_\mathfrak{p}$, where $\mathfrak{p}$ runs through all the minimal prime ideals of $F\langle M\rangle$. Each such ring of quotients is a valuation ring of A whose associated valuation is a divisor of A. Extend each of these divisors in all possible ways to divisors of E, and all the divisors of the first kind of C have been found. If V_B is a divisor of the first kind of C, the field $H_B(A \cap B)$ is, according to statement* 1.2, *a finite algebraic extension of the field of the center of V_B on C.*

In order to get all the divisors of the first kind of C, we can obviously restrict the construction of statement 1.4 to a subset C_0 of C which has the property expressed in the following definition.

DEFINITION 1.3. *A subset C_0 of a projective class C of modules of E is said to cover C if, for any valuation V_B of E, B contains at least one module of C_0.*

For example, C covers C and, if $M \in C$ where $M = (a_1, \ldots, a_m)$ and each $a_j \neq 0$, the finite set of modules $(1/a_1)M, \ldots, (1/a_m)M$ covers C; we can see from statement 1.5 that, even if the generators $a_1, \ldots, a_m$ of M are linearly independent, a proper subset of these m modules $(1/a_j)M$ may already cover C. (See *FIII*, statement 1.2, for another characterization of a covering C_0 of C.)

Let $C_1, \ldots, C_h$ be h projective classes of modules of E. The products $M_1 \cdot \ldots \cdot M_h$, where $M_j \in C_j$ for $j = 1, \ldots, h$, clearly all belong to one and the same projective class of modules of E. This class is called *the product of the classes* $C_1, \ldots, C_h$ and is

denoted by $C_1 \cdot \ldots \cdot C_h$. The products $M_1 \cdot \ldots \cdot M_h$, where $M_j \in C_j$, do not necessarily exhaust the modules of $C_1 \cdot \ldots \cdot C_h$, but they clearly cover $C_1 \cdot \ldots \cdot C_h$. In general, when the subset $C_0^{(j)}$ of C_j covers C_j for $j = 1, \ldots, h$, the products $M_1 \cdot \ldots \cdot M_h$, where $M_j \in C_0^{(j)}$, cover $C_1 \cdot \ldots \cdot C_h$. This multiplication of projective classes is obviously commutative and associative and, consequently, the powers C^h, for $h \geqq 0$, are well defined; of course, C^0 consists of the modules M^0 where $M \in C$, i.e. according to our convention that $M^0 = F$, C^0 is the projective class which consists of F only. We see immediately that the field $F(C_1 \cdot \ldots \cdot C_h)$ of the product class $C_1 \cdot \ldots \cdot C_h$ is the compositum of the fields $F(C_j)$ of the individual classes $C_1, \ldots, C_h$.

STATEMENT 1.5. *Let C be a projective class of modules of E. Then, when $h \geqq 1$, the field of C^h is the same as the field of C. If the set of modules $\{M_j\}$ covers C, the set of Modules $\{M_j^h\}$ covers C^h, for $h \geqq 0$; here, the set $\{M_j\}$ can have any cardinality. If $\dim(C) \geqq 1$ and $h \geqq 1$, C^h and C have the same sets of divisors of the first kind.*

PROOF. The field $F(C^h)$ of C^h is the compositum of the field $F(C)$ of C, h times with itself, if $h \geqq 1$; hence then $F(C^h) = F(C)$. Let the set of modules $\{M_j\}$ cover C; then $M_j \in C$ and hence certainly, $M_j^h \in C^h$ for each $M_j \in \{M_j\}$. Furthermore, if V_B is a valuation of E, $M_j \subset B$ for some $M_j \in \{M_j\}$, and hence $M_j^h \subset B$; this shows that the set of modules $\{M_j^h\}$ covers C^h when $h \geqq 0$. It now follows from statement 1.4 that, when $\dim(C) \geqq 1$ and $h \geqq 1$, we obtain the divisors of the first kind of C from the minimal prime ideals of the rings $F\langle M_j \rangle$ and those of C^h from the minimal prime ideals of the rings $F\langle M_j^h \rangle$, where $M_j \in \{M_j\}$. Since $1 \in M_j$, clearly $F[M_j] = F[M_j^h]$ and hence certainly $F\langle M_j \rangle = F\langle M_j^h \rangle$, and we are done.

REMARK 1.3. If the projective class C_j arises, as in remark 1.1, from the affine models of an irreducible algebraic variety S_j, the product class $C_1 C_2$ arises in the same way from the affine models of the graph of the algebraic correspondence between S_1 and S_2.

We like to point out that Krull discusses a field-theoretic definition of divisor of the first kind in section 50 of [3]. We did not make use of Krull's definition, because it is easy to construct an algebraic variety whose set of divisors of the first kind does not fall under his definition. Precisely, from geometric point of view, Krull restricts himself to a very special kind of algebraic variety.

2. *Extension of divisors.* In order to discuss the connection between the notions of „divisors of the first kind of a projective class of modules" and of „integral closure of a module" we need some results concerning extensions of divisors.

We denote by H an intermediate field of our field extension E/F, i.e. $F \subset H \subset E$, and by s the degree of transcendency of E/H; consequently, $r-s$ is the degree of transcendency of H/F. We assume that $s \geqq 1$ and that the element e of E is transcendental over H. Let W be a valuation of H/F (i.e. W induces the trivial valuation on F), which is either a divisor of H/F or is the trivial valuation of H; of course, the first case can only occur if $s < r$.

STATEMENT 2.1. *W can be extended to a divisor V_B of E in such a way that $e \notin B$.*

PROOF. *Case* 1. *W is a divisor of H/F.* We choose an element $k \in H'$ such that $W(k) > 0$; this can be done since W is not the trivial valuation of H. We then choose elements $t_2, \ldots, t_s$ in E, such that $ke = t_1, t_2, \ldots, t_s$ is a transcendence base for E/H. For each polynomial $f(t_1, \ldots, t_s) \in H[t_1, \ldots, t_s]$, we define that $W^*(f(t_1, \ldots, t_s))$ is the minimum of $W(c)$, where c runs through all the coefficients of $f(t_1, \ldots, t_s)$. When $f, g \in H[t_1, \ldots, t_s]$ and $g \neq 0$, we define that $W^*(f/g) = W^*(f) - W^*(g)$ and so obtain a valuation of the field $H(t_1, \ldots, t_s)$; for a proof, see [4], pages 134—137. It is clear that W^* is an extension of W and we now show that *W^* is a divisor of $H(t_1, \ldots, t_s)$*. Hereto, let $B_1, \mathfrak{P}_1$ be the valuation ring and place of W and $B_2, \mathfrak{P}_2$ those of W^*; when $b \in B_2$, we denote by $\bar{b}$ the image of b under the natural homomorphism from B_2 onto $B_2/\mathfrak{P}_2$. All we have to show is that the elements $\bar{t}_1, \ldots, \bar{t}_s$ of $B_2/\mathfrak{P}_2$ are algebraically independent over $B_1/\mathfrak{P}_1$. If $\Sigma \bar{a}_{i_1 \ldots i_s} \bar{t}_1^{i_1} \ldots \bar{t}_s^{i_s} = 0$ where $\bar{a}_{i_1 \ldots i_s} \in B_1/\mathfrak{P}_1$, then $W^*(\Sigma a_{i_1 \ldots i_s} t_1^{i_1} \ldots t_s^{i_s}) > 0$ where $a_{i_1 \ldots i_s} \in B_1$; the definition of W^* implies that actually each $a_{i_1 \ldots i_s} \in \mathfrak{P}_1$ and hence that each $\bar{a}_{i_1 \ldots i_s} = 0$, which proves that $\bar{t}_1, \ldots, \bar{t}_s$ are algebraically independent over $B_1/\mathfrak{P}_1$. We now extend W^* to a valuation V_B of E. Clearly, V_B is a divisor of E and $V_B(e) = W^*(e) = W^*((1/k)t_1) < 0$ and case 1 is finished.

CASE 2. *W is the trivial valuation of H.* Consider the minimal prime ideal $\mathfrak{p} = (1/e)$ of the ring $H[1/e]$. The ring of quotients $(H[1/e])_{\mathfrak{p}}$ is a valuation ring of the field $H(e)$ whose associated valuation is a divisor W_1 of $H(e)$. We extend W_1 to any divisor V_B of E, using the technique of case 1; clearly, $V_B(e) < 0$ and V_B is an extension of W and hence we are done.

Let M be a module of E which is not algebraic over F and let

$F[M]$, $F\langle M\rangle$, A, etc. have the same meanings as in section 1. We denote by $\{V_{B_u}\}$ the set of divisors of E which have the following two porperties: (1) $M \subset B_u$; (2) $\mathfrak{q}_u = F[M] \cap \mathfrak{P}_u$ is a minimal prime ideal of $F[M]$. We know that property (2) is equivalent with property (2'): $\mathfrak{p}_u = F\langle M\rangle \cap \mathfrak{P}_u$ is a minimal prime ideal of $F\langle M\rangle$.

STATEMENT 2.2. $\bigcap_u B_u = F\langle M\rangle$.

PROOF. Since each B_u is integrally closed in E, all we have to show is that $\bigcap_u B_u \subset F\langle M\rangle$. Hereto, let e be an element of E which does not belong to $F\langle M\rangle$; we have to prove that there exists a B in the set $\{B_u\}$ which does not contain e. We assume first that $e \notin A$. In this case, we choose any minimal prime ideal $\mathfrak{p}$ of $F\langle M\rangle$ and consider the divisor W of A whose valuation ring is the ring of quotients $(F\langle M\rangle)_{\mathfrak{p}}$. Since e is transcendental over A, we conclude from statement 2.1 that W can be extended to a divisor V_B of E which is such that $e \notin B$; clearly, $F\langle M\rangle \cap \mathfrak{P} = \mathfrak{p}$ and hence $B \in \{B_u\}$. If $e \in A$, we choose a minimal prime ideal $\mathfrak{p}$ in $F\langle M\rangle$ in such a way that $e \notin (F\langle M\rangle)_{\mathfrak{p}}$; this can be done since $F\langle M\rangle = \cap (F\langle M\rangle)_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through the minimal prime ideals of $F\langle M\rangle$. We then extend the divisor of A whose valuation ring is $(F\langle M\rangle)_{\mathfrak{p}}$ to any divisor V_B of E; clearly, $e \notin B$ and, since $F\langle M\rangle \cap \mathfrak{P} = \mathfrak{p}$, $B \in \{B_u\}$.

If M is algebraic over F, $F\langle M\rangle = F^*$ and every divisor V_B of E has the property that $M \subset B$; we also know that the ideals $\mathfrak{q} = F[M] \cap \mathfrak{P}$ and $\mathfrak{p} = F\langle M\rangle \cap \mathfrak{P}$ are then 0-ideals. It is indeed true that, if $\{V_{B_t}\}$ denotes the set of all the divisors of E, $F^* = \bigcap_t B_t$. Again, all we have to show is that, if e is an element of E which does not belong to F^*, there exists a B in the set $\{B_t\}$ which does not contain e. This however follows from the fact that, according to statement 2.1, the trivial valuation of F^* can be extended to a divisor V_B of E in such a way that $e \notin B$. We will see in statement 2.3 that the equality $F^* = \bigcap_t B_t$ is valid for much smaller sets of divisors than the set $\{V_{B_t}\}$ just used.

We will use statement 2.2 in the following form.

STATEMENT 2.3. *Let C be a projective class of modules of E, where $\dim(C) \geqq 1$, and let $M \in C$. We denote by $\{V_{B_u}\}$ the set of divisors of the first kind of C which have the property that $M \subset B_u$. Then $\bigcap_u B_u = F\langle M\rangle$. If $\{V_{B_t}\}$ denotes the set of all divisors of the first kind of C, $\bigcap_t B_t = F^*$.*

PROOF. According to statement 1.3, the sets of divisors $\{V_{B_u}\}$

of statements 2.2 and 2.3 coincide, and hence $\bigcap_u B_u = F\langle M\rangle$. It follows that $\bigcap_t B_t = \bigcap F\langle M_j\rangle$, where M_j runs through all the modules of C. Now $M_j = (1/a_j)M$, for some $a_j \in M'$, and consequently, $\bigcap_t B_t = \bigcap F\langle(1/a_j)M\rangle$, where a_j runs through the elements of M'. When we apply statement 5.2 of *FI* for the case $h = 0$, we see that $\bigcap F\langle(1/a_j)M\rangle = F^*$ and we are done.

3. *The connection between the integral closure of a module and the divisors of the first kind of a projective class.* Let $\mathfrak{V}$ be any nonempty set of valuations V_B of E and M a module of E. We denote by $|M;\mathfrak{V}|$ the subset of E which consists of those elements $e \in E$ which have the property that $V_B(e) \geqq V_B(M)$ for all $V_B \in \mathfrak{V}$. Obviously, $M \subset |M;\mathfrak{V}|$ and $|M;\mathfrak{V}|$ is a subgroup of the additive group of E which is closed under multiplication by elements of F^* and hence certainly under multiplication by elements of F; we can only call $|M;\mathfrak{V}|$ a module, when $\mathfrak{V}$ is so large that the additive group $|M;\mathfrak{V}|$ has a finite number of generators over F. It is clear that, for any $a \in E$, $|aM;\mathfrak{V}| = a|M;\mathfrak{V}|$. We denote by $R(M;\mathfrak{V})$ the ring which is the intersection of all those valuation rings B which satisfy the following two properties: (1) $V_B \in \mathfrak{V}$; (2) $M \subset B$; following the usual conventions of set theory, $R(M;\mathfrak{V}) = E$ if no such valuation ring B exists. Clearly, $F[M] \subset R(M;\mathfrak{V})$ and hence, since $R(M;\mathfrak{V})$ is integrally closed in E, $F\langle M\rangle \subset R(M;\mathfrak{V})$. Again $|M|_i$ denotes the integral closure of M in E.

STATEMENT 3.1. $|M|_i \subset |M;\mathfrak{V}| \subset R(M;\mathfrak{V})$.

PROOF. If $e \in |M|_i$, there exists a nonzero module L such that $eL \subset LM$. We conclude that for any valuation V_B of E, $V_B(e) + V_B(L) \geqq V_B(L) + V_B(M)$ and hence, since $V_B(L) \neq \infty$, that $V_B(e) \geqq V_B(M)$; this shows that $|M|_i \subset |M;\mathfrak{V}|$. Now suppose that $e \in |M;\mathfrak{V}|$ and that the valuation ring B is such that $M \subset B$ and $V_B \in \mathfrak{V}$. Then, $V_B(e) \geqq V_B(M) \geqq 0$ and hence $e \in B$, and we are done.

When M is a nonzero module of E, we denote by $C(M)$ the projective class which consists of the modules $(1/a)M$, where $a \in M'$. The valuation-theoretic characterization of $|M|_i$, mentioned in section 5 of *FI*, is the following.

STATEMENT 3.2. *Let M be a nonzero module of E, where $\dim(C(M)) \geqq 1$. Then, if $\mathfrak{V}$ denotes the set of divisors of the first kind of $C(M)$, $|M|_i = |M;\mathfrak{V}|$.*

PROOF. All we have to prove, according to statement 3.1, is

that $|M; \mathfrak{V}| \subset |M|_i$. We know from *FI*, statement 5.2, that $|M|_i = \cap aF\langle(1/a)M\rangle$, where a runs through the set M' of nonzero elements of M. Hence we assume that $e \in |M; \mathfrak{V}|$ and that $a \in M'$, and we prove that then $e \in aF\langle(1/a)M\rangle$. Now, since $e \in |M; \mathfrak{V}|$, $e/a \in |(1/a)M; \mathfrak{V}| \subset R((1/a)M; \mathfrak{V})$. We conclude from statement 2.3 that $R((1/a)M; \mathfrak{V}) = F\langle(1/a)M\rangle$ and we are done.

If M is a nonzero module of E and $\dim(C(M)) = 0$, then for any $a \in M'$, $F \subset (1/a)M \subset F^*$; hence $|(1/a)M|_i = F^*$, from which we conclude that $|M|_i = aF^*$. In that case, if $\mathfrak{V}$ denotes the set of all the divisors V_B of E, we see from the observation made after the proof of statement 2.2, that also $|(1/a)M; \mathfrak{V}| = F^*$ and hence $|M|_i = |M; \mathfrak{V}| = aF^*$. Finally, if $M = 0$ and $\mathfrak{V}$ is any nonempty set of valuations V_B of E, $|M|_i = |M; \mathfrak{V}| = 0$.

It follows immediately from statement 3.2 that, if $\mathfrak{V}$ denotes the set of all valuations of E, $|M|_i = |M; \mathfrak{V}|$ for any module M whatsoever.

For the theory of linear systems we need the following generalization of statement 3.2.

THEOREM 3.1. *Let M be a nonzero module of E and $\mathfrak{V}$ the set of divisors of the first kind of the class $C(M)$, where $\dim(C(M)) \geqq 1$. Let $\mathfrak{W}$ be any set of valuations V_B of E, where $\mathfrak{V} \subset \mathfrak{W}$. Then, for $h \geqq 0$, $|M^h|_i = |M^h; \mathfrak{W}|$.*

PROOF. Again, all we have to show is that $|M^h; \mathfrak{W}| \subset |M^h|_i$. If $a \in M'$, $a^h \in M^h$ and $(1/a^h)M^h \in C(M^h)$; since $(1/a^h)M^h = ((1/a)M)^h$, we see that $C(M^h) = (C(M))^h$. It now follows from statement 1.5 that, if $h \geqq 1$, $\mathfrak{V}$ is also the set of divisors of the first kind of the class $C(M^h)$. Hence we conclude from statement 3.2 that $|M^h|_i = |M^h; \mathfrak{V}|$, while $|M^h; \mathfrak{W}| \subset |M^h; \mathfrak{V}|$ follows from $\mathfrak{V} \subset \mathfrak{W}$; our theorem is now proved for $h \geqq 1$. If $h = 0$, $|M^0|_i = |F|_i = F^*$, while $|F; \mathfrak{W}| = \bigcap_u B_u$ where V_{B_u} runs through all the valuations of $\mathfrak{W}$. It follows that $\bigcap_u B_u \subset \bigcap_t B_t$, where V_{B_t} runs only through the valuations of $\mathfrak{V}$. We know from statement 2.3 that already $\bigcap_t B_t = F^*$, and hence $|F; \mathfrak{W}| = F^*$; our theorem is now completely proved.

In the applications, $\mathfrak{W}$ will usually be the set of divisors of the first kind of some other projective class C of modules of E, where $\dim(C) \geqq 1$. Of course, we may choose $\mathfrak{W} = \mathfrak{V}$.

The following material is needed for the next two sections and for *FIII*.

Let C be a projective class of modules of E. It is clear that the modules $|M|_i$, where $M \in C$, all belong to one and the same pro-

jective class of modules of E; this class is denoted by $|C|_i$. The analogue of statement 1.5 is the following.

STATEMENT 3.3. *Let C be a projective class of modules of E. The field of $|C|_i$ has the same algebraic closure A in E as the field $F(C)$ of C. If the set of modules $\{M_j\}$ covers C, the set of modules $\{|M_j|_i\}$ covers $|C|_i$; here, the set $\{M_j\}$ can have any cardinality. If $\dim(C) \geqq 1$, $|C|_i$ and C have the same sets of divisors of the first kind.*

PROOF. If $M \in C$, $F(M) = F(C)$ and $F(|M|_i)$ is the field of $|C|_i$. Since $|M|_i \subset F\langle M\rangle$, $F\langle |M|_i\rangle \subset F\langle M\rangle$; we conclude from *FI*, statement 5.1 and remark 5.1, that $F\langle M\rangle = \bigcup_{s=0}^{\infty} |M^s|_i$ and hence that $F\langle |M|_i\rangle = F\langle M\rangle$. This proves the first sentence of statement 3.3. If the set $\{M_j\}$ covers C, obviously the set $\{|M_j|_i\}$ is a subset of $|C|_i$. Furthermore, if V_B is a valuation of E, $M_j \subset B$ for some $M_j \in \{M_j\}$. This implies that $|M_j|_i \subset B$, since $|M_j|_i \subset F\langle M_j\rangle \subset B$, and hence the set $\{|M_j|_i\}$ covers $|C|_i$. Finally, if $\dim(C) \geqq 1$, the divisors of the first kind of C arise from the minimal prime ideals of $F\langle M_j\rangle$ and those of $|C|_i$ from the minimal prime ideals of $F\langle |M_j|_i\rangle$, where $M_j \in \{M_j\}$. We saw above that $F\langle |M_j|_i\rangle = F\langle M_j\rangle$ and hence we are done.

We can now prove the following corollary of theorem 3.1.

STATEMENT 3.4. *Let M, $\mathfrak{V}$, C, $\mathfrak{W}$ be as in theorem 3.1. Then, for $h \geqq 0$, $|M^h|_i = ||M^h|_i; \mathfrak{W}|$.*

PROOF. If $a \in M'$, $a^h \in M^h \subset |M^h|_i$ and hence $(1/a^h)|M^h|_i \in C(|M^h|_i)$. Furthermore, $(1/a^h)|M^h|_i = |((1/a)M)^h|_i$ and consequently, $C(|M^h|_i) = |(C(M))^h|_i$. We now conclude from statements 1.5 and 3.3 that $\mathfrak{V}$ is also the set of divisors of the first kind of the class $C(N)$, where $N = |M^h|_i$, if $h \geqq 1$. Theorem 3.1 then implies that $|N|_i = |N; \mathfrak{W}|$ and since $||M^h|_i|_i = |M^h|_i$, we are done for $h \geqq 1$. When $h = 0$, we have again that $|M^0|_i = F^*$ and that $||M^0|_i; \mathfrak{W}| = F^*$, and the statement is completely proved.

4. *The $(r-1)$-dimensional cycles of an r-dimensional projective class of modules. In the remainder of this paper, C denotes a fixed, projective class of modules of E, where $\dim(C) = r$.* Consequently, continuing the notations of the previous sections, $A = E$ and, when $M \in C$, E is the field of quotients of $F\langle M\rangle$. We denote *by* $\mathfrak{V}$ the set of divisors of the first kind of C. *We are going to study the linear systems which arise from this fixed set of divisors* $\mathfrak{V}$; all our notions as zero, pole, zero-cycle, pole-cycle, linear equivalence, linear system, etc. are relative notions and depend on the choice of $\mathfrak{V}$.

REMARK 4.1. We only postulated that E is an algebraic extension of the field $F(C)$ or our class C and not that $E = F(C)$. Hence, if S is again an algebraic variety which is defined over F as groundfield and whose field of rational functions is E, we are studying the linear systems inside E which arise from an algebraic variety S^*, where S^* is a rational transform of S which has the same dimension as S. The only reason for not insisting upon $F(C) = E$, is that we won't have an opportunity to profit from this further restriction.

We will base our proof of the existence of complete linear systems on the following statement.

STATEMENT 4.1. *If N is a module of E, $|N; \mathfrak{B}|$ has a finite number of generators over F;* i.e., *$|N; \mathfrak{B}|$ is a module.*

PROOF. Let $N = (e_1, \ldots, e_n)$. If $M \in C$, E is the field of quotients of $F\langle M\rangle$ and hence $e_j = a_j/b_j$, where $a_j, b_j \in F\langle M\rangle$ and $b_j \neq 0$ for $j = 1, \ldots, n$; consequently, $bN \subset F\langle M\rangle$, where $b = b_1 \cdot \ldots \cdot b_n$. We then conclude from FI, statement 5.1 and remark 5.1, that $bN \subset |M_h|_i$ for large enough h. This implies that $|bN; \mathfrak{B}| \subset ||M^h|_i; \mathfrak{B}|$ and, using statement 3.4 with $\mathfrak{W} = \mathfrak{B}$, that $|bN; \mathfrak{B}| \subset |M^h|_i$. Since $b \neq 0$, it follows that $|N; \mathfrak{B}| \subset (1/b)|M^h|_i$ and the finite dimensionality of $(1/b)|M^h|_i$ tells us that $|N; \mathfrak{B}|$ is finite dimensional.

The author does not know whether statement 4.1 remains valid when the condition that $\dim(C) = r$ is dropped, and is replaced by merely $\dim(C) \geqq 1$.

In order to define zeros and poles of elements of E', we add to our conventions that, *when V_B is a divisor of E, $V_B(e) = 1$ for at least one $e \in E'$.* When $e \in E'$ and $V_B \in \mathfrak{B}$, we say that *V_B is a zero of e of order m if $V_B(e) = m > 0$* and that *V_B is a pole of e of order n if $V_B(e) = n < 0$; observe that zeros and poles are defined only with respect to divisors of $\mathfrak{B}$.*

STATEMENT 4.2 *Let $e \in E'$. Then e has at most a finite number of zeros and a finite number of poles. The following three statements are equivalent:* (1) *e has no zeros;* (2) *e has no poles;* (3) *$e \in F^*$.*

PROOF. Let $M \in C$ and let $\{V_{B_u}\}$ denote the set of divisors of $\mathfrak{B}$ which have the property that $M \subset B_u$; we begin by showing that e has at most a finite number of zeros in the set $\{V_{B_u}\}$. Hereto, we assume that $V_B \in \{V_{B_u}\}$ and that V_B is a zero of e, and we choose two nonzero elements $a, b \in F\langle M\rangle$ which are such that $e = a/b$. Then, $V_B(e) = V_B(a) - V_B(b) > 0$ and $V_B(b) \geqq 0$, and hence $V_B(a) > 0$. Consequently, a belongs to the minimal prime ideal $\mathfrak{p} = F\langle M\rangle \cap \mathfrak{P}$ of $F\langle M\rangle$. Since $B = (F\langle M\rangle)_{\mathfrak{p}}$ is

completely determined by $\mathfrak{p}$ and a is contained in only a finite number of minimal prime ideals of $F\langle M\rangle$, e can indeed have at most a finite number of zeros in the set $\{V_{B_u}\}$. We now apply this remark to each module of a finite set of modules $M_1, \ldots, M_n$ of C which cover C (see section 1), and we conclude immediately that e can have at most a finite number of zeros. In particular, this is true for $1/e$, and hence e can also have at most a finite number of poles. Furthermore, e has no poles if and only if $e \in \bigcap_t B_t$, where B_t runs through the valuation rings of all the divisors of $\mathfrak{V}$. Since $\bigcap_t B_t = F^*$ (see statement 2.3), we see that (2) and (3) of statement 4.1 are equivalent. Finally, e has no zeros if and only if $1/e$ has no poles, and hence (1) and (3) are also equivalent, and we are done.

It follows immediately from the methods of section 2 that statement 4.2 is false if the condition that $\dim(C) = r$ is dropped.

We now consider the additive group $\mathfrak{G}$ of the finite, integral, linear combinations of the elements of $\mathfrak{V}$. The precise construction of $\mathfrak{G}$ is a routine matter and an element $G \in \mathfrak{G}$ can be written as $G = n_1 V_{B_1} + \ldots + n_s V_{B_s}$, where $V_{B_j} \in \mathfrak{V}$ and n_j is a rational integer for $j = 1, \ldots, s$; the 0-element of $\mathfrak{G}$ is denoted by O. We consider $\mathfrak{V}$ as a subset of $\mathfrak{G}$, i.e. we identify the divisor V_B of $\mathfrak{V}$ with the linear combination $1V_B$. *We call the elements of* $\mathfrak{G}$ *the* $(r-1)$*-dimensional cycles of* C *and we refer to* $\mathfrak{G}$ *as the* $(r-1)$*-dimensional cycle group of* C. Hence the divisors of $\mathfrak{V}$ are a special kind of $(r-1)$-dimensional cycles of C. Since in this paper and in *FIII* all cycles which occur have dimension $r-1$, *whenever we say "cycle", we always mean "*$(r-1)$*-dimensional cycle."*

REMARK 4.2. Of course, by now, the usual fight is on. Should we perhaps have written $\mathfrak{G}$ multiplicatively instead of additively? Since the reasonings of these papers come from geometry rather than from number theory, the author has chosen the additive notation. In this connection, Professor Zariski has pointed out that, if one uses the additive notation, it is better to use the term "$(r-1)$-dimensional cycle" then "divisor", since the latter term definitely reminds one of a multiplicative theory. This then is the reason why we do not refer to the elements of $\mathfrak{G}$ as the divisors of C. Those readers who prefer to write $\mathfrak{G}$ multiplicatively would probably rather use the place $\mathfrak{P}$ of V_B in their notation, and they would write $\mathfrak{P}_1^{n_1} \cdot \ldots \cdot \mathfrak{P}_s^{n_s}$ when we write $n_1 V_{B_1} + \ldots + n_s V_{B_s}$.

When $G_1, G_2 \in \mathfrak{G}$, we say that $G_1 \leqq G_2$ when, for each $V_B \in \mathfrak{V}$,

the coefficient of V_B in G_1 does not exceed the coefficient of V_B in G_2; of course, $G_1 < G_2$ means that $G_1 \leqq G_2$ and $G_1 \neq G_2$. The relation $\leqq$ makes $\mathfrak{G}$ into a partially ordered group; when $O \leqq G$, we say that G is nonnegative, and when $O < G$ that G is positive. When G_1 and G_2 are positive cycles and V_B of $\mathfrak{V}$ occurs in both cycles with positive coefficient, we say that "G_1 and G_2 have V_B in common." When $G \neq O$, we can split this cycle in the obvious way into two positive cycles $G = Z - P$, where Z and P have no divisor in common; we refer to these uniquely determined cycles Z and P as respectively the zero-cycle and pole-cycle of G.

Let M be a nonzero module of E. We conclude immediately from statement 4.2 that, when $V_B \in \mathfrak{V}$, $V_B(M) = 0$ except for at most a finite number of divisors of $\mathfrak{V}$. Consequently, the sum $\Sigma V_B(M) V_B$, where V_B runs through all the divisors of $\mathfrak{V}$, is a cycle of C and we call it *the cycle* $G(M)$ *of* M. Statement 4.2 also shows that $G(M) = O$ if and only if M is algebraic over F. When M is not algebraic over F, we call the zero-cycle and pole-cycle of $G(M)$ respectively *the zero-cycle* $Z(M)$ *and pole-cycle* $P(M)$ of M; hence $G(M) = Z(M) - P(M)$. If $M = (e)$, the cycle, zero-cycle and pole-cycle of M are also called the cycle $G(e)$, zero-cycle $Z(e)$ and pole-cycle $P(e)$ of e.

Of course, not every cycle G is the cycle $G(M)$ of some nonzero module M of E (see statement 6.1).

STATEMENT 4.3. *If G is a cycle of C, there always exists a nonnegative cycle G_0 of C and a nonzero module M of E, such that* $G + G_0 = -G(M)$.

PROOF. *Case* 1. *G is a divisor $V_B \in \mathfrak{V}$.* Let e be a nonzero element of the place $\mathfrak{P}$ of V_B and consider the module $M = (1/e, 1)$. Clearly, $Z(e) = -G(M)$ and since $Z(e) \geqq G$, we are done.

CASE 2. *G is nonnegative*, say $G = n_1 V_{B_1} + \ldots + n_s V_{B_s}$ where each $n_j \geqq 0$. Let the nonzero modules $M_1, \ldots, M_s$ be such that $V_{B_j} + G_0^{(j)} = -G(M_j)$ and $G_0^{(j)} \geqq 0$, for $j = 1, \ldots, s$. Then, $-G(M_1^{n_1} \cdot \ldots \cdot M_s^{n_s}) = G + n_1 G_0^{(1)} + \ldots + n_s G_0^{(s)}$ and we are done.

CASE 3. *G is arbitrary.* We add, if necessary, a positive cycle G_0 to G, which is so large that $G + G_0 \geqq O$; we then apply the result of case 2 to $G + G_0$.

When G is a cycle of C, *we denote by* $[G]$ *the set of elements $e \in E'$ which have the property that $G(e) + G \geqq O$, together with the* 0*-element of* E. Clearly, $[G]$ is a subgroup of the additive group of E, which is closed under multiplication by elements of F^* and hence certainly under multiplication by elements of F. If $G_0 \geqq O$, evidently $[G] \subset [G + G_0]$.

THEOREM 4.1. *Let G be a cycle of C. The additive group $[G]$ has a finite number of generators over F,* i.e., *$[G]$ is a module.*

PROOF. CASE 1. *There exists a nonzero module M of E which is such that $G = -G(M)$.* In that case, it follows immediately from the definition of $[G]$ and $|M; \mathfrak{V}|$, that $[G] = |M; \mathfrak{V}|$; we conclude from statement 4.1 that $[G]$ is then a module. Observe that in this case $[G]$ is not the zero module, which already shows that not every divisor can be considered as a $G(M)$.

CASE 2. *G is arbitrary.* Let the cycle $G_0 \geqq O$ and the module $M \neq 0$ be as in statement 4.3, i.e. $G+G_0 = -G(M)$. Then, $[G+G_0]$ is a module and $[G] \subset [G+G_0]$, and hence $[G]$ is a module; done.

The next section contains a reformulation of theorem 4.1 in terms of linear systems.

We finish this section with the following remarks regarding the modules $[hG]$, where h is a nonnegative rational integer and G is a cycle of C. It is evident that the sum $\Sigma_{h=1}^{\infty}[hG]$ of these modules is a ring, and we denote this ring by $R[G]$. If $G \geqq O$, $[hG] \subset [kG]$ when $h \leqq k$, and hence then $R[G] = \bigcup_{h=1}^{\infty}[hG]$. Zariski has conjectured in his, as yet unpublished, manuscript entitled "Algebra-geometric interpretations of the 14th problem of Hilbert" that, when $G \geqq O$, the ring $R[G]$ is finitely generated over F. He has shown in the same manuscript, which we designate by ZH, that the proof of his conjecture would constitute a solution of Hilbert's 14th problem. (In ZH, $F(C) = E$, but that is evidently immaterial.)

Of course, if $G = O$, $R[G] = F^*$ and we are done. Let then $G = m_1 V_{B_1} + \ldots + m_s V_{B_s}$ where $V_{B_1}, \ldots, V_{B_s} \in \mathfrak{V}$ and $m_1, \ldots, m_s > 0$. If $e \in R[G]$ where $e \neq o$, $G(e)+hG \geqq O$ for some $h \geqq 1$; this is clearly equivalent to saying that, if $P(e)$ denotes the pole-cycle of e, $P(e) \leqq hG$, i.e. that the poles of e occur among the divisors $V_{B_1}, \ldots, V_{B_s}$. We conclude from this immediately that $R[G] = \bigcap_u B_u$, where V_{B_u} runs through all the elements of $\mathfrak{V}$ which are distinct from $V_{B_1}, \ldots, V_{B_s}$. Hence, *$R[G]$ is integrally closed in E* and Zariski's conjecture can be formulated as follows. *Let $V_{B_1}, \ldots, V_{B_s}$ be a finite set of elements of $\mathfrak{V}$; then, the ring $\bigcap_u B_u$ where V_{B_u} runs through the elements of $\mathfrak{V}$ which are distinct from $V_{B_1}, \ldots, V_{B_s}$, is finitely generated over F.* This formulation of the conjecture brings out the fact that the ring $R[G]$ is not so much associated with the cycle G, but rather only with the divisors $V_{B_1}, \ldots, V_{B_s}$. We come back to the ring $R[G]$ in section 6.

5. *Linear systems.* We continue all conventions of the previous sections; the letter G always denotes a cycle of C.

It is clear that the mapping $e \to G(e)$ is a homomorphism from the multiplicative group E' of E into the additive group $\mathfrak{G}$, and that the multiplicative group of F^* is the kernel of this homomorphism. The subgroup of $\mathfrak{G}$ which is the image of E' under this homomorphism, induces in the usual way a congruence relation $\equiv$ in $\mathfrak{G}$; precisely, $G_1 \equiv G_2$ means that $G_1 - G_2 = G(e)$, for some $e \in E'$. *When $G_1 \equiv G_2$, we say that the cycles G_1 and G_2 of C are linearly equivalent.*

Let M again be a nonzero module of E. Clearly, a cycle G of C has the property that $O \leqq G(e)+G$, for every $e \in M'$, if and only is $O \leqq G(M)+G$.

DEFINITION 5.1. *Let M be a nonzero module of E and let G be a cycle of C which has the property that $O \leqq G(M)+G$. The set of cycles $G(e)+G$, where $e \in M'$, is called a linear system of C and is denoted by $g(M; G)$.*

We observe that linear systems consist of nonnegative, linearly equivalent cycles. Precisely, if $G_0 \in g(M; G)$, G_0 has the following two properties: (1) $G_0 \geqq O$; (2) $G_0 \equiv G$. Clearly, the set of all cycles of C which possess these two properties consists exactly of the cycles of the form $G(e)+G$, where $e \in [G]$ and $e \neq 0$. We know from theorem 4.1 that $[G]$ is a module and, since $N \subset [G]$, this module is in the present case a nonzero module. Consequently, *the cycles of C which have the above two properties, constitute the linear system $g([G]; G)$ and we designate this system by $|G|$.* We call a linear system *complete* if it is not contained in a larger linear system and we see that $|G|$ is complete. Precisely, we have found the following formulation of theorem 4.1. *Every linear system $g(M; G)$ is contained in a unique complete linear system $|G|$; here, $|G|$ consists of all the nonnegative cycles of C which are linearly equivalent to G.*

When we choose a cycle G of C at random, it may be that $[G] = 0$, i.e. that no nonnegative cycles exist which are linearly equivalent to G. The question when $[G] \neq 0$ is much too closely related to theorems of the type of Riemann-Roch theorems to be treated by the elementary methods of this paper.

We now turn our attention to the notion of the dimension of a linear system. It may very well be that $g_1(M_1; G_1) = g_2(M_2; G_2)$, where $M_1 \neq M_2$ and $G_1 \neq G_2$. Hence, we can only use the dimension of M to define the dimension of $g(M; G)$, if we know that the set of cycles $g(M; G)$ determines at least the dimension of M.

Now, this is simply not true, unless we postulate that $F = F^*$; for example, $g(F; O) = g(F^*; O) = O$. Consequently, *in the remainder of this section we regretfully assume that F is algebraically closed in E*, i.e. *that* $F = F^*$.

STATEMENT 5.1. *Let* $g_1(M_1; G_1)$ *and* $g_2(M_2; G_2)$ *be two linear systems of C, where* $g_1(M_1; G_1) \subset g_2(M_2; G_2)$. *Then,* $G_1 \equiv G_2$ *and* $M_1 \subset eM_2$, where $e \in E'$ *is such that* $G_2 = G(e) + G_1$; *consequently, dim* $(M_1) \leqq$ *dim* (M_2). *If* $g_1(M_1; G_1) = g_2(M_2; G_2)$, *dim* $(M_1) =$ *dim* (M_2) *and* $M_1 = eM_2$.

PROOF. Let $g_1(M_1; G_1) \subset g_2(M_2; G_2)$. If $e_1 \in M_1'$, $G(e_1) + G_1 \in g_2(M_2; G_2)$ and hence $G(e_1) + G_1 = G(e_2) + G_2$ for some $e_2 \in M_2'$. This shows already that $G_1 \equiv G_2$, say $G_2 = G(e) + G_1$, where $e \in E'$. It is obvious that $g_2(M_2; G_2) = g_2(eM_2; -G(e) + G_2) = g_2(eM_2; G_1)$. Now, let $a \in M_1'$; then $G(a) + G_1 \in g_2(eM_2; G_1)$ and consequently $G(a) + G_1 = G(eb) + G_1$, where $b \in M_2'$. We now conclude from $G(a) = G(eb)$ that $a = ebc$, where $c \in F$ (here we use that $F = F^*$), and hence $a \in eM_2$. This shows that $M_1 \subset eM_2$ which implies that $\dim(M_1) \leqq \dim(M_2)$. It now follows that, when $g_1(M_1; G_1) = g_2(M_2; G_2)$, $\dim(M_1) = \dim(M_2)$ and still $M_1 \subset eM_2$; but this can only be when $M_1 = eM_2$, and we are done.

We define that *the dimension of the linear system* $g(M; G)$ *is* $-1 + dim(M)$.

REMARK 5.1. Again, the definition of the dimension of a linear system has been chosen in such a way as to agree with the terminology used in the theory of algebraic varieties. If $F(C) = E$, our $g(M; G)$ can be interpreted in the usual way as a linear system L of $(r-1)$-dimensional cycles, cut out on a suitably chosen model S of E by a linear system of hypersurfaces of the ambient projective space. The customary dimension of this linear system L is $-1 + \dim(M)$.

6. *Well-behaved linear systems. We drop the assumption that* $F = F^*$. The importance of the following definition is that, whenever we are able to prove that a given linear system has the properties we expect all linear systems to have, we can do this because the given linear system can be reduced, in one way or another, to well-behaved linear systems.

DEFINITION 6.1. *A linear system* $g(M; G)$ *of C is said to be well-behaved if* $[hG] = |M^h|_i$ for all $h \geqq 0$.

Of course, when $h = 0$, $[0G] = |M^0|_i = F^*$ and hence $g(M; G)$ is well-behaved as soon as $[hG] = |M^h|_i$, for $h \geqq 1$. Clearly, the property of being well-behaved is a property only of the cycles

of $g(M; G)$ and does not depend on the choice of M and G.

The properties of well-behaved linear systems we expect, on grounds of Zariski's work and conjectures, to be properties of all linear systems are stated in the following theorem, which is an immediate corollary of theorem 4.1 and statement 5.1 of *FI*.

THEOREM 6.1. *Let $g(M; G)$ be a well-behaved linear system of C. Then, the ring $R[G]$ is finitely generated over F. If furthermore $F = F^*$, we can associate with the cycle G a rational polynomial $f(x) = a_0\binom{x}{d} + a_1\binom{x}{d-1} + \ldots + a_d$ and a rational integer h_0, which are such that $\dim(|hG|) = f(h)$, when $h \geqq h_0$. The coefficients $a_0, \ldots, a_d$ are rational integers and a_0 is positive. Finally, the degree d of $f(x)$ is equal to the $\dim(C(M))$.*

We now go over to the discussion of an important example of well-behaved linear systems; this is needed for *FIII*.

Let $g(M; G)$ be a linear system of C. We see immediately that the nonnegative cycle $G(M)+G$ is the largest (in the sense of the partial ordering of $\mathfrak{G}$) cycle G^* with the property that $G' \geqq G^*$ for all $G' \in g(M; G)$. When $G(M)+G > O$, we call $G(M)+G$ *the fixed cycle of* $g(M; G)$; when $G(M)+G = O$, we say that "$g(M; -G(M))$ has no fixed cycle." Clearly, the notion of the fixed cycle of $g(M; G)$ is independent of the choice of M and G, and depends only on the cycles of the linear system.

STATEMENT 6.1. *Let G be a cycle of C. There exists a nonzero module M of E which is such that $G = -G(M)$, if and only if G has the following two properties*: (1) $[G] \neq 0$; (2) $|G|$ *has no fixed* cycle.

PROOF. Let $G = -G(M)$. Then $M \subset [G]$ and hence $[G] \neq 0$. Furthermore, the linear system $g(M; G)$ has then no fixed cycle and, since $g(M; G) \subset |G|$, certainly $|G|$ has no fixed cycle. Conversely, let G have the above two properties. Then, $|G| = g([G]; G)$ and since $|G|$ has no fixed cycle, $G = -G([G])$ and we are done.

REMARK 6.1. Let, as in remark 5.1, $F(C) = E$ and let $g(M; G)$ again be interpreted as a linear system L of $(r-1)$-dimensional cycles of a suitably chosen model S of E. Then, the fixed cycle of L in the sense of algebraic geometry corresponds to the fixed cycle $G(M)+G$ of $g(M; G)$. Of course, if L has no fixed cycle and $r \geqq 2$, L may still very well have base points. If L has no base points, the rational transformation T of S which is associated with L has no fundamental points on S. This implies that the divisors of the first kind of $T(S)$, if the dimension of $T(S)$ is the same as of S, are also of the first kind for S. This is the geometric

background of the two conditions which occur in theorem 6.2.

If M is a nonzero module of E which is such that $\dim(C(M)) \geqq 1$, we designate the set of divisors of the first kind of the projective class $C(M)$ by $\mathfrak{V}(M)$.

THEOREM 6.2. *Let $g(M; G)$ be a linear system of C which has the following two properties*: (1) *$g(M; G)$ has no fixed cycle*, i.e. $G = -G(M)$; (2) *either $\dim(C(M)) = 0$ or $\mathfrak{V}(M) \subset \mathfrak{V}$. Then, $g(M; G)$ is well-behaved.*

PROOF. Since $g(M; G)$ has no fixed cycle, $G = -G(M)$. It follows that $hG = -G(M^h)$, which implies that $[hG] = |M^h; \mathfrak{V}|$. If $\dim(C(M)) \geqq 1$, we conclude from $\mathfrak{V}(M) \subset \mathfrak{V}$ and theorem 3.1, that $[hG] = |M^h|_i$; if $\dim(C(M)) = 0$, $M = eN$ for some $e \in E'$ and $N \subset F^*$, and it is obvious that then $|M^h|_i = e^h F^*$ and $[hG] = [-G(e^h)] = e^h F^*$; done.

In order to tie up the modules $[hG]$ and the linear systems $|hG|$ with the multiples of the linear system $g(M; G)$, we say a few words about the notion of the sum of linear systems; this is also needed for *FIII*. When $g_1(M_1; G_1)$ and $g_2(M_2; G_2)$ are linear systems, we define that *the sum $g_1(M_1; G_1)+g_2(M_2; G_2)$ is the linear system $g(M_1M_2; G_1+G_2)$*; this definition is permissible since $G(M_1M_2)+(G_1+G_2) = (G(M_1)+G_1)+(G(M_2)+G_2) \geqq O$. Clearly, the sum is the smallest linear system which contains the cycles $G_1'+G_2'$, where $G_1' \in g_1(M_1; G_1)$ and $G_2' \in g_2(M_2; G_2)$; this shows in particular that the sum is independent of the choice of M_1, G_1, M_2, G_2, but depends only on the cycles of $g_1(M_1; G_1)$ and $g_2(M_2; G_2)$. The summation of linear systems is a commutative and associative operation which has the linear system $g(F; O)$, which consists of O only, as 0-element. Finally, it is clear that the fixed cycle of a sum of linear systems is the sum of the fixed cycles of the individual systems.

It follows from the above that the multiples $hg(M; G)$ of a linear system are well defined for $h \geqq 0$ and that $hg(M; G) = g(M^h; hG)$. Each such multiple is contained in the unique complete linear system $|hG|$.

REMARK 6.2. *Linear systems without base points.* We assume that S is a normal model of E which is defined over F as ground-field; here, "normal" means "locally normal" in the sense of Zariski. We denote by C the projective class of modules of E which arises from the affine models of S (see remark 1.1) and by $\mathfrak{V}$ the set of divisors of the first kind of C. We know that there exists a $(1-1)$ correspondence between the elements of $\mathfrak{V}$ and the irreducible, $(r-1)$-dimensional subvarieties of S. In

particular, a linear system $g(M; G)$ of C can be interpreted as a linear system L of cycles on S. The main purpose of this remark is to prove that, if L has no base points on S, $g(M; G)$ is well-behaved.

STATEMENT 6.2. *Let L have no base points on S. Then, $[G] \subset F\langle M\rangle$.*

PROOF. We have seen already several times that $F\langle M\rangle$ is the intersection of all the valuation rings of E/F which contain M. Hence, let V_B be a valuation of E/F where $M \subset B$; all we have to show is that then $[G] \subset B$. If $e \in M'$, the cycle $Y = G(e)+G$ is nonnegative and $g(M; G) = g((1/e)M; Y)$. Furthermore, $e[Y] = [G]$ and hence, if $[Y] \subset B$, also $[G] \subset B$. We may consequently assume for our proof that $G \geqq O$. If $G = O$, clearly $[G] = F^* \subset B$ and we are done. So, let $G = m_1 V_{B_1} + \ldots + m_s V_{B_s}$, where $m_j > 0$ and $V_{B_j} \in \mathfrak{V}$ for $j = 1, \ldots, s$. We denote by D_j the center of V_{B_j} on S and by Q the center of V_B on S. We show that $Q \notin D_j$, for $j = 1, \ldots, s$, by means of the following argument which occurs in section 4 of ZH. Since L has no base points, there exists a cycle $Y \in L$ which is such that $Q \notin Y$. This means exactly the following. There exists an $e \in M'$ such that, if $G(e) + G = n_1 W_{B_1} + \ldots + n_t W_{B_t}$ where $W_{B_j} \in \mathfrak{V}$ and $n_j > 0$ for $j = 1, \ldots, t$, then $Y = n_1 Y_1 + \ldots + n_t Y_t$ where Y_j is the center of W_{B_j} on S for $j = 1, \ldots, t$; $Q \notin Y$ means that $Q \notin Y_j$ for $j = 1, \ldots, t$. Since $G(e) = n_1 W_{B_1} + \ldots + n_t W_{B_t} - (m_1 V_{B_1} + \ldots + m_s V_{B_s})$, the cycle of e on S is $n_1 Y_1 + \ldots + n_t Y_t - (m_1 D_1 + \ldots + m_s D_s)$. Now suppose that $Q \subset D_u$ for some $1 \leqq u \leqq s$. Then, none of the cycles $Y_1, \ldots, Y_t$ could be equal to D_u, and hence Q would then be a subvariety of the polar cycle D_u of e and not of any null-cycle of e. This would imply that $e \notin B$, against the assumption that $M \subset B$; hence indeed, $Q \notin D_j$ for $j = 1, \ldots, s$. Now let $e \in [G]'$, i.e. let $G(e)+G \geqq O$. If $P(e)$ is the pole-cycle of e, we conclude from $P(e) \leqq G$, that the poles of e must occur among the divisors $V_{B_1}, \ldots, V_{B_s}$; hence the polar cycles of e on S must occur among the cycles $D_1, \ldots, D_s$. This proves that Q is not contained in any polar cycle of e and hence that $e \in B$. The proof of statement 6.2 is now complete.

THEOREM 6.3. *Let L have no base points on S. Then, $g(M; G)$ is well-behaved.*

PROOF. If $e \in |M|_i'$, $eN \subset NM$ for some nonzero module N, from which we conclude that $G(e) \geqq G(M)$; hence, since $G(M)+G \geqq O$, certainly $G(e)+G \geqq O$, which shows that $e \in [G]$ and hence that $|M|_i \subset [G]$. So far, we did not use that L has no base points.

We now use the absence of base points to show that $[G] \subset |M|_i$. Again $|M|_i = \cap aF\langle(1/a)M\rangle$, where a runs through M'; consequently, all we have to show is that, when $a \in M'$, $[G] \subset aF\langle(1/a)M\rangle$. Now $g(M; G)=g((1/a)M; G(a)+G)$ and hence, according to statement 6.2, $[G(a)+G] \subset F\langle(1/a)M\rangle$. We observe that $[G(a)+G] = (1/a)[G]$ and hence indeed $[G] \subset aF\langle(1/a)M\rangle$, which shows that $[G] = |M|_i$. When $h \geqq 0$, the linear system $hg(M; G) = g(M^h; hG)$ corresponds to the linear system hL on S. Evidently, hL has no base points and hence, according to the just proven result, $[hG] = |M^h|_i$ for $h \geqq 0$, and theorem 6.3 is proved.

According to theorem 6.3, we may apply theorem 6.1 to a linear system $g(M; G)$ whose corresponding linear system L has no base points on S. When we do this, we obtain a statement concerning the dimension of L and a statement concerning the ring $R[G]$. The first statement occurs in Z and the second one in ZH.

Let us now return to an arbitrary linear system $g(M; G)$ of C whose corresponding linear system L on S is not necessarily free of base points. We can always consider the linear system L_i of S which corresponds to the linear system $g(|M|_i; G)$ of C; namely, the cycles $G(M)$ of M and $G(|M|_i)$ of $|M|_i$ are the same and hence, since $G(M)+G \geqq O$, also $G(|M|_i)+G \geqq O$. Evidently, $L \subset L_i$ and we see from theorem 6.3 that, if L has no base points on S, L_i is the complete linear system $|L|$ which contains L. If L has base points, L_i may or may not be equal to $|L|$. The author conjectures that the relationship between L and L_i is always the following. The linear system L_i is the largest linear system which satisfies the following two conditions: (1) $L \subset L_i$; (2) every base variety of L, if L has any such varieties, is a base variety *of the same type* of L_i. Accepting this, the following geometrical interpretation is associated with the strictly algebraic notion of a well-behaved linear system. A linear system L of S is well-behaved if and only if, for all h, every base variety of hL is a base variety of the same type of the complete linear system $|hL|$. This makes theorem 6.3 geometrically self-evident.

REFERENCES

E. SNAPPER

[1] *Higher-dimensional field theory*; I. *The integral closure of a module*, Compositio Mathematica. This issue.

E. ARTIN and J. T. TATE

[2] *A note on finite ring extensions*, Journal of the Mathematical Society of Japan vol. 3 (1951) pp. 74—77.

W. KRULL

[3] *Idealtheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Chelsea Publishing Company, New York (1948).

H. HASSE

[4] Zahlentheorie, Akademie Verlag, Berlin (1949).

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