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On Sums and Differences of Primes and Squares

by

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1. Introduction

1.1. This paper is concerned with representations of an integer q in terms of primes p_σ belonging to given arithmetic progressions and squares g_τ^2 with given coefficients b_τ, i.e. with representations of q in the form

$$(1.1.1)\qquad q = \sum_{\sigma=1}^{s'} p_\sigma - \sum_{\sigma=s'+1}^{s} p_\sigma + \sum_{\tau=1}^{t'} |b_\tau| g_\tau^2 - \sum_{\tau=t'+1}^{t} |b_\tau| g_\tau^2 .$$

One or two of the four sums on the right hand side may be empty, but it is assumed that $0 < s'+t' < s+t$ so that at least one positive and one negative term are present.

We shall prove that if $s+t/2 > 2$ and if q satisfies certain trivial conditions, then there are infinitely many such representations of q. Moreover we shall prove an asymptotic formula for the number $N(q, n)$ of those representations (1.1.1) in which the sum of the positive and the sum of the negative terms do not both exceed n.

1.2. The method of investigation is a generalization of the method employed in my paper [12] which, in turn, is a generalization of Linnik's and Čudacov's modification of the classical Hardy-Littlewood method in Additive Number Theory (cf. [9], [2], [5], and [3]).

The notation is as much as possible the same as that of my paper [12], and results from this paper are quoted as they are required.

1.3. The special case $s = 3$, $t = 0$, i.e. the problem of representations of q in the form

$$q = p_1 \pm p_2 \pm p_3 ,$$

has already been dealt with by J. G. van der Corput [1] who, however, used his own variant of a method due to Vinogradov for the proofs. Vinogradov's method was also employed by H. E.

Richert [10] for an investigation of the more general case $s \geqq 3$, $t = 0$.

The cases $s-s' = t-t' = 0$, excluded from the present paper, have been discussed in my papers [11], [12], [13], and [14].

2. Notation

2.1. Let q, s, s', t, t', $a_1, \ldots, a_s$, $K_1, \ldots, K_s$, $b_1, \ldots, b_t$, be given integers, and

$0 \leqq s' \leqq s$, $s \geqq 1$, $0 \leqq t' \leqq t$,

$\omega = s+t/2 \geqq 5/2$, $\omega' = s'+t'/2 \geqq 1/2$, $\omega'' = \omega-\omega' \geqq 1/2$,

$(a_\sigma, K_\sigma) = 1 (\sigma = 1, \ldots, s)$,

$\eta_\tau = \operatorname{sgn} b_\tau = +1$, or -1, according as $\tau \leqq t'$, or $\tau > t'$.

2.2. Let p_σ denote an odd prime number satisfying

$p_\sigma \equiv \varepsilon_\sigma a_\sigma \pmod{K_\sigma}$

where

$\varepsilon_\sigma = +1$, or -1, according as $\sigma \leqq s'$, or $\sigma > s'$.

Let g_τ denote an integer.

Let q' and q'' denote integers of the form

$$q' = \sum_{\sigma=1}^{s'} p_\sigma + \sum_{\tau=1}^{t'} b_\tau g_\tau^2,$$

$$q'' = \sum_{\sigma=s'+1}^{s} p_\sigma + \sum_{\tau=t'+1}^{t} |b_\tau| g_\tau^2$$

(one or two of the four sums on the right hand sides may be empty).

2.3. Let n be a sufficiently large integer, and let

$$N(q, n) = \sum_{\substack{q'-q''=q \\ \min(q', q'') \leqq n}} 1$$

denote the number of representations of q in the form

$$q = q'-q'', \ \min(q', q'') \leqq n.$$

Let further

$$\nu(q, n) = \sum_{\substack{q'-q''=q \\ \min(q', q'') \leqq n}} \prod_{\sigma=1}^{s} \log p_\sigma$$

and

$$\hat{\nu}(q, n) = \sum_{q'-q''=q} r^Q \prod_{\sigma=1}^{s} \log p_\sigma,$$

where

$$r = e^{-1/n}, \ Q = q'+q''-|q|.$$

2.4. Let A' be a given positive constant, and

$$A = 2A'+4(s+2),\ A_1 = 4A+38.$$

2.5. The symbols $0(..)$ and $\sim$ refer to the limiting process $n \to \infty$, i.e. $r \to 1$, unless otherwise stated. Instead of $g(n) = 0(G(n))$ we shall occasionally write $g(n) \ll G(n)$.

The constants involved in the 0-notation may, and usually will, depend on the parameters defined in subsections 2.1 and 2.4.

2.6. Other notations will be explained at their first appearance.

3. Basic Identity

3.1. Let

$$F_\sigma(r, \alpha) = \varphi(K_\sigma) \sum_{p_\sigma} (\log p_\sigma) r^{p_\sigma} e^{i\alpha p_\sigma}$$

where φ denotes Euler's function.

Let

$$\vartheta_\tau(r, \alpha) = (|b_\tau|/\pi)^{\frac{1}{2}} \sum_{g_\tau} r^{|b_\tau| g_\tau^2} e^{i\alpha |b_\tau| g_\tau^2}.$$

3.2. On expansion of the integrand we obtain easily

$$P\hat{\nu}(q, n) = \frac{1}{2\pi} r^{-|q|} \int_C \prod_{\sigma=1}^{s} F_\sigma(r, \varepsilon_\sigma \alpha) \times \prod_{\tau=1}^{t} \vartheta_\tau(r, \eta_\tau \alpha) \times e^{-i\alpha q} d\alpha$$

where

$$P = \pi^{-t/2} \prod_{\sigma=1}^{s} \varphi(K_\sigma) \times \prod_{\tau=1}^{t} |b_\tau|^{\frac{1}{2}} \tag{3.2.1}$$

and C is an interval of length 2π.

3.3. We now dissect C into Farey intervals C_ϱ of order $T = \log^{A_1} n$ To each of these intervals C_ϱ there corresponds one, and only one, of the numbers

$$\varrho = \exp(2\pi i l/k) \text{ with } 1 \leqq k \leqq T,\ (l, k) = 1,$$

and $2\pi l/k$ is an inner point of C_ϱ (cf. subsection 1.12 of my paper [12]).

3.4. We thus obtain the formula

$$P\hat{\nu}(q, n) = \sum_{\varrho} \frac{1}{2\pi} r^{-|q|} \int_{C_\varrho} \prod_{\sigma=1}^{s} F_\sigma(r, \varepsilon_\sigma \alpha) \times \prod_{\tau=1}^{t} \vartheta_\tau(r, \eta_\tau \alpha) \times e^{-i\alpha q} d\alpha$$

which henceforth will be referred to as the *basic identity*. ($\sum_\varrho$ means, of course, $\sum_{1 \leqq k \leqq T} \sum'_{l(k)}$ where l runs through all positive integers less than and prime to k, or any set congruent to this set to modulus k.)

4. Lemmas

4.1.1. Let

$$\xi(r, \beta) = \frac{1}{n} - i\beta$$

and

$$\Xi_\gamma(r,\ \beta) = \frac{1}{\Gamma(\gamma)} \sum_{m=1}^{\infty} m^{\gamma-1} r^m e^{im\beta}.$$

4.1.2. Lemma 1. *When $\beta \leqq \pi$ we have*

$$\xi^{-\gamma}(r, \beta) = \Xi_\gamma(r, \beta) + 0(1)$$

where γ is any real number.

Proofs of this lemma were given by Lindeloef ([8], nr. 66) and Kluyver [7] (cf. also Kloosterman [6], p. 25).

4.2. We now put

$$\beta = \beta_\varrho(\alpha) = \alpha - 2\pi l/k.$$

4.3.1. Let $d_\sigma = d_\sigma(k) = (K_\sigma,\ k)$, $Q_\sigma = Q_\sigma(k)$ a solution of $kx/d_\sigma \equiv 1 \pmod{d_\sigma}$, $R_\sigma(k) = kQ_\sigma/d_\sigma$, and $N_\sigma(k) = +1$, or $= 0$, according as $(K,\ k/d_\sigma) = 1$, or > 1.

For $\varrho = \exp(2\pi il/k)$ let

$$f_{\varrho\sigma} = N_\sigma(k) \frac{\mu}{\varphi} \left(\frac{k}{d_\sigma(k)}\right) \varrho^{a_\sigma R_\sigma(k)}$$

where μ and φ denote Moebius' and Euler's functions.

4.3.2. Lemma 2. *When $\alpha \in C_\varrho$ we have*

$$\psi^*_{\varrho\sigma}(r, \alpha) = F_\sigma(r, \varepsilon_\sigma \alpha) - \psi_{\varrho\sigma}(r, \alpha) \ll n \log^{-A} n$$

where

$$\psi_{\varrho\sigma}(r, \alpha) = f_{\varrho\sigma} \xi^{-1}(r, \varepsilon_\sigma \beta).$$

For $\varepsilon_\sigma = +1$ this lemma was proved in subsection 2.1.2 of my paper [12]. When $\varepsilon_\sigma = -1$ the result still holds since we then have $\varepsilon_\sigma \alpha \in C_{\varrho^{-1}}$, and $p_\sigma \equiv -a_\sigma \pmod{K_\sigma}$ in the sum for F_σ.

4.4.1. Let

$$S_{K,L} = \sum_{j=1}^{K} \exp(2\pi iLj^2/K)$$

and

$$g_{\varrho\tau} = k^{-1} S_{k, b_\tau l}.$$

4.4.2. Lemma 3. *When $\alpha \in C_\varrho$ we have*

$$\theta^*_{\varrho\tau}(r, \alpha) = \vartheta_\tau(r, \eta_\tau \alpha) - \theta_{\varrho\tau}(r, \alpha) \ll n^{1/2} T^{-1/2}$$

where

$$\theta_{\varrho\tau}(r, \alpha) = g_{\varrho\tau}\,\xi^{-\frac{1}{2}}(r, \eta_\tau \beta).$$

For $\eta_\tau = +1$ and, consequently, $b_\tau = |b_\tau|$ this lemma was proved in subsection 2.2.2 of my paper [12]. When $\eta_\tau = -1$ the result still holds since we then have $\eta_\tau \alpha \in C_{\varrho^{-1}}$ and $|b_\tau| = -b_\tau$.

5. Approximation

5.1. Using the last three lemmas, we shall now deduce an approximate formula from the basic identity for $\hat{\nu}(q, n)$. The deduction is divided into five main steps. We begin by modifying in three steps the integrals occurring in the basic identity. In the last two steps, which will follow in section 6, we shall evaluate the modified integrals.

5.2. *First step.* In the basic identity we replace the functions F_σ and ϑ_τ by $\psi_{\varrho\sigma}$ and $\theta_{\varrho\tau}$ and estimate the error which, as is easily seen, equals

$$\sum_{\lambda=1}^{s} \frac{1}{2\pi} r^{-|q|} \int_{C_\varrho} U_\varrho^{\langle\lambda\rangle}(r, \alpha) e^{-i\alpha q}\, d\alpha + \sum_{\lambda'=1}^{t} \frac{1}{2\pi} r^{-|q|} \int_{C_\varrho} V^{\langle\lambda'\rangle}(r, \alpha) e^{-i\alpha q}\, d\alpha$$

where

$$U_\varrho^{\langle\lambda\rangle} = \prod_{\sigma=1}^{s-\lambda} \psi_{\varrho\sigma} \times \psi^*_{\varrho(s-\lambda+1)} \times \prod_{\sigma'=s-\lambda+2}^{s} F_{\sigma'}(r, \varepsilon_{\sigma'}\alpha) \times \prod_{\tau=1}^{t} \theta_{\varrho\tau},$$

$$V_\varrho^{\langle\lambda'\rangle} = \prod_{\sigma=1}^{s} F_\sigma(r, \varepsilon_\sigma \alpha) \times \prod_{\tau=1}^{t-\lambda'} \theta_{\varrho\tau} \times \theta^*_{\varrho(t-\lambda'+1)} \times \prod_{\tau'=t-\lambda'+2}^{t} \vartheta_{\tau'}(r, \eta_{\tau'}\alpha)$$

(for brevity the arguments are shown only if they differ from (r, α)).

Since

$$|F_\sigma(r, \varepsilon_\sigma \alpha)| = |F_\sigma(r, \alpha)|, \quad |\vartheta_\tau(r, \eta_\tau \alpha)| = |\vartheta_\tau(r, \alpha)|,$$

$$|\psi_{\varrho\sigma}(r, \alpha)| = |f_{\varrho\sigma}\,\xi^{-1}(r, \beta)|, \quad |\theta_{\varrho\tau}(r, \alpha)| = |g_{\varrho\tau}\,\xi^{-\frac{1}{2}}(r, \beta)|,$$

the analysis of section 3 of my paper [12] shows that

$$\int_{C_\varrho} |U_\varrho^{\langle\lambda\rangle}(r, \alpha)|\, d\alpha \ll n^{\omega-1} \log^{-A'} n \qquad (\lambda = 1, \ldots, s),$$

$$\int_{C_\varrho} |V_\varrho^{\langle\lambda'\rangle}(r, \alpha)|\, d\alpha \ll n^{\omega-1} \log^{-A'} n \qquad (\lambda' = 1, \ldots, t).$$

Hence

$$P\hat{\nu}(q, n) = \sum_\varrho \frac{1}{2\pi} r^{-|q|} \int_{C_\varrho} \prod_{\sigma=1}^{s} \psi_{\varrho\sigma}(r, \alpha) \times \prod_{\tau=1}^{t} \theta_{\varrho\tau}(r, \alpha) \times e^{-i\alpha q}\, d\alpha$$
$$+ 0(n^{\omega-1} \log^{-A'} n).$$

Putting

$$\Pi_\varrho = \prod_{\sigma=1}^{s} f_{\varrho\sigma} \times \prod_{\tau=1}^{t} g_{\varrho\tau}$$

we obtain thus

$$P\hat{\nu}(q, n) = \sum_\varrho \frac{1}{2\pi} r^{-|q|} \Pi_\varrho \int_{C_\varrho} \xi^{-\omega'}(r, \beta)\xi^{-\omega''}(r, -\beta)e^{-i\alpha q}\, d\alpha$$
$$+ 0(n^{\omega-1} \log^{-A'} n).$$

5.3. *Second step.* Next we change the ranges of integration from C_ϱ to $I_\varrho = [-\pi+2\pi l/k, +\pi+2\pi l/k]$. The error thus made is clearly

$$\ll \sum_\varrho |\Pi_\varrho| \int_{(I_\varrho - C_\varrho)} |\xi^{-\omega}(r, \beta)| d\alpha$$

which is

$$\ll T^{\omega+2} \log n \ll n^{\omega-1} \log^{-A'} n$$

as was shown in subsection 4.3 of my paper [12] (note that there is a misprint in formula (4.3.1) of [12]).

5.4. *Third step.* We now replace the integrands by

$$\Xi_{\omega'}(r, \beta)\, \Xi_{\omega''}(r, -\beta)e^{-i\alpha q}$$

with error

$$\{\xi^{-\omega'}(\xi^{-\omega''} - \Xi_{\omega''}) + \xi^{-\omega''}(\xi^{-\omega'} - \Xi_{\omega'})$$
$$-(\xi^{-\omega'} - \Xi_{\omega'})(\xi^{-\omega''} - \Xi_{\omega''})\}\, e^{-i\alpha q}$$

which, by Lemma 1, is

$$\ll |\xi(r, \beta)|^{1/2-\omega}$$

since $\omega' \leqq \omega - 1/2$, $\omega'' \leqq \omega - 1/2$, and $|\xi(r, \beta)| = |\xi(r, -\beta)| \gg 1$.

Now

$$\int_{I_\varrho} |\xi(r, \beta)|^{1/2-\omega} d\alpha = \int_{-\pi}^{+\pi} \left(\frac{1}{n^2} + \beta^2\right)^{1/4-\omega/2} d\beta$$
$$\ll n^{\omega-3/2} \int_0^\infty (1+\gamma^2)^{1/4-\omega/2} d\gamma \ll n^{\omega-3/2}$$

since $\omega - 1/2 \geqq 2$. By formula (4.1.4) of my paper [12] we also have that

$$\Pi_\varrho \ll k^{-\omega+1/4} \leqq k^{-9/4}. \tag{5.4.1}$$

The total error made in the third step is therefore

$$\ll n^{\omega-3/2} \sum_\varrho |\Pi_\varrho| \ll n^{\omega-3/2} \sum_\varrho k^{-9/4}$$
$$\ll n^{\omega-3/2} \sum_{k \leqq T} k^{-5/4} \ll n^{\omega-3/2} \ll n^{\omega-1} \log^{-A'} n.$$

5.5. Collecting the results of subsections 5.2, 5.3, and 5.4, and noting that $e^{-i\alpha} = \varrho^{-1} e^{-i\beta}$, we obtain the approximate formula

$$(5.5.1)\quad P\hat{\nu}(q, n) = \sum_{\varrho} \frac{1}{2\pi} r^{-|q|} \varrho^{-q} \Pi_\varrho \int_{-\pi}^{+\pi} \Xi_{\omega'}(r, \beta)\, \Xi_{\omega''}(r, -\beta)\, e^{-i\beta q}\, d\beta + 0(n^{\omega-1} \log^{-A'} n).$$

6. Singular Series

6.1. *Fourth step.* We now proceed to evaluate the integrals occurring in (5.5.1). On multiplying the series defining $\Xi_{\omega'}(r, \beta)$ and $\Xi_{\omega''}(r, -\beta)$, we obtain easily

$$\Gamma(\omega')\Gamma(\omega'') \frac{1}{2\pi} r^{-|q|} \int_{-\pi}^{+\pi} \Xi_{\omega'}(r, \beta) \Xi_{\omega''}(r, -\beta) e^{-i\beta q}\, d\beta$$

$$= \sum_{\substack{m_1,\, m_2=1 \\ m_1-m_2=q}}^{\infty} m_1^{\omega'-1}\, m_2^{\omega''-1}\, r^{m_1+m_2-|q|}$$

$$= \sum_{m=1}^{\infty} (m+q_1)^{\omega'-1} (m+q_2)^{\omega''-1} r^{2m}$$

where $q_1 = (|q|+q)/2$, $q_2 = (|q|-q)/2$.

Now, as $m \to \infty$,

$$(m+q_1)^{\omega'-1}(m+q_2)^{\omega''-1} = m^{\omega-2} + 0(m^{\omega-3})$$

and hence

$$\sum_{m=1}^{\infty} (m+q_1)^{\omega'-1} (m+q_2)^{\omega''-1} r^{2m}$$

$$= \sum_{m=1}^{\infty} m^{\omega-2} r^{2m} + 0 \left(\sum_{m=1}^{\infty} m^{\omega-3} r^{2m} \right)$$

$$= \Gamma(\omega-1)\Xi_{\omega-1}(r^2, 0) + 0(\Xi_{\omega-2}(r^2, 0))$$

$$= \Gamma(\omega-1)(n/2)^{\omega-1} + 0(n^{\omega-2}),$$

by Lemma 1.

But, by (5.4.1),

$$\sum_{\varrho} |\Pi_\varrho| \ll \sum_{\varrho} k^{-9/4} \leqq \sum_{k \leqq T} k^{-5/4} \ll 1.$$

The approximate formula (5.5.1) gives therefore

$$(6.1.1.)\quad P\hat{\nu}(q, n) = \frac{\Gamma(\omega-1)}{\Gamma(\omega')\Gamma(\omega'')} \left(\frac{n}{2}\right)^{\omega-1} \sum_{\varrho} \varrho^{-q} \Pi_\varrho + 0(n^{\omega-1} \log^{-A'} n).$$

6.2. *Fifth step.* Finally we extend the range of summation to infinity, i.e. we replace

$$\sum_{\varrho} = \sum_{1 \leqq k \leqq T} \sum_{l(k)}' \text{ by } \sum_{k=1}^{\infty} \sum_{l(k)}'$$

The error thus made is, by (5.4.1), obviously

$$\ll n^{\omega-1} \sum_{k>T} \sum_{l(k)}' k^{-9/4} \ll n^{\omega-1} \sum_{k>T} k^{-5/4}$$
$$\ll n^{\omega-1} T^{-1/4} \ll n^{\omega-1} \log^{-A'} n.$$

Putting

$$\mathfrak{S}(q) = \sum_{k=1}^{\infty} \sum_{l(k)}' \varrho^{-q} \Pi_{\varrho} \qquad (\varrho = \exp(2\pi i l/k)) \tag{6.2.1}$$

we obtain thus

$$P\hat{v}(q, n) = \frac{\Gamma(\omega-1)}{\Gamma(\omega')\Gamma(\omega'')} \left(\frac{n}{2}\right)^{\omega-1} \mathfrak{S}(q) + 0(n^{\omega-1} \log^{-A'} n). \tag{6.2.2}$$

6.3. The series $\mathfrak{S}(q)$, known as *singular series*, has been discussed in my paper [13]. Here the following remarks may suffice.

An integer q is called *suitable*, i.e. suitable for representation in the form

$$q = q' - q'' = \sum_{\sigma=1}^{s} \varepsilon_{\sigma} p_{\sigma} + \sum_{\tau=1}^{t} b_{\tau} g_{\tau}^2, \tag{6.3.1}$$

if four trivial, but rather complicated, conditions are satisfied by q. For suitable integers q we have

$$\mathfrak{S}(q) > 0,$$

and it follows therefore from (6.2.2) that

$$\hat{v}(q, n) \sim \frac{\Gamma(\omega-1)}{P\Gamma(\omega')\Gamma(\omega'')} \mathfrak{S}(q) \left(\frac{n}{2}\right)^{\omega-1}. \tag{6.3.2}$$

For non-suitable integers, however, we have $\mathfrak{S}(q) = 0$. But these are of no interest since it can be shown by elementary methods that a non-suitable integer q is either not representable at all in the form (6.3.1), or $q - \varepsilon_{\sigma^*} p_{\sigma^*}^*$ is suitable in the above sense for representation in the form

$$\sum_{\sigma \neq \sigma^*} \varepsilon_{\sigma} p_{\sigma} + \sum_{\tau} b_{\tau} g_{\tau}^2$$

where σ^* and $p_{\sigma^*}^*$ have certain well-defined values (cf. subsection 8.2 of my paper [13]).

7. Theorem.

7.1. With the notation, and under the conditions, of section 2 we have the following

THEOREM. *The integer q can in an infinity of ways be represented in the form*

$$q = q' - q'' = \sum_{\sigma=1}^{s} \varepsilon_\sigma p_\sigma + \sum_{\tau=1}^{t} b_\tau g_\tau^2 \tag{7.1.1}$$

provided that $s + t/2 > 2$ and that q is suitable in the sense of section 6.3.

Moreover, if $N(q, n)$ denotes the number of representations (7.1.1) *with the restriction* $\min(q', q'') \leqq n$, *then, as $n \to \infty$,*

$$N(q, n) \sim \frac{1}{P\Gamma(\omega')\Gamma(\omega'')(\omega-1)} \mathfrak{S}(q) \frac{n^{\omega-1}}{\log^s n}$$

where P is defined by (3.2.1) *and $\mathfrak{S}(q)$ by* (6.2.1).

7.2. *Proof of the theorem.* Putting

$$B_m = \sum_{q'=m+(|q|+q)/2} \sum_{q''=m+(|q|-q)/2} \prod_{\sigma=1}^{s} \log p_\sigma$$

we can write

$$\hat{\nu}(q, n) = \sum_{m=1}^{\infty} B_m e^{-m(2/n)}, \quad \nu(q, n) = \sum_{m=1}^{n} B_m.$$

By a well known Tauberian theorem (cf. Hardy [4], Theorem 108) it follows thus from (6.3.2) that for suitable q

$$\nu(q, n) \sim \frac{\Gamma(\omega-1)}{P\Gamma(\omega')\Gamma(\omega'')} \mathfrak{S}(q) \frac{1}{\Gamma(\omega)} n^{\omega-1}.$$

This proves the theorem since it is an easy deduction that

$$N(q, n) \sim (\log n)^{-s} \nu(q, n)$$

(cf. subsection 5.1 of my paper [12]).

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