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On Entire Functions of Infinite Order

by

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1. **Introduction.** The purpose of this paper is to extend to a class of entire functions of infinite order some theorems on entire functions of finite order.

Theorems 1 and 2 are formal analogues of two theorems [1] and [2] of Shah. Theorems 3, 4 and 5 are new; but they are closely connected with some theorems [3] of Shah. Theorem 6 is an analogue of a theorem of Lindelöf [4].

2. DEFINITIONS. We define the k-th order and the k-th lower order of an entire or meromorphic function as

$$\varrho_k = \overline{\lim_{r\to\infty}} \frac{l_k T(r)}{\log r}$$

and

$$\lambda_k = \underline{\lim_{r\to\infty}} \frac{l_k T(r)}{\log r}.$$

Similarly, we define the k-th order and the k-th lower order of the zeros of $f(z)$ as

$$\sigma_k = \overline{\lim_{r\to\infty}} \frac{l_k n(r)}{\log r}$$

and

$$\delta_k = \underline{\lim_{r\to\infty}} \frac{l_k n(r)}{\log r},$$

where $T(r)$, $n(r)$ have their usual meanings and $l_1 x = \log x$, $l_2 x = \log\log x$, and so on.

3. LEMMA (i) If $\chi(x)$ is a positive function continuous almost every where in every interval (r_0, r); and if

$$\overline{\lim_{r\to\infty}} \frac{l_k \xi(r)}{\log r} = \sigma_k$$

then

$$\underline{\lim_{r\to\infty}} \frac{\xi(r) l_1\xi(r) l_2\xi(r) \dots l_{k-1}\xi(r)}{\chi(r)} \leqq \frac{1}{\sigma_k},$$

where
$$\xi(r) = \int_{r_0}^{r} \frac{\chi(x)}{x}\, dx.$$

Lemma (ii) If $\chi(x)$ and $\xi(r)$ are the same functions as before; and if
$$\lim_{r\to\infty} \frac{l_k \xi(r)}{\log r} = \delta_k,$$
then
$$\varliminf_{r\to\infty} \frac{\chi(r)}{\xi(r) l_1 \xi(r) \ldots l_{k-1}\xi(r)} \leqq \delta_k .$$

Proof. If $f(x)$ and $g(x)$ are two positive functions which tend to infinity with x; and if each of the functions is differentiable almost every where in every interval (r_0, r), such that their derivatives $f'(x)$ and $g'(x)$ have a definite finite value at every point of this interval, then
$$\varlimsup_{r\to\infty} \frac{f(r)}{g(r)} \leqq \varlimsup_{r\to\infty} \frac{f'(r)}{g'(r)}$$
and
$$\varliminf_{r\to\infty} \frac{f(r)}{g(r)} \geqq \varliminf_{r\to\infty} \frac{f'(r)}{g'(r)}.$$

Now, putting $f(r) = l_k \xi(r)$ and $g(r) = \log r$, we get the required results.

4. Theorem 1. If $f(z)$ is an entire function of infinite order; and if the k-th lower order of its zeros is δ_k, then
$$\text{(i)}\quad \varliminf_{r\to\infty} \frac{n(r)}{l_1 M(r) l_2 M(r) \ldots l_k M(r)} \leqq \delta_k$$
and
$$\text{(ii)}\quad \varliminf_{r\to\infty} \frac{n(r)}{l_1 M(r) l_1 n(r) l_2 n(r) \ldots l_{k-1} n(r)} \leqq \delta_k,$$
provided that
$$\varliminf_{r\to\infty} \frac{\log n(r)}{l_2 r} = \infty.$$

These can be proved easily by putting $\xi(r) = \int_{r_0}^{r} \frac{n(x)}{x}\, dx$ in Lemma (ii).

Theorem 2. If $f(z)$ is an entire function of finite k_1-th order but of infinite (k_1-1)-th lower order, then

$$\varliminf_{r\to\infty} \frac{l_1M(r)\cdot l_2M(r)\ldots l_kM(r)}{\nu(r)} \leqq \frac{1}{\varrho_k},$$

where ϱ_k is the k-th order of $f(z)$.

Proof. Since, by hypothesis, $f(z)$ is of finite k_1-th order but of infinite (k_1-1)-th lower order, we can very easily prove, by using the inequalities

$$u(r) \leqq M(r) \leqq 3u(r)\nu(2r) \tag{1}$$

that

$$\varlimsup_{r\to\infty} \frac{l_{k_1}\nu(r)}{\log r} < \infty$$

and

$$\varliminf_{r\to\infty} \frac{l_{k_1-1}\nu(r)}{\log r} = \infty.$$

Now, we can very easily show that

$$\lim_{r\to\infty} \frac{l_{k+1}\nu(2r)}{l_k\nu(\alpha r)} = 0, \tag{2}$$

where k is any positive integer or zero; and α is any fixed positive number.

Also, putting $\xi(r) = \log u(r)$ in Lemma (i); and using (1), we have

$$\varliminf_{r\to\infty} \frac{l_1u(r)l_2u(r)\ldots l_ku(r)}{\nu(r)} \leqq \frac{1}{\varrho_k} \tag{3}$$

ϱ_k being the k-th order of $f(z)$.

Lastly, by using (1), (2) and (3), we can easily prove the required result.

Theorem 3. If $f(z)$ is an entire function of finite k_1-th order but of infinite (k_1-1)-th lower order, then

$$\varliminf_{r\to\infty} \frac{T(r)l_1T(r)\ldots l_{k-1}T(r)}{n(r, f-f_1)} \leqq \frac{2}{\varrho_k}$$

for every entire function $f_1(z)$ of finite (k_1-1)-th order, with one possible exception, where $T(r)$ refers to $f(z)$, ϱ_k is the k-th order of $f(z)$; and $n(r, f-f_1)$ denotes the number of zeros of $f(z)-f_1(z)$ in the region $|z| \leqq r$, every zero being counted according to its order.

Proof. By the second fundamental theorem of Nevanlinna [5, § 34], we have

$$T(r, \varphi) = T(r) < N(r, 0)+N(r, 1)+N(r, \infty) + 8\log T(cr) + 0(\log r) \tag{4}$$

for all sufficiently large r, where c is a fixed number greater than 1.

Putting $\varphi(z) = \dfrac{f(z)-f_1(z)}{f(z)-f_2(z)}$ in (4), we have

$$T(r, f) = T(r)<N(r, f-f_1)+N(r, f-f_2)+8 \log T(cr) \\ +aT(r, f_1)+bT(r, f_2)+0 (\log r) \quad 5)$$

for all $r>r_0$, where a and b are certain positive constants.

Since, by hypothesis, $f(z)$ is of finite k_1-th order but of infinite (k_1-1)-th lower order; and each of the functions $f_1(z)$ and $f_2(z)$ is of finite (k_1-1)-th order, we have

$$\lim_{r\to\infty} \frac{\log T(cr)}{T(r)} = 0$$

and

$$\lim_{r\to\infty} \frac{T(r, F)}{T(r)} = 0,$$

where F denotes each of the functions $f_1(z)$ and $f_2(z)$. Consequently, we have

$$l_k\{T(r)-8\log T(cr)-aT(r, f_1)-bT(r, f_2)\} < l_k\{N(r, f-f_1) \\ +N(r, f-f_2)\} \quad (6)$$

Now, putting $\xi(r) = N(r, f-f_1)+N(r, f-f_2)$ in Lemma (i), we get

$$\varrho_k \leqq \overline{\lim_{r\to\infty}} \frac{n(r, f-f_1)+n(r, f-f_2)}{\xi(r)l_1\xi(r)\ldots l_{k-1}\xi(r)}. \quad (7)$$

Combining (6) and (7), we have

$$\varrho_k \leqq \overline{\lim_{r\to\infty}} \frac{n(r, f-f_1)+n(r, f-f_2)}{T(r)l_1T(r)\ldots l_{k-1}T(r)}.$$

Therefore

$$\underline{\lim_{r\to\infty}} \frac{T(r)l_1T(r)\ldots l_{k-1}T(r)}{n(r, f-f_1)+n(r, f-f_2)} \leqq \frac{1}{\varrho_k} \quad (8)$$

The required result follows easily from (8).

THEOREM 4. If $f(z)$ is an entire function of finite k_1-th order but of infinite (k_1-1)-th lower order, for which the deficiency sum (excluding $\alpha=\infty)\Sigma\delta(\alpha) = \sigma>0$; and if $n'(r, \alpha)$ denotes the number of simple zeros of the function $f(z)-\alpha$ in the region $|z|\leqq r$, then

$$\underline{\lim_{r\to\infty}} \frac{T(r)l_1T(r)\ldots l_{k-1}T(r)}{n'(r, \alpha)} \leqq \frac{2}{\varrho\cdot\sigma_k}$$

for every finite value of α, with one possible exception, where ϱ_k is the k-th order of $f(z)$.

Proof. If $N'(r, \alpha)$ and $N'(r, \beta)$ refer to $n'(r, \alpha)$ and $n'(r, \beta)$ respectively, we have

$$N(r, \alpha)+N(r, \beta)<N'(r, \alpha)+N'(r, \beta)+2N_1(r)+0\ (\log r).$$

Also, by the theorem of Nevannlina (loc. cit.), we have

$$\begin{aligned} T(r, f) &< N(r, \alpha)+N(r, \beta)-N_1(r)+8\log T(cr)+0\ (\log r) \\ &< N'(r, \alpha)+N'(r, \beta)+N_1(r)+8\log T(cr)+0\ (\log r) \end{aligned} \tag{9}$$

for all sufficiently large r, where $N_1(r)$ has the same meaning as in [6, § 33, (16)].

Further, by the same theorem, we have

$$\sum \delta(\alpha)+\overline{\lim_{r\to\infty}}\frac{N_1(r)}{T(r)} \leqq 1+\overline{\lim_{r\to\infty}}\frac{\log T(cr)}{T(r)}.$$

But, under the conditions of the theorem, we have

$$\lim_{r\to\infty}\frac{\log T(cr)}{T(r)}=0.$$

Therefore

$$\overline{\lim_{r\to\infty}}\frac{N_1(r)}{T(r)} \leqq 1-\sigma. \tag{10}$$

By (9), we have

$$l_k\{T(r)-N_1(r)-\log T(cr)-0\ (\log r)\}<l_k\{N'(r, \alpha)+N'(r, \beta)\}.$$

The rest of the proof, now, depends on (10) and follows the same lines as that of the preceding theorem.

Theorem 5. If $f(z)$ is a meromorphic function of finite k_1-th order but of infinite (k_1-1)-th lower order, then

$$\underline{\lim_{r\to\infty}}\frac{T(r)l_1(Tr)\ldots l_{k-1}T(r)}{n(r, f-f_1)} \leqq \frac{3}{\varrho_k}$$

for every meromorphic function $f_1(z)$ of finite (k_1-1)-th order, with two possible exceptions, where $n(r, f-f_1)$ and ϱ_k have the same meanings as before.

The proof of this is similar.

4. We define the type of an entire function $f(z)$ of finite k-th order as

$$T_k=\overline{\lim_{r\to\infty}}\frac{l_kM(r)}{r^{\varrho_k}}.$$

LEMMA. If $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is an entire function of finite k-th order ϱ_k, $k>1$, then

$$T_k = \overline{\lim_{n\to\infty}}\, l_{k-1}n \cdot |a_n|^{\frac{\varrho_k}{n}}.$$

PROOF. Let

$$\nu_k = \overline{\lim_{n\to\infty}}\, l_{k-1}n \cdot |a_n|^{\frac{\varrho_k}{n}}.$$

We have

$$|a_n| > \left(\frac{\nu_k-\varepsilon}{l_{k-1}n}\right)^{\frac{n}{\varrho_k}}$$

for an infinity of n.

Therefore, by Cauchy's inequality, we have

$$M(r) \geqq \left(\frac{\nu_k-\varepsilon}{l_{k-1}n}\right)^{\frac{n}{\varrho_k}} \cdot r^n$$

for an infinity of n. Choose r such that

$$r^{\varrho_k} = \frac{a\cdot l_{k-1}n}{\nu_k-\varepsilon},$$

where a is any fixed number greater than 1.

Consequently, we have

$$\begin{aligned} M(r) &\geqq \left(\frac{\nu_k-\varepsilon}{l_{k-1}n}\right)^{\frac{n}{\varrho_k}} \left(\frac{a\cdot l_{k-1}n}{\nu_k-\varepsilon}\right)^{\frac{n}{\varrho_k}} \\ &= a^{\frac{n}{\varrho_k}} \\ &= a^{\frac{1}{\varrho_k}\cdot e_{k-1}\left\{\frac{(\nu_k-\varepsilon)r^{\varrho_k}}{a}\right\}} \end{aligned}$$

Proving thereby that

$$aT_k \geqq \nu_k - \varepsilon.$$

Making a and ε tend to unity and zero respectively, we have

$$T_k \geqq \nu_k. \tag{11}$$

Also, we have $$|a_n| \leqq \left(\frac{\nu_k+\varepsilon}{l_{k-1}n}\right)^{\frac{n}{\varrho_k}}$$

for all sufficiently large n.

Therefore

$$\begin{aligned} |f(z)| &\leqq \sum_{n=o}^{\infty} |a_n| r^n \\ &\leqq \sum_{n=n_0}^{\infty} r^n \left(\frac{\nu_k+\varepsilon}{l_{k-1}n}\right)^{\frac{n}{\varrho_k}} + 0(r^{n_0}). \end{aligned}$$

Now, $r^x\left(\dfrac{\nu_k+\varepsilon}{l_{k-1}x}\right)^{\frac{x}{\varrho_k}}$ is maximum for a value of x, say x_1, which satisfies the equation

$$(\nu_k+\varepsilon)r^{\varrho_k} = l_{k-1}x_1 \cdot e^{\overline{l_{k-1}x_1\cdot l_{k-2}x_1\cdots l_1x_1}}.$$

We can take x_1 sufficiently large, by choosing r to be large. Therefore, we have

$$e_{k-1}\left\{\frac{(\nu_k+\varepsilon)r^{\varrho_k}}{1+\varepsilon_1}\right\} \leqq x_1 \leqq e_{k-1}\left\{\frac{(\nu_k+\varepsilon)r^{\varrho_k}}{1-\varepsilon}\right\},$$

where ε_1 is arbitrarily small.

Let $m = e_{k-1}\{(\nu_k+2\varepsilon)r^{\varrho_k}\}$. We have

$$\begin{aligned}|f(z)| &\leqq \sum_{n\leqq m}|a_n|r^n + \sum_{n>m}|a_n|r^n\\ &\leqq e_{k-1}\{(\nu_k+2\varepsilon)r^{\varrho_k}\}\,(1+\varepsilon_1)^{\frac{1}{\varrho_k}e_{k-1}\left\{\frac{(\nu_k+\varepsilon)r^{\varrho_k}}{1-\varepsilon_1}\right\}} + \sum_{n=0}^{\infty}\left(\frac{\nu_k+\varepsilon}{\nu_k+2\varepsilon}\right)^{\frac{n}{\varrho_k}}\\ &= e_{k-1}\{(\nu_k+2\varepsilon)r^{\varrho_k}\}\,(1+\varepsilon_1)^{\frac{1}{\varrho_k}e_{k-1}\left\{\frac{(\nu_k+\varepsilon)r^{\varrho_k}}{1+\varepsilon_1}\right\}} + 0(1).\end{aligned}$$

Therefore, we have

$$T_k \leqq \nu_k. \tag{12}$$

Hence, combining (11)and (12), we have

$$T_k = \nu_k.$$

THEOREM 6. If $P(z) = \prod_1^{\infty} E\left(\dfrac{z}{z_n}, p_n\right)$ is a product of primary factors of finite k-th order, having zeros (z_n) $n = 1, 2, 3, \ldots$, where $p_n \leqq \log n < p_n+1$; and if

$$L_k = \overline{\lim_{r\to\infty}}\,\frac{l_{k-1}n(r)}{r^{\varrho_k}},$$

then

$$L_k \leqq T_k \leqq AL_k,$$

where $n(r)$ has its usual meaning and A is a constant.

PROOF. When $p_n > 0$ and $|z| \geqq \frac{1}{2}$, we have

$$\begin{aligned}\operatorname{Log}|E(z, p_n)| &\leqq \log(1+|z|) + |z| + \frac{|z|^2}{2} + \ldots + \frac{|z|^{p_n}}{p_n}\\ &\leqq 2|z| + \frac{|z|^2}{2} + \ldots\ldots\ldots\ldots\ldots \frac{|z|^{p_n}}{p_n}\\ &\leqq 2(2|z|)^{p_n}.\end{aligned}$$

Similarly, we have

$$\log|E(z, p_n)| \geqq \log|1-z| - |z| - \frac{|z|^2}{2} - \dots - \frac{|z|^{p_n}}{p_n}$$
$$\geqq \log|1-z| - 2(2|z|)^{p_n}.$$

Let N be a positive integer such that $|z_N| \leqq 2|z| < |z_{N+1}|$. The product of primary factors is

$$P(z) = \prod_1^N E\left(\frac{z}{z_n}, p_n\right) \cdot \prod_{N+1}^{\infty} E\left(\frac{z}{z_n}, p_n\right) = \Pi_1 \cdot \Pi_2, \tag{13}$$

say. We denote $|z|$, $|z_n|$, $\left|\frac{z}{z_n}\right|$ by r, r_n, u_n respectively.

If $p_n > 0$, when $n > n_0$, we have

$$\sum_{n_0+1}^{N} \log\left|1 - \frac{z}{z_n}\right| - 2\sum_{n_0+1}^{\infty}(2u_n)^{p_n} \leqq \log\left|\prod_{n_0+1}^{N} E\left(\frac{z}{z_n}, p_n\right)\right|$$
$$\leqq 2\sum_{n_0+1}^{N}(2u_n)^{p_n}$$

since $u_n \geqq \frac{1}{2}$ in $\prod_1$.

In Π_2, we have $u_n < \frac{1}{2}$ and so

$$|\log|\Pi_2|| \leqq |\log \Pi_2| \leqq \sum_{N+1}^{\infty}\left|\log E\left(\frac{z}{z_n}, p_n\right)\right| \leqq 2\sum_{N+1}^{\infty} u_n^{p_n+1}.$$

Combining the two inequalities, we have

$$\sum_1^N \log\left|1 - \frac{z}{z_n}\right| - 2\sum_{n_0+1}^{N}(2u_n)^{p_n} - 2\sum_{N+1}^{\infty} u_n^{p_n+1} \leqq \log|P(z)|$$
$$\leqq 2\sum_{n_0+1}^{N}(2u_n)^{p_n} + 2\sum_{N+1}^{\infty} u_n^{p_n+1} + 0(\log r). \tag{14}$$

Let us suppose that the second order of $P(z)$ is ϱ_2, where ϱ_2 is finite; and let $L_2 = \overline{\lim_{r\to\infty}} \frac{\log n(r)}{r^{\varrho_2}} < \infty$. We have

$$r_n > \left(\frac{\log n}{H}\right)^a,$$

when $n > n_1$, where $a = 1/\varrho_2$; and H is any fixed positive number greater than L_2.

If m denotes the greater of the two numbers n_0 and n_1, we have

$$
\begin{aligned}
I &= 2\sum_{m+1}^{N}(2u_n)^{p_n} + 2\sum_{N+1}^{\infty} u_n^{p_n+1} \\
&= 2\sum_{m+1}^{N} 2^{p_n} u_n^{\log n}\cdot u_n^{p_n-\log n} + 2\sum_{N+1}^{\infty} u_n^{\log n}\cdot u_n^{p_n+1-\log n} \\
&< 2\sum_{m+1}^{N}(2u_n)^{\log n} + 2\sum_{N+1}^{\infty} u_n^{\log n} \\
&< 2\sum_{m+1}^{N}\frac{(2rH^a)^{\log n}}{(\log n)^{a\log n}} + \sum_{N+1}^{\infty}\frac{(rH^a)^{\log n}}{(\log n)^{a\log n}}.
\end{aligned}
$$

We can easily see that the function $\dfrac{r^x}{x^{ax}}$ is steadily increasing or steadily decreasing, according as $x < \dfrac{Hr^{\frac{1}{a}}}{e}$ or $x > \dfrac{Hr^{\frac{1}{a}}}{e}$. Putting $R = e^{\frac{H(2r)^{\frac{1}{a}}}{e}}$, $R_1 = e^{\frac{Hr^{\frac{1}{a}}}{e}}$, we have

$$
\begin{aligned}
I < 2\sum_{m+1}^{n<R}\frac{(2rH^a)^n}{n^{an}} + 2\frac{(2rH^a)^{\log R}}{(\log R)^{a\log R}} + 2\sum_{n>R}^{N}\frac{(2rH^a)^{p_n}}{p_n^{ap_n}} \\
+ 2\sum_{N+1}^{n<R_1}\frac{(rH^a)^n}{n^{an}} + 2\frac{(rH^a)^{\log R_1}}{(\log R_1)^{a\log R_1}} + 2\sum_{n>R_1}^{\infty}\frac{(rH^a)^{p_n}}{p_n^{ap_n}}.
\end{aligned}
$$

Now, if $[x]$ denotes the integral part of the positive number x; and if $s_1 = \left[\dfrac{s}{e}\right]$, where s is a positive integer, not less than e, we have

$$
\begin{aligned}
p_{3s} &= [\log 3s] \geqq [\log s] + 1 \\
p_{s_1} &= [\log s_1] = [\log s] - 1.
\end{aligned}
$$

Therefore, the number of times an integer p_s can be repeated is less than $\dfrac{s(3e-1)}{e}$; and this is less than $(3e-1)e^{p_s}$. Consequently, we have

$$
\begin{aligned}
I < \sum_{1}^{\infty}\frac{(2rH^a)^n}{n^{an}} + 2\frac{(2rH^a)^{\log R}}{(\log R)^{a\log R}} + 2(3e-1)\sum_{1}^{\infty}\frac{(2eH^ar)^n}{n^{an}} \\
+ 2\sum_{1}^{\infty}\frac{(rH^a)^n}{n^{an}} + 2\frac{(rH^a)^{\log R_1}}{(\log R_1)^{a\log R_1}} + 2(3e-1)\sum_{1}^{\infty}\frac{(erH^a)^n}{n^{an}} \\
< A\cdot\sum_{1}^{\infty}\frac{(2eH^ar)^n}{n^{an}} + 2\frac{(2rH^a)^{\log R}}{(\log R)^{a\log R}} + \frac{2(rH^a)^{\log R_1}}{(\log R_1)^{a\log R_1}},
\end{aligned}
$$

where A is a constant.

Since the type [7, § 2.2.9] of the entire function $\sum_{1}^{\infty} \frac{(2eH^a r)^n}{n^{an}}$ is $(2e)^{\varrho_2} \cdot H$, we have proved that

$$I \leqq e^{A_2 H r^{\varrho_2}} \tag{15}$$

for all sufficiently large r, where A is an absolute constant.

By (14) and (15), we can easily show that

$$T_2 \leqq A_2 L_2.$$

But, by Jensen's theorem, we have

$$L_2 \leqq T_2.$$

Combining the two, we have

$$L_2 \leqq T_2 \leqq A_2 L_2.$$

Next, let us suppose that the 3rd. order of $f(z)$ is ϱ_3, where ϱ_3 is finite; and let

$$L_3 = \overline{\lim_{r\to\infty}} \frac{l_2 n(r)}{r^{\varrho_3}} < \infty.$$

We have

$$r_n > \left(\frac{l_2 n}{H}\right)^a,$$

when $n > n_2$, where H is any fixed positive number greater than L_3 and $a = 1/\varrho_3$.

If m_1 be a positive integer greater than n_0 and n_2, such that $\log\log m_1 > 1$, we have

$$\begin{aligned} I &= 2\sum_{m_1+1}^{N} (2u_n)^{p_n} + 2\sum_{N+1}^{\infty} u_n^{p_n+1} \\ &< 2\sum_{m_1+1}^{N} (2u_n)^{\log n} + 2\sum_{N+1}^{\infty} u_n^{\log n} \\ &< 2\sum_{m_1+1}^{N} \frac{(2H^a r)^{\log n}}{(\log\log n)^{a\log n}} + 2\sum_{N+1}^{\infty} \frac{(H^a r)^{\log n}}{(\log\log n)^{a\log n}}. \end{aligned}$$

Now, the function $\frac{r^x}{(\log x)^{ax}}$ is steadily increasing or steadily decreasing, according as

$$\log r \;\substack{> \\ \text{or} \\ <}\; a\log\log x + \frac{a}{\log x}.$$

Let $r > 1$. If $n = R_2$ be a root of the equation

$$\log(rH^a) = a l_3 n + \frac{a}{l_2 n},$$

when $n > m_1$; and $n = R_3$ be a root of the same equation with r replaced by $2r$, then $\log n < e^{Hr^{\frac{1}{a}}}$, when $n = R_2$ and $\log n < e^{H(2r)^{\frac{1}{a}}}$, when $n = R_3$.

Consequently, if E_r be the set of values of r, at which the inequality

$$\log (rH^a) > al_3 n + \frac{a}{l_2 n}$$

holds; and S_r the set at which the reverse inequality holds, then we have

$$I < 2\sum_{E_{2r}} \frac{(2rH^a)^n}{(\log n)^{an}} + 2e_2\{H(2r)^{\frac{1}{a}}\} \cdot (2rH^a)^{\frac{H(2r)^{\frac{1}{a}}}{e}} +$$

$$+ 2\sum_{S_{2r}} \frac{(2rH^a)^{p_n}}{(\log p_n)^{ap_n}} + 2\sum_{E_r} \frac{(rH^a)^n}{(\log n)^{an}} + 2e_2(Hr^{\frac{1}{a}}) \cdot (rH^a)^{e^{\frac{Hr^{\frac{1}{a}}}{e}}} + 2\sum_{S_r} \frac{(rH^a)^{p_n}}{(\log p_n)^{ap_n}}$$

$$< 2\sum_{m_1+1} \frac{(2rH^a)^n}{(\log n)^{an}} + 2e_2\{H(2r)^{\frac{1}{a}}\} \cdot (2rH)^{e^{H(2r)^{\frac{1}{a}}}}$$

$$+ 2\sum_{m_1+1}^{\infty} \frac{(2rH^a)^{x_n}}{(\log p_n)^{ap_n}} + \sum_{N+1}^{\infty} \frac{(rH^a)^n}{(\log n)^{an}} + 2e_2(Hr^{\frac{1}{a}}) \cdot r^{e^{Hr^{\frac{1}{a}}}} + 2\sum_{N+1}^{\infty} \frac{(rH^a)^{p_n}}{(\log p_n)^{ap_n}}$$

$$< 2\sum_{3}^{\infty} \frac{(2rH^a)^n}{(\log n)^{an}} + 2(3e-1)\sum_{3}^{\infty} \frac{(2erH^a)^n}{(\log n)^{an}} +$$

$$+ 2\sum_{3}^{\infty} \frac{(rH^a)^n}{(\log n)^{an}} + 2(3e-1)\sum_{3}^{\infty} \frac{(erH^a)^n}{(\log n)^{an}} +$$

$$+ 2e_2\{H(2r)^{\frac{1}{a}}\} \cdot (2rH^a)^{e^{H(2r)^{\frac{1}{a}}}} + 2e_2(Hr^{\frac{1}{a}}) \cdot (rH^a)^{e^{Hr^{\frac{1}{a}}}}$$

$$< A\sum_{3}^{\infty} \frac{(2erH^a)^n}{(\log n)^{an}} + 4e_2\{H(2r)^{\frac{1}{a}}\} \cdot (2rH^a)^{e^{H(2r)^{\frac{1}{a}}}}, \tag{16}$$

where A is a constant.

It is easily seen, by putting $k = 2$ in the lemma, that the type of the series on the right-hand side is $H(2e)^{\varrho_3}$.

Therefore, by (14) and (16), we have

$$T_3 \leqq A_3 L_3.$$

Now, let us suppose that the k-th order of $P(z)$ is ϱ_k, where ϱ_k is finite; and let

$$L_k = \overline{\lim_{r\to\infty}} \frac{l_{k-1}n(r)}{r^{\varrho_k}} < \infty.$$

We have

$$r_n > \left(\frac{l_{k-1}n}{H}\right)^a,$$

when $n > n_3$, where H is any fixed positive number greater than L_k and $a = 1/\varrho_k$.

Let m_2 be a positive integer greater than n_0 and n_3, such that $l_{k-2}m_2 > 1$.

Proceeding in the same way as before, we can prove that

$$I < A \sum_{m_2}^{\infty} \frac{(2erH^a)^n}{(l_{k-2}n)^{an}} + e_{k-1}(Br^{\frac{1}{a}}H),$$

where A and B are absolute constants.

The rest of the proof follows easily, if we put $(k-1)$ for k in the lemma.

Corollary 1. If $f(z) = P(z)e^{Q(z)}$ is an entire function of finite k-th order, where $P(z)$ is the product of primary factors of Theorem 6 formed with the zeros of $f(z)$; and $Q(z)$ is an entire function, then $Q(z)$ is of finite or zero type, finite $(k-1)$-th order, if $f(z)$ is of finite or zero type.

Proof. By a slight modification of the proof of Theorem 6, it can be easily shown that the k-th order of the product of primary factors $P(z)$ is equal to the k-th order of its zeros.

By (14), we have

$$\log |P(z)| \geqq \sum_{1}^{N} \log \left|1 - \frac{z}{z_n}\right| - I,$$

where

$$I = 2\sum_{n_0+1}^{N} (2u_n)^{p_n} + 2\sum_{N+1}^{\infty} u_n^{p_n+1}.$$

If $f(z)$ is of finite type, L_k is finite.

Consequently, by Theorem (6), we have

$$I < e_{k-1}(Ar^{\varrho_k})$$

for all sufficiently large values of r, where A is a constant.

Now, when $r_n \leqq 1$, we have $\left|1 - \dfrac{z}{z_n}\right| > 1$, provided that $r > 2$, and so

$$\log \prod_{r_n \leqq 1} \left|1 - \frac{z}{z_n}\right| > 0.$$

But, when $1 < r_n \leqq 2r$ and z lies outside all the small circles $|z - z_n| = e^{-he_{k-2}(r_n^{\varrho_k+\varepsilon})}$ for which $r_n = |z_n| > 1$, h being any fixed number greater than 1, we have

$$\left|1-\frac{z}{z_n}\right|=\frac{|z-z_n|}{r_n}\geqq\frac{1}{r_n}\cdot e^{-he_{k-2}(r_n{}^{\varrho_k+\varepsilon})}$$
$$\geqq\frac{1}{2r}\cdot e^{-he_{k-2}(2r)^{\varrho_k+\varepsilon}}$$

Therefore

$$\log\prod_{1>r_n\leqq 2r}\left|1-\frac{z}{z_n}\right|\geqq -N[he_{k-2}(2r)^{\varrho_k+\varepsilon}+\log 2r]$$

Since L_k is finite, we have

$$N<e_{k-1}(Br^{\varrho_k})$$

for all sufficiently large r, where B is a constant.

Combining these results, we have

$$\log\prod_1^N\left|1-\frac{z}{z_n}\right|>-e_{k-1}(Br^{\varrho_k})\cdot[he_{k-2}(2r)^{\varrho_k+\varepsilon}+\log 2r].$$

Consequently, we have

$$\log|P(z)|>-e_{k-1}(Br^{\varrho_k})[he_{k-2}(2r)^{\varrho_k+\varepsilon}+\log 2r]-e_{k-1}(Ar^{\varrho_r})$$
$$>-2e_{k-1}(cr^{\varrho_k})\cdot e_{k-2}(2r)^{\varrho_k+\varepsilon}$$

for all sufficiently large r such that the circle $|z|=r$ intersects none of the small circles containing the zeros of $f(z)$, c being any fixed number greater than each of A and B.

Also, since $f(z)$ is of finite type, we have

$$|f(z)|<e_k(Mr^{\varrho_k})$$

for all sufficiently large r, M being a constant.

Combining the two inequalities, we have

$$|e^{Q(z)}|=\left|\frac{f(z)}{P(z)}\right|<e_k(Mr^{\varrho_k})\cdot e^{2e_{k-2}(cr^{\varrho_k})\cdot e_{k-2}(2r)^{\varrho_k+\varepsilon}}$$
$$<e^{e_{k-1}(c_1r^{\varrho_k})\cdot e_{k-2}(2r)^{\varrho_k+\varepsilon}}$$

for a certain set of arbitrarily large values of r, c_1 being an absolute constant.

Consequently, by the principle of the maximum modulus, it can be easily proved that

$$|e^{Q(z)}|<e^{e_{k-1}(c_1r^{\varrho_k})e_{k-2}(2r)^{\varrho_k+\varepsilon}}$$

for all sufficiently large values of r. Hence it follows that $Q(z)$ is of finite type.

The proof for zero type follows the same lines.

CORALLARY 2(i). If $f(z) = P(z)e^{Q(z)}$ is an entire function of finite 2nd. order, then a necessary and sufficient condition that $f(z)$ be of finite or zero type is that L_2 be finite or zero and $Q(z)$ satisfy the conditions of a theorem of Lindelöf (loc-cit.).

(ii) If $f(z) = P(z)e^{Q(z)}$ is an entire function of finite 3rd. order, then a necessary and sufficient condition that $f(z)$ be of finite or zero type is that L_3 be finite or zero and $Q(z)$ satisfy the conditions of (i).

(iii) If $f(z) = P(z)e^{Q(z)}$ is an entire function of finite k-th order, then a necessary and sufficient condition that $f(z)$ be of finite or zero type is that L_k be finite or zero; and $Q(z)$ satisfy the conditions for an entire function of finite $(k-1)$-th order to be of finite or zero type, where $P(z)$ is a product of primary factors of Theorem 6, formed with the zeros of $f(z)$.

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