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Bounds of matrices with regard to an hermitian metric


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Bounds of matrices with regard to an Hermitian metric \(^1\)

by

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§ 1. The bounds \(\Omega_{H,K}, \omega_{H,K}\).

1. Introduction. In various questions concerning the solutions of systems of equations and the errors made by rounding off, the following definition of upper and lower bounds \(\Omega(A), \omega(A)\) of a matrix \(A\) has frequently been used:

\[
\Omega(A) = \max_v \sqrt{\varphi_v}, \quad \omega(A) = \min_v \sqrt{\varphi_v},
\]

where \(\varphi_v\) denote the eigenvalues of \(A'A\) (cf. e.g. [8] p. 1042 ff., or [9], p. 787, for the special case in which \(\varphi, \chi\) are Euclidean lengths).

In this paper we will discuss a generalization of this definition introducing as “parameters” two positive definite Hermitian matrices \(H, K\). If \(H, K\) vary independently, the generalized bounds \(\Omega_{H,K}(A), \omega_{H,K}(A)\) can in general take values in the whole range \((0, \infty)\) (cf. § 1, section 3(vi)); to obtain appropriate values one has to couple \(H, K\) in some way. This can be done very naturally when \(A\) is an \(n \times n\) matrix, by taking \(K = H\). The bounds \(\Omega_{H,H} \equiv \Omega_H, \omega_{H,H} \equiv \omega_H\) are in fact often more favourable for \(A\) than (1), but at the same time their actual calculation is considerably more difficult, as is shown by the examples given in § 2. If, however, \(A\) contains only a few non-vanishing elements, \(\Omega_{H,K}(A)\) can fairly well be estimated from above by means of our theorem 2 in § 3, section 1, which generalizes a theorem due to W. Ledermann [7]. We will also make use of the theorem 2 in § 3, section 2, to determine both \(\inf_{H} \Omega_H(A)\) and \(\sup_{H} \omega_H(A)\), where \(H\) runs through all positive definite matrices. In sections 2 and 3 of § 1 we give the exact definitions and a few elementary properties of \(\Omega_{H,K}(A)\) and \(\omega_{H,K}(A)\), while in section 4 of § 1 a property not quite so trivial is proved.

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The idea of relating lengths of vectors to a positive definite Hermitian matrix $H$ has recently been applied to the solution of linear equations $Ax = b$ by M. R. Hestenes and M. L. Stein [6]. Their main problem is to minimize the "$H$-length" of the residual vectors $r(x) = b - Ax$. Our definition of $\Omega_{H,K}(A)$, $\omega_{H,K}(A)$ involves a similar extremum problem, but (in contrast to [6]) with a side condition.

In defining the $H$-length of a vector we make use of the "scalar product" $(x, y)$ with regard to $H$ of two vectors $x$, $y$, as given e.g. in H. L. Hamburger and M. E. Grimshaw [4], p. 153. Such products $(x, y)$ have also recently been used by W. Givens [2] to obtain theorems on the fields of values of a square matrix, which considerably extend the well known results due to O. Toeplitz [12] and F. Hausdorff [5].

I am very much indebted to Prof. Dr. A. Ostrowski for having most kindly allowed me to see through the manuscript of the yet unpublished book [10] from which I received many suggestions. In particular a chapter of [10] on the bounds (1) was the starting point of our investigations, which rather closely follow the disposition of this chapter.

2. Notations and definitions. Let $A = (a_{\mu\nu})$ ($\mu = 1, \ldots, m$; $\nu = 1, \ldots, n$) be an $m \times n$ matrix with real or complex elements $a_{\mu\nu}$. By $A^* = A'$ we denote its conjugate-transpose and by $A^{(p)}$ ($p = 1, \ldots, \text{Min}(m, n)$) its $p$th compound matrix, i.e. the $\binom{m}{p} \times \binom{n}{p}$ matrix consisting of all minors of $A$ of order $p$. The groups of $p$ rows and columns which form the minors are supposed to be arranged in lexicographical order. We have to use the following rules concerning $A^{(p)}$:

$$ (AB)^{(p)} = A^{(p)}B^{(p)}, \quad (A^*)^{(p)} = (A^{(p)})^*, $$

(2)

if the product $AB$ exists (cf. e.g. [1], p. 90ff). The first relation in (2) (the so called Binet-Cauchy theorem) is readily extended to more than two factors. Further if $m = n$ and $A^{-1}$ exists, from (2) (with $B = A^{-1}$) it follows that

$$ (A^{-1})^{(p)} = (A^{(p)})^{-1}. $$

(3)

$\text{tr} A$ will denote the trace $\sum a_{\mu\nu}$ of a square matrix $A$, $\lambda_A$ an eigenvalue of $A$ and $|\lambda_A|_{\text{max}}$, $|\lambda_A|_{\text{min}}$ respectively the maximal, minimal modulus of the eigenvalues of $A$.

By $x, y$ etc. we denote column-vectors of a $k$-dimensional complex Euclidean space, by $x^*$ the conjugate-transposed row-vector
and by \(|x|\) the Euclidean length of \(x\). In order to introduce an Hermitian metric we define the scalar product \((x, y)\) of two vectors \(x, y\) by
\[
(x, y) = y^* H x \quad (H > 0),
\]
where \(H\) is an Hermitian matrix of order \(k\); the meaning of the relation \(H > 0\) is that \(H\) is positive definite. In particular \((x, x)\) is real and \(\geq 0\) with \((x, x) = 0\) only when \(x = 0\). We therefore define
\[
\|x\| = \sqrt{(x, x)}
\]
as the norm of \(x\) with regard to \(H\). Sometimes we add the subscript \(H\) and write \(\|x\|_H\) instead of \(\|x\|\). By routine arguments (cf. e.g. [11], p. 5, [3], p. 90—92, or [4], p. 4—5) the following three properties of \(\|x\|\) are obtained:
\[
\begin{align*}
\|x\| &\geq 0 \text{ with equality if and only if } x = 0 \\
\|\gamma x\| &= |\gamma| \|x\| \quad (\gamma \text{ any complex scalar}) \\
\|x + y\| &\leq \|x\| + \|y\|.
\end{align*}
\]

Now let \(A\) be an \(m \times n\) matrix and \(H > 0, K > 0\) be Hermitian matrices of orders \(m, n\) respectively; we then define the upper and lower bounds \(\Omega_{H,K}(A), \omega_{H,K}(A)\) of \(A\) by
\[
\begin{align*}
\Omega_{H,K}(A) &= \max_{\|x\|_K = 1} \|Ax\|_H = \left(\max_{\|x\|_K = 1} x^* A^* H A x\right)^{1/2}, \\
\omega_{H,K}(A) &= \min_{\|x\|_K = 1} \|Ax\|_H = \left(\min_{\|x\|_K = 1} x^* A^* H A x\right)^{1/2}.
\end{align*}
\]

If in particular \(m = n\) and \(H = K\) we write \(\Omega_{H,H} = \Omega_H, \omega_{H,H} = \omega_H\). The definition (7) can also (partly) be expressed in terms of Euclidean lengths: Let \(K\) be transformed to a diagonal matrix by the unitary matrix \(U\):
\[
D = U^* K U = \text{Diag} (k_1, \ldots, k_n), \quad U^* U = I_n,
\]
where \(I_n\) is the \(n \times n\) unity matrix. Since \(k_v > 0\) \(v = 1, \ldots, n\) \(D\) can further be reduced to \(I_n\) by multiplying on the right and left by \(\Delta = \text{Diag} \left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right)\):
\[
\Delta U^* K U \Delta = I_n, \quad \Delta = \text{Diag} \left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right).
\]

If we now apply to \(x\) the substitution \(x = U \Delta y\) we get
\[
\|Ax\|_H^2 = x^* A^* H A x = y^* \Delta U^* A^* H A U \Delta y \quad (x = U \Delta y)
\]
and by (9)
\[ x^* K x = y^* \Delta U^* K U \Delta y = y^* y. \]

Hence \( \| x \|_K = 1 \) implies \( |y| = 1 \) and vice versa; we therefore have
\[ \| x \|_K^{-1} \frac{F_y}{F_x} \| Ax \|_H^2 = \frac{F_y}{F_x} \| y B y \|, \quad B = \Delta U^* A^* H A U \Delta, \quad (10) \]

where \( F_x, F_y \) denote the fields of values over the sets of vectors
\( \| x \|_K^{-1}, \| y \|^{-1} \)

\( x, y \) with \( \| x \|_K = 1, \| y \| = 1 \) respectively. Since \( B \) is non-negative definite, from (10) we see that both Max, Min in (7) actually exist and

\[ \Omega^2_{H, K}(A) = \lambda^\max_B, \quad \omega^2_{H, K}(A) = \lambda^\min_B, \quad B = \Delta U^* A^* H A U \Delta. \quad (11) \]

If in particular we take \( H = I_m, K = I_n \), so that clearly \( U = \Delta = I_n \), we obtain the bounds defined in (1).

Throughout this paper we denote respectively by \( h_1, \ldots, h_m > 0, k_1, \ldots, k_n > 0 \) the eigenvalues (not necessarily distinct and arranged in any order) of \( H, K \) and we put \( h' = \max h_\mu, k' = \max k_\nu, k'' = \min k_\nu. \)

3. Elementary properties of \( \Omega_{H, K}, \omega_{H, K} \). If not otherwise stated in this section \( A, H > 0, K > 0 \) are respectively \( m \times n, m \times m, n \times n \) matrices.

(i) The following properties of \( \Omega_{H, K}, \omega_{H, K} \) are immediate consequences of (6) and (7):

\[ \Omega_{H, K}(\gamma A) = |\gamma| \Omega_{H, K}(A), \quad \omega_{H, K}(\gamma A) = |\gamma| \omega_{H, K}(A) \quad (\gamma \text{ any complex scalar}) \]

\[ \Omega_{H, K}(A + B) \leq \Omega_{H, K}(A) + \Omega_{H, K}(B), \quad \omega_{H, K}(A + B) \geq \omega_{H, K}(A) - \Omega_{H, K}(B) \]

\[ \Omega_{H, K}(A) = 0 \text{ if and only if } A = 0 \]

\[ \omega_{H, K}(A) = 0 \text{ if and only if the rank of } A \text{ is } < n. \quad (13) \]

(ii) Obviously we can also write

\[ \Omega_{H, K}(A) = \max_{x \neq 0} \frac{\| Ax \|_H}{\| x \|_K}, \quad \omega_{H, K}(A) = \min_{x \neq 0} \frac{\| Ax \|_H}{\| x \|_K}. \quad (14) \]

so that for any \( m \times n \) matrix \( C \):

\[ \| C x \|_H \leq \Omega_{H, K}(C) \| x \|_K, \quad \| C x \|_H \geq \omega_{H, K}(C) \| x \|_K. \quad \]

Hence, if \( A, B, L > 0 \) respectively are \( m \times l, l \times n, l \times l \) matrices, we have

\[ \Omega_{H, K}(A B) \leq \Omega_{H, L}(A) \Omega_{L, K}(B), \quad \omega_{H, K}(A B) \geq \omega_{H, L}(A) \omega_{L, K}(B). \quad (15) \]
On the other hand, for any vector \( x \) with \( Bx \neq 0 \)

\[
\frac{\| ABx \|_H}{\| x \|_K} = \frac{\| A(Bx) \|_H}{\| Bx \|_L} \frac{\| Bx \|_L}{\| x \|_K} (Bx \neq 0). \tag{16}
\]

Suppose now that for the vector \( x : \frac{\| Bx \|_L}{\| x \|_K} = \Omega_{L,K}(B) \). Then by (16) and (14)

\[
\Omega_{H,K}(AB) \subseteq \frac{\| ABx \|_H}{\| x \|_K} \subseteq \frac{\| A(Bx) \|_H}{\| Bx \|_L} \Omega_{L,K}(B) \subseteq \omega_{H,L}(A) \Omega_{L,K}(B).
\]

Similarly, if \( B \) is of rank \( n \), from (16) we deduce \( \omega_{H,K}(AB) \leq \Omega_{H,L}(A) \omega_{L,K}(B) \). If \( B \) is of rank \( < n \), then the same holds for \( AB \) (cf. e.g. [1], p. 96–97) and therefore \( \omega_{H,K}(AB) = \omega_{L,K}(B) = 0 \). Thus we can extend (15) as follows:

\[
\Omega_{H,K}(AB) \supseteq \omega_{H,L}(A) \Omega_{L,K}(B), \omega_{H,K}(AB) \leq \Omega_{H,L}(A) \omega_{L,K}(B). \tag{17}
\]

(iii) Suppose that \( m = n \) and \( A^{-1} \) exists; then putting \( x = A^{-1}y \) we see that

\[
\frac{\| A x \|_H}{\| x \|_K} = \frac{\| y \|_H}{\| y \|_K} = \left( \frac{\| A^{-1}y \|_K}{\| y \|_H} \right)^{-1} (x = A^{-1}y).
\]

Hence in using (14) we get

\[
\Omega_{H,K}(A) = \frac{1}{\omega_{K,H}(A^{-1})}, \quad \omega_{H,K}(A) = \frac{1}{\Omega_{K,H}(A^{-1})}. \tag{18}
\]

(iv) Let \( S, T \) be two nonsingular matrices of orders \( m, n \) respectively; then we have

\[
\Omega_{H,K}(A) = \Omega_{S*HS, T*KT}(S^{-1}AT), \omega_{H,K}(A) = \omega_{S*HS, T*KT}(S^{-1}AT). \tag{19}
\]

If in particular \( m = n \), \( \Omega_{H,K}, \omega_{H,K} \) do not change, if a unitary transformation \( S \) is applied both to \( A, H \) and \( K \).

Indeed, putting \( x = Ty \) we see that the field of values \( x*A^*HAx \) over the set of vectors \( x \) with \( x*Kx = 1 \) coincides with the field of values

\[
y*T*A^*HATy = y*(T*A*(S^*)^{-1})(S*HS)(S^{-1}AT)y
\]

taken for all vectors \( y \) with \( y*T*KTy = 1 \). Hence (19) follows at once from the definition (7).

(v) Suppose that both \( A, H \) and \( K \) are respectively the "direct sums" of \( A_1, \ldots, A_s, H_1, \ldots, H_s \) and \( K_1, \ldots, K_s \), i.e. that in an obvious notation

\[
A = \text{Diag}(A_1, \ldots, A_s), \quad H = \text{Diag}(H_1, \ldots, H_s), \quad K = \text{Diag}(K_1, \ldots, K_s),
\]
where $A_\sigma$ is an $m_\sigma \times n_\sigma$, $H_\sigma > 0$ an $m_\sigma \times m_\sigma$ and $K_\sigma > 0$ an $n_\sigma \times n_\sigma$ matrix ($\sigma = 1, \ldots, s$). Then we have

$$\Omega_{H,K}(A) = \max_{\sigma=1,\ldots,s} \Omega_{H_\sigma,K_\sigma}(A_\sigma), \quad \omega_{H,K}(A) = \min_{\sigma=1,\ldots,s} \omega_{H_\sigma,K_\sigma}(A_\sigma).$$

In fact, let $K_\sigma$ be transformed to a diagonal matrix by the unitary matrix $U_\sigma$ ($\sigma = 1, \ldots, s$) and put $U = \text{Diag}(U_1, \ldots, U_s)$, so that clearly (8) holds. Put $A = \text{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right) = \text{Diag}(A_1, \ldots, A_s)$, $A_\sigma$ being of the same order as $K_\sigma$ ($\sigma = 1, \ldots, s$). Then obviously the matrix $B$ in (10) is the direct sum of $A_\sigma U_\sigma^* H_\sigma A_\sigma U_\sigma A_\sigma$ ($\sigma = 1, \ldots, s$), whence (20) follows from (11).

(vi) For every $H > 0$, $K > 0$ we have

$$\sqrt{\frac{h'}{k'}} \omega(A) \leq \omega_{H,K}(A) \leq \Omega_{H,K}(A) \leq \sqrt{\frac{h'}{k'}} \Omega(A),$$

where $\omega(A)$; $\Omega(A)$ are the bounds defined in (1).

Indeed, (21) follows from (14) by putting $y = Ax$ in

$$h'' |y|^2 \leq ||y||_H^2 \leq h' |y|^2, \quad k'' |x|^2 \leq ||x||_K^2 \leq k' |x|^2.$$

(vii) We have for any eigenvalue $\lambda_A$ of a square matrix $A$:

$$\omega_H(A) \leq |\lambda_A| \leq \Omega_H(A).$$

In fact, let $x$ be an eigenvector corresponding to $\lambda_A$ with $||x||_H = 1$. Then $Ax = \lambda_A x$, $||Ax||_H = |\lambda_A|$, whence (22) follows directly from (7).

4. For the proof of our first theorem we need the following

**Lemma 1.** Let $S$ be an $n \times m$ matrix and $T$ an $m \times n$ matrix. Then, if $m < n$, we have

$$(ST)^{(p)} = 0 \quad (p > m).$$

**Proof.** Put $S_0 = (SO_1)$, $T_0 = \begin{pmatrix} T \\ O_2 \end{pmatrix}$, where $O_1$, $O_2$ are $n \times (n-m)$, $(n-m) \times n$ zero-matrices respectively. Obviously both $S_0$ and $T_0$ are $n \times n$ matrices and $ST = S_0 T_0$. Hence by (2) $(ST)^{(p)} = S_0^{(p)} T_0^{(p)}$, and (23) follows from $S_0^{(p)} = T_0^{(p)} = 0$ ($p > m$).

**Lemma 2.** Suppose that $D = \text{Diag}(k_1, \ldots, k_n)$ ($k_\nu > 0$),

$$A = \text{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right) \quad \text{and} \quad G = \text{Diag}(h_1, \ldots, h_m) \quad (h_\mu > 0),$$

$$F = \text{Diag}\left(\frac{1}{\sqrt{h_1}}, \ldots, \frac{1}{\sqrt{h_m}}\right).$$

Further let $R = (r_{\mu\nu})$ be an $m \times n$
matrix and put \( B = \Gamma RDR* \Gamma, C = \Delta^{-1} R^* G^{-1} R \Delta^{-1}. \) Then, if 
\[ \varphi(\lambda) = |\lambda I_m - B|, \psi(\lambda) = |\lambda I_n - C| \]
are the characteristic polynomials of \( B, C, \) we have 
\[ \psi(\lambda) = \lambda^{n-m} \varphi(\lambda). \]

**Proof.** Without loss of generality we may assume \( m \leq n. \) Put \( B = (b_{\mu\nu}) (\mu, \nu = 1, \ldots, m), C = (c_{\mu\nu}) (\mu, \nu = 1, \ldots, n); \) by direct multiplication we get
\[
\begin{align*}
\varphi(\lambda) &= \lambda^{n-m} \varphi(\lambda) \\
\psi(\lambda) &= \lambda^{n-m} \psi(\lambda)
\end{align*}
\]

Hence
\[
\begin{align*}
\text{tr} B &= \sum_{\mu=1}^{m} b_{\mu\mu} = \sum_{\mu=1}^{m} \sum_{\sigma=1}^{n} \frac{1}{\sqrt{h_{\mu}}} |r_{\mu\sigma}|^2 = \sum_{v=1}^{n} \sum_{\tau=1}^{m} \frac{1}{r_{\tau\mu}} r_{\tau v} c_{\mu\nu} = \text{tr} C.
\end{align*}
\]

We now form the \( p^{th} \) compound matrices \( B^{(p)}, C^{(p)} \) of \( B, C; \) from (2), (3) it follows that 
\[
\begin{align*}
B^{(p)} &= \Gamma^{(p)} R^{(p)} D^{(p)} (R^{(p)})^* \Gamma^{(p)} \\
C^{(p)} &= (\Delta^{(p)})^{-1} (R^{(p)})^* (G^{(p)})^{-1} R^{(p)} (\Delta^{(p)})^{-1} \quad (p = 1, \ldots, m).
\end{align*}
\]

Evidently \( B^{(p)}, C^{(p)} \) are built analogously to \( B, C. \) Therefore our first conclusion again is applicable and we get
\[
\text{tr} B^{(p)} = \text{tr} C^{(p)} \quad (p = 1, \ldots, m). \tag{24}
\]

If \( m < n \) by the lemma 1 with \( S = \Delta^{-1} R^*, T = G^{-1} R \Delta^{-1} \) we have
\[
C^{(p)} = 0 \quad (p > m). \tag{25}
\]

Since generally \((-1)^p \text{tr} A^{(p)} \) is the coefficient of \( \lambda^{n-p} \) in the characteristic polynomial \( |\lambda I_n - A| \) of an \( n \times n \) matrix \( A, \) (cf. e.g. [1], p. 88), our assertion now follows immediately from (24) and (25).

**Theorem 1.** Let \( A \) be an \( m \times n \) matrix and \( H > 0, K > 0 \) be respectively of orders \( m, n; \) then we have
\[
\Omega_{K, H}(A^*) = \Omega_{H^{-1}, K^{-1}}(A), \tag{26}
\]
and, if \( m = n, \)
\[
\omega_{K, H}(A^*) = \omega_{H^{-1}, K^{-1}}(A). \tag{27}
\]

**Proof.** Let
\[
G = V^* H V = \text{Diag} (h_1, \ldots, h_m), \quad V^* V = I_m, \quad (28)
\]
\[
\Gamma = \text{Diag} \left( \frac{1}{\sqrt{h_1}}, \ldots, \frac{1}{\sqrt{h_m}} \right) \quad (29)
\]
be the equations corresponding to (8), (9), applied to the matrix $H$. Then in using (8), (28) we have

$$D = U^*KU, \quad D^{-1} = U^*K^{-1}U; \quad G = V^*HV, \quad G^{-1} = V^*H^{-1}V, \quad (30)$$

$$K = UDU^*, \quad K^{-1} = U^{-1}GU^*; \quad H = VGV^*, \quad H^{-1} = VG^{-1}V^*. \quad (31)$$

According to (11) and (28)—(30) we have to examine the eigenvalues of

$$B = IV^*A^*V^*V, \quad C = \Delta^{-1}U^*A^*H^{-1}AU\Delta^{-1}.$$  

By means of (31) we can write

$$B = \Gamma(V^*AU)D(U^*A^*V)\Gamma = \Gamma RDR^*\Gamma$$
$$C = \Delta^{-1}(U^*A^*V)G^{-1}(V^*AU)\Delta^{-1} = \Delta^{-1}R^*G^{-1}R\Delta^{-1},$$

putting $R = V^*AU$. If we now apply the lemma 2, our assertion follows at once.

**Corollary 1.** For any square matrix $A$ and $H > 0$ we have

$$\Omega_H(A^*) = \Omega_H^{-1}(A), \quad \omega_H(A^*) = \omega_H^{-1}(A). \quad (32)$$

**Corollary 2.** If $A$ is an Hermitian matrix, then for any $H > 0$

$$\Omega_H(A) = \Omega_H^{-1}(A), \quad \omega_H(A) = \omega_H^{-1}(A). \quad (33)$$

§ 2. Examples.

For the sake of simplicity in this section we only consider square $n \times n$ matrices $A$ and we take $H = K$. As to the selection of examples we follow very closely the arrangement given by A. Ostrowski in [10].

(i) Let $A = (a_{\mu\nu})$ be a matrix the only non-vanishing element of which is $a_{ik} = a$. Put $H = (h_{\mu\nu}), \quad B = (b_{\mu\nu}), \quad U = (u_{\mu\nu})$, where $U$ satisfies (8) and $B$ is the matrix defined in (10). By direct multiplication we get

$$b_{\mu\nu} = |a|^2 h_{ii}\bar{u}_{k\mu} u_{k\nu} \frac{1}{\sqrt{h_{\mu}}} \frac{1}{\sqrt{h_{\nu}}}. $$

If by $v$ we denote the row-vector $\left(\frac{u_{k1}}{\sqrt{h_1}}, \ldots, \frac{u_{kn}}{\sqrt{h_n}}\right)$, $B$ can be considered as the product $|a|^2 h_{ii}v^*v$ and is therefore of rank 1. Hence by (11) we have

$$\Omega^2_H(A) = \lambda_B = \text{tr} B = |a|^2 h_{ii} \sum_{\nu=1}^{n} \frac{|u_{k\nu}|^2}{h_{\nu}}.$$
On the other hand, by (31), \( h_{ii} = \sum_{y=1}^{n} h_{y} \mid u_{iy} \mid^2 \) and so

\[
\Omega_H^2(A) = \mid a \mid^2 \sum_{y=1}^{n} h_{y} \mid u_{iy} \mid^2 \sum_{y=1}^{n} \frac{|u_{ky}|^2}{h_y}.
\] (34)

If in particular \( H \) is a diagonal matrix, and therefore \( U = I_n \), we get

\[
\Omega_H(A) = \mid a \mid \sqrt{\frac{h_i}{h_k}}, \quad H = \text{Diag}(h_1, \ldots, h_n).
\] (35)

Let us in this example discuss, to what extent \( \Omega_H(A) \) is determined by the eigenvalues of \( H \). Clearly all Hermitian matrices having the fixed eigenvalues \( h_1, \ldots, h_n > 0 \) are obtained by letting \( U \) in \( H = U D U^* \), \( D = \text{Diag}(h_1, \ldots, h_n) \), run through all unitary \( n \times n \) matrices. In the case \( i \neq k \), from (34) we can derive the following bounds, between which \( \Omega_H^2(A) \) varies:

\[
\frac{h''}{h'} \leq \frac{1}{\mid a \mid^2} \Omega_H^2(A) \leq \frac{h'}{h''},
\]

where the upper and lower bounds are attained by taking in (34) for \( (u_{i1}, \ldots, u_{in}), (u_{k1}, \ldots, u_{kn}) \) suitable unit vectors. Similarly, if \( i = k \), from (34) we see that \( \frac{1}{\mid a \mid^2} \Omega_H^2(A) \) takes values in a certain closed interval, the left-hand end point of which by (22) is equal to 1.

If on the other hand we let \( H \) run through all diagonal matrices, (35) shows that in the case \( i \neq k \) the range of \( \Omega_H(A) \) is the whole interval \( (0, \infty) \), while \( \Omega_H(A) \) for \( i = k \) is always equal to \( \mid a \mid \).

Evidently in this example \( \omega_H(A) = 0 \) by (13).

(ii) Let \( A = (a_{\mu \nu}) \) be a matrix all elements of which are zero except those lying in the \( i \)th row, and put \( (a_{i1}, \ldots, a_{in}) = (a_1, \ldots, a_n) = a \). We suppose that \( H = \text{Diag}(h_1, \ldots, h_n) \), i.e. \( U = I_n \). Then for the matrix \( B \) in (10) we have \( B = \Delta A^* H A A \),

\[
b_{\mu \nu} = h_{\mu} \delta_{\mu} a_{\nu} \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}}
\]

and as in our example (i) the rank of \( B \) is equal to 1, so that

\[
\Omega_H^2(A) = \text{tr} B = h_i \sum_{y=1}^{n} \frac{|a_y|^2}{h_y} = |a_i|^2 + h_i \sum_{y \neq i}^{n} \frac{|a_y|^2}{h_y}.
\] (36)

Clearly we have always \( \omega_H(A) = 0 \), and, by a suitable choice of \( H \), \( \Omega_H(A) \) can take values arbitrarily near to \( \mid a_i \mid = \mid \lambda_A \mid^\text{max} \).
(iii) If all elements of the matrix $A$ are equal to $a \neq 0$ and if we take $H = \text{Diag} \ (h_1, \ldots, h_n)$, we have $b_{\mu\nu} = \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}} \ |a|^2 \sum_{\kappa=1}^n h_\kappa$ and therefore by the Cauchy-Schwarz inequality

$$Q_H^n (A) = \text{tr} \ B = |a|^2 \left( \sum_{\nu=1}^n h_\nu \right) \left( \sum_{\nu=1}^n \frac{1}{h_\nu} \right) \geq |a|^2 n^2. \quad (37)$$

The lower bound for $Q_H (A)$, $|a| \ n = |\lambda_A|^{\text{max}}$, is attained for $H = I_n$, while $Q_H (A)$ is not bounded at all from above. On the other hand $\omega_H (A) = 0$.

(iv) Let $A = \text{Diag} \ (a_1, \ldots, a_n)$, $H = \text{Diag} \ (h_1, \ldots, h_n)$. Then $B = \text{Diag} \ (|a_1|^2, \ldots, |a_n|^2)$, so that

$$Q_H (A) = \max_v |a_v| = |\lambda_A|^{\text{max}}, \ \omega_H (A) = \min_v |a_v| = |\lambda_A|^{\text{min}}. \quad (38)$$

(v) Let $A = (a_{\mu\nu})$ be a matrix all elements of which are zero except those lying in the $i^{th}$ row and $k^{th}$ column, while we have also $a_{ik} = 0$. We further assume $H$ to be a diagonal matrix. In applying to both $A$ and $H$ the same permutation to the rows and columns, whereby in virtue of §1, section 3(iv), $Q_H (A)$, $\omega_H (A)$ are not changed, we can make $k = 1$. Having carried through this transformation we denote by $a = (a_2, \ldots, a_n)$ the $i^{th}$ $(n - 1)$-dimensional row-vector of $A$ (without its first element), by $\beta = (b_1, \ldots, b_n)$ the first (n-dimensional) column-vector of $A$ (where $b_i = 0$) and we put $H = \text{Diag} \ (h_1, \ldots, h_n)$, $\Delta_1 = \text{Diag} \left( \frac{1}{\sqrt{h_2}}, \ldots, \frac{1}{\sqrt{h_n}} \right)$. For the matrix $B$ of (10) we then obtain by direct multiplication (observing that $U = I_n$)

$$B = \begin{pmatrix} \frac{1}{h_1} \beta^* H \beta & 0 \\ 0 & h_i \Delta_1^* \alpha \Delta_1 \end{pmatrix}. \quad (39)$$

Since again the $(n - 1) \times (n - 1)$ matrix in the lower right-hand corner of $B$ is of rank 1, it follows from (11) that

$$Q_H (A) = \max (Q_{1}, Q_{2}), \quad \omega_H (A) = 0 \ \ (n > 2).$$
§ 3. A generalization of Ledermann’s theorem and the determination of $\inf \omega_H$, $\sup \omega_H$.

1. The reason we succeeded to calculate directly $\omega_H$, $\omega_H$ in the examples given in § 2 was that the matrices $A$ contained a sufficiently large number of zeros. We now prove a general theorem which in similar cases always yields an upper bound for $\Omega_{H,K}(A)$ and which is a generalization of a theorem due to W. Ledermann [7]. More precisely:

**Theorem 2.** Let $A = (a_{\mu \nu})$ be an $m \times n$ matrix and denote by $a_{\mu}$ its $\mu$th row-vector; then, if $H = \text{Diag} \,(h_1, \ldots, h_m)$ ($h_\mu > 0$), $K = \text{Diag} \,(k_1, \ldots, k_n)$ ($k_\nu > 0$) and if every column-vector of $A$ contains at most $s$ non-vanishing elements, we have

$$\Omega_{H,K}^2(A) \leq \sum_{\sigma=1}^s h_{\mu_\sigma} \| \alpha_{\mu_\sigma} \|_{K^{-1}}^2,$$

where the sum on the right-hand side has to be taken over the $s$ largest numbers $h_{\mu_\sigma} \| \alpha_{\mu_\sigma} \|_{K^{-1}}^2$ ($\sigma = 1, \ldots, s$) among $h_{\mu} \| \alpha_{\mu} \|_{K^{-1}}^2$ ($\mu = 1, \ldots, m$).

**Proof.** Our proof is essentially the same as that given for the case $H = I_m$, $K = I_n$ by A. Ostrowski in [10].

Without loss of generality we may assume that

$$h_1 \| a_1 \|_{K^{-1}} \geq h_2 \| a_2 \|_{K^{-1}} \geq \ldots \geq h_m \| a_m \|_{K^{-1}}. \tag{41}$$

Indeed, let a permutation $P$ be applied to the rows of $A$; if we further permute the rows and the columns of $H$ according to $P$, by (19) (with $T = I_n$) $\Omega_{H,K}(A)$ does not change and the numbers $h_{\mu} \| \alpha_{\mu} \|_{K^{-1}}^2$ ($\mu = 1, \ldots, m$) are arranged as required.

Let $\text{Max} \, \| A x \|_H^2$ be attained for the vector $x = (x_1, \ldots, x_n)$ and put $y = A x$, $y = (y_1, \ldots, y_m)$. For every $\mu$ ($\mu = 1, \ldots, m$) replace the coordinates $x_\nu$ of $x$ for which $a_{\mu \nu} = 0$ by zeros and denote the vector so obtained by $x^{(\mu)}$. Then we have

$$\Omega_{H,K}^2(A) = \| y \|_H^2 = \sum_{\mu=1}^m h_\mu \| y_\mu \|^2 = \sum_{\mu=1}^m h_\mu \| \alpha_\mu x^{(\mu)} \|^2. \tag{42}$$

We further put $x^{(\mu)} = (x_1^{(\mu)}, \ldots, x_n^{(\mu)})$ ($\mu = 1, \ldots, m$). Since by the Cauchy-Schwarz inequality

$$| \alpha_\mu x^{(\mu)} |^2 = \left| \sum_{\nu=1}^n a_{\mu \nu} x_\nu^{(\mu)} \right|^2 \leq \left( \sum_{\nu=1}^n \frac{1}{k_\nu} \right) \left( \sum_{\nu=1}^n k_\nu \ | x_\nu^{(\mu)} |^2 \right) \leq \left( \sum_{\nu=1}^n \frac{1}{k_\nu} \right) \left( \sum_{\nu=1}^n k_\nu \ | x_\nu^{(\mu)} |^2 \right) \leq \left( \sum_{\nu=1}^n \frac{1}{k_\nu} \right) \sum_{\nu=1}^n k_\nu \ | x_\nu^{(\mu)} |^2,$$
from (42) we get

\[ \Omega^2_{H,K}(A) \leq \sum_{\mu=1}^{m} h_\mu \| x_\mu \|_{K^{-1}}^{2} \sum_{\nu=1}^{n} k_\nu \| x_\nu^{(\mu)} \|^2 = \]

\[ = \sum_{\nu=1}^{n} k_\nu \left( \sum_{\mu=1}^{m} \| x_\nu^{(\mu)} \|^2 h_\mu \| x_\mu \|_{K^{-1}}^{2} \right). \tag{43} \]

If \( x_\nu^{(\mu)} \neq 0 \) then

\[ x_\nu^{(\mu)} = x_\nu \quad (x_\nu^{(\mu)} \neq 0) \tag{44} \]

and \( a_{\mu\nu} \neq 0 \). From this and the hypothesis it follows that for any fixed \( \nu \) at most \( s \) of the \( x_\nu^{(\mu)} \) are \( \neq 0 \). Therefore taking the sum in brackets on the right-hand side of (43) only over the terms with \( x_\nu^{(\mu)} \neq 0 \) and using (44), (41) we see that

\[ \sum_{\mu=1}^{m} \| x_\nu^{(\mu)} \|^2 h_\mu \| x_\mu \|_{K^{-1}}^{2} \leq \| x_\nu \|^2 \sum_{\sigma=1}^{s} h_\sigma \| x_\sigma \|_{K^{-1}}^{2} \quad (\nu = 1, \ldots, n), \]

whence by (43)

\[ \Omega^2_{H,K}(A) \leq \left( \sum_{\nu=1}^{n} k_\nu \| x_\nu \|^2 \right) \left( \sum_{\sigma=1}^{s} h_\sigma \| x_\sigma \|_{K^{-1}}^{2} \right). \]

This proves our assertion, since \( \sum_{\nu=1}^{n} k_\nu \| x_\nu \|^2 = \| x \|_{K^{-1}}^2 = 1 \).

REMARKS. The theorem of Ledermann is obtained by taking \( H = \text{Im}, K = I_n \). If in particular we apply (40) with \( H = K \) to our examples (i), (ii) and (iv) we obtain respectively as upper bounds

\[ h_1 |a_1|^2, h_i \| x \|_{K^{-1}}^2, \text{Max } |a_\nu|^2, \text{ which all coincide with the corresponding } \Omega^2_{H}. \]

Even if \( s = m \) the theorem 2 is often useful. Take e.g.

\[ A = \begin{pmatrix} 2 & 1 & 8 \\ 7 & 0 & 5 \\ 0 & 1 & 3 \end{pmatrix}, \]

where the elements of the second column are comparatively small.

In order to get favourable bounds for \( \Omega_H(A) \) in applying (40), we choose \( h_2 \) relatively small. With \( H = \text{Diag} (4,1,20) \) we obtain

\[ \Omega^2_{H}(A) \leq 63.3, \text{ while } H = I \text{ gives } \Omega^2(A) \leq 158. \]

2. We now use our theorem 2 to give a refinement of (22):

\[ \text{Theorem 3. For any } n \times n \text{ matrix } A \text{ we have} \]

\[ \text{Inf}_{H>0} \Omega_H(A) = | \lambda_A |^{\text{max}}, \text{ Sup}_{H>0} \omega_H(A) = | \lambda_A |^{\text{min}}. \tag{45} \]

If in particular \( A \) has only simple elementary divisors both Inf and Sup in (45) are attained for suitable matrices \( H > 0 \).
PROOF. Since for a nonsingular matrix $|\lambda_{A^{-1}}|^{\text{max}} = 1/|\lambda_A|^{\text{min}}$ and by (18) $\omega_H(A) = \frac{1}{\Omega_H(A^{-1})}$, it is sufficient to prove the relation concerning $\inf \Omega_H$. Let $A$ be transformed to Jordan's canonical form by the nonsingular matrix $S$:

$$S^{-1}AS = A + C,$$

where $A$ is a diagonal matrix the elements of which are the eigenvalues of $A$, and $C$ denotes a matrix consisting of zeros except possibly some elements $c_{\mu \nu} = 1$ with $\nu = \mu + 1$. If in (19) we take $T = S$ we have by (22), (12)

$$|\lambda_A|^{\text{max}} \leq \Omega_H(A) = \Omega_K(A + C) \leq \Omega_K(A) + \Omega_K(C),$$

where $K = S^*HS$. It suffices to show that for a suitable choice of $K$ the sum on the right-hand side of (46) is arbitrarily near to $|\lambda_A|^{\text{max}}$. Take $K = \text{Diag}(k_1, \ldots, k_n)$; then by (38) $\Omega_K(A) = |\lambda_A|^{\text{max}}$; if on the other hand $\gamma_v$ is the $v^{\text{th}}$ row-vector of $C$, then $\|\gamma_v\|_{K^{-1}}^2 = \left\{ \begin{array}{ll} 0 & (\gamma_v = 0) \\ 1/k_{v+1}(\gamma_v \neq 0) & \end{array} \right.$

Hence by the theorem 2

$$\Omega_K^2(C) \leq \max_{v=1, \ldots, n-1} \frac{k_v}{k_{v+1}},$$

which obviously can be made as small as we please.

If all elementary divisors of $A$ are simple we have in (46) $C = 0$, $\Omega_K(C) = 0$, so that $\Omega_H(A)$ attains the value $|\lambda_A|^{\text{max}}$ for a suitable matrix $H > 0$.

It is natural to ask whether we could in (45) take Inf, Sup only over the set of all diagonal matrices $H > 0$. This is however not true as the following example shows: Take

$$A = \begin{pmatrix} 0 & i & 1 \\ i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where $|\lambda_A|^{\text{max}} = 0$. If $H = \text{Diag}(h_1, h_2, h_3)$, it follows from § 2, Ex. (v), that

$$\Omega_H(A) = \max (\Omega_1, \Omega_2),$$

where

$$\Omega_1 = \sqrt{\frac{h_2 + h_3}{h_1}}, \quad \Omega_2 = \sqrt{\frac{h_1(\frac{1}{h_2} + \frac{1}{h_3})}{h_2h_3}} = \sqrt{\frac{h_1(h_2 + h_3)}{h_2h_3}}.$$

But by the inequality of the arithmetic and geometric mean

$$\Omega_1 \Omega_2 = \frac{h_2 + h_3}{\sqrt{h_2h_3}} \geq 2,$$

so that certainly $\Omega_H(A) \geq \sqrt{2}$ for all diagonal matrices $H > 0$. 

The second statement in theorem 3 can be made more precise by the following

**Theorem 4.** In order that for some matrix $H > 0$

$$Q_H(A) = |\lambda_A|^{\max}$$

it is necessary and sufficient that the elementary divisors corresponding to the eigenvalues of $A$ with maximal modulus are simple. Similarly, if $A$ is a non-singular matrix, we have

$$\omega_H(A) = |\lambda_A|^{\min} \quad (|\lambda_A|^{\min} > 0)$$

for some $H > 0$, if and only if all elementary divisors associated with the eigenvalues of $A$ of minimal modulus are simple.

**Proof.** Necessity: let $\lambda$ be an eigenvalue of $A$ of either maximal or minimal modulus having multiple elementary divisors. It then suffices to show that, given a matrix $H > 0$, there always exists a vector $x$ for which

$$\|Ax\|_H < |\lambda_A|^{\min}, \text{ if } |\lambda| = |\lambda_A|^{\min}$$

$$\|x\|_H > |\lambda_A|^{\max}, \text{ if } |\lambda| = |\lambda_A|^{\max}.$$  \hspace{1cm} (49)

From Jordan's canonical form of $A$ it is easily seen that under our hypothesis on $\lambda$ there exist two linearly independent vectors $u_1$, $u_2$ such that $Au_1 = \lambda u_1$, $Au_2 = \lambda u_2 + u_1$. Put $v_1 = u_1$, $v_2 = \alpha u_1 + u_2$; in order to make $v_1$, $v_2$ orthogonal with respect to $H$, using the notation (4) we must have

$$(v_2, v_1) = (\alpha u_1 + u_2, u_1) = \alpha(u_1, u_1) + (u_2, u_1) = 0, \quad \alpha = -(u_2, u_1)/(u_1, u_1).$$

Clearly $Av_1 = \lambda v_1$, $Av_2 = \alpha \lambda u_1 + \lambda u_2 + u_1 = \lambda v_2 + v_1$, and so the vectors $w_1 = v_1/\|v_1\|_H$, $w_2 = v_2/\|v_2\|_H$ satisfy

$$Aw_1 = \lambda w_1$$

$$Aw_2 = \lambda w_2 + \beta w_1$$

(\|w_1\|_H = \|w_2\|_H = 1, \ (w_2, w_1) = 0), \hspace{1cm} (50)$$

where $\beta = \|w_1\|_H/\|w_2\|_H > 0$. We now take

$$x = \gamma w_1 + w_2$$

(51) and determine the scalar $\gamma$ in such a way that (49) holds. In fact, by (50)

$$Ax = \gamma \lambda w_1 + \lambda w_2 + \beta w_1 = \lambda x + \beta w_1,$$

$$\|Ax\|^2_H = x^*A^*HAx = (\bar{\lambda}x^* + \beta w_1^*)(\bar{\lambda}Hx + \beta Hw_1) =$$

$$= |\lambda|^2 \|x\|^2_H + 2Re(\beta \bar{\lambda}w_1^*Hx) + \beta^2.$$  \hspace{1cm} (52)

Substituting the expression (51) for $x$ in $w_1^*Hx$ we obtain

$$\frac{\|Ax\|^2_H}{\|x\|^2_H} = \left|\lambda\right|^2 + \frac{\beta}{\|x\|^2_H} \left\{2Re(\gamma \lambda) + \beta \right\} \quad (\beta > 0).$$
Now (49) certainly holds, if in the case $|\lambda| = |\lambda_\alpha|^\text{min}$ we choose $\gamma$ such that $Re(\gamma \lambda) < -\frac{\beta}{2}$ and $\gamma = 0$ if $|\lambda| = |\lambda_\alpha|^\text{max}.$

**Sufficiency:** suppose that all eigenvalues with maximal modulus have simple elementary divisors. Let $A$ be transformed to Jordan’s canonical form $S^{-1} A S = J = \text{Diag}(J_1, J_2)$, where $J_1$ is a diagonal matrix containing the eigenvalues $\lambda$ with $|\lambda| = |\lambda_\alpha|^\text{max}$. By (19) we have $\Omega_H(A) = \Omega_K(J)$ ($K = S^*HS$). To show that for a suitable matrix $K > 0$: $\Omega_K(J) = |\lambda_\alpha|^\text{max}$, take $K = \text{Diag}(K_1, K_2)$, where $K_1, K_2 > 0$ are matrices of the same order as $J_1$, $J_2$ respectively and $K_1$ is a unity matrix. Since $\Omega_{K_1}(J_1) = |\lambda_\alpha|^\text{max}$, from (20) we get

$$\Omega_K(J) = \text{Max} \{ |\lambda_\alpha|^\text{max}, \Omega_{K_1}(J_2) \}.$$

On the other hand $|\lambda_{J_1}|^\text{max} < |\lambda_\alpha|^\text{max}$, whence, by theorem 3, $K_2$ can be chosen such that $\Omega_{K_2}(J_2) < |\lambda_\alpha|^\text{max}$.

A similar argument shows that (48) holds for some $H > 0$, if all eigenvalues of $A$ with minimal modulus are simple.

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(P. 91, line 10: instead of $|y|^2$ read $||y||^2$

p. 91, line 15: instead of $R[\alpha\beta(x, y)]$ read $R[\alpha\beta(x, y)]$.)

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