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Bounds of matrices with regard to an hermitian metric

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Bounds of matrices with regard to an Hermitian metric [1])

by

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§ 1. The bounds $\Omega_{H,K}$, $\omega_{H,K}$.

1. *Introduction.* In various questions concerning the solutions of systems of equations and the errors made by rounding off, the following definition of upper and lower bounds $\Omega(A)$, $\omega(A)$ of a matrix A has frequently been used:

$$\Omega(A) = \operatorname*{Max}_{\nu} \sqrt{\varrho_\nu}, \quad \omega(A) = \operatorname*{Min}_{\nu} \sqrt{\varrho_\nu}, \tag{1}$$

where ϱ_ν denote the eigenvalues of $\bar{A}'A$ (cf. e.g. [8] p. 1042 ff., or [9], p. 787, for the special case in which φ, χ are Euclidean lengths). In this paper we will discuss a generalization of this definition introducing as "parameters" two positive definite Hermitian matrices H, K. If H, K vary independently, the generalized bounds $\Omega_{H,K}(A)$, $\omega_{H,K}(A)$ can in general take values in the whole range $(0, \infty)$ (cf. § 1, section 3(vi)); to obtain appropriate values one has to couple H, K in some way. This can be done very naturally when A is an $n \times n$ matrix, by taking $K = H$. The bounds $\Omega_{H,H} \equiv \Omega_H$, $\omega_{H,H} \equiv \omega_H$ are in fact often more favourable for A than (1), but at the same time their actual calculation is considerably more difficult, as is shown by the examples given in § 2. If, however, A contains only a few non-vanishing elements, $\Omega_{H,K}(A)$ can fairly well be estimated from above by means of our theorem 2 in § 3, section 1, which generalizes a theorem due to W. Ledermann [7]. We will also make use of the theorem 2 in § 3, section 2, to determine both $\operatorname*{Inf}_{H} \Omega_H(A)$ and $\operatorname*{Sup}_{H} \omega_H(A)$, where H runs through all positive definite matrices. In sections 2 and 3 of § 1 we give the exact definitions and a few elementary properties of $\Omega_{H,K}(A)$ and $\omega_{H,K}(A)$, while in section 4 of § 1 a property not quite so trivial is proved.

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The idea of relating lengths of vectors to a positive definite Hermitian matrix H has recently been applied to the solution of linear equations $Ax = b$ by M. R. Hestenes and M. L. Stein [6]. Their main problem is to minimize the "H-length" of the residual vectors $r(x) = b - Ax$. Our definition of $\Omega_{H,K}(A)$, $\omega_{H,K}(A)$ involves a similar extremum problem, but (in contrast to [6]) with a *side condition.*

In defining the H-length of a vector we make use of the "scalar product" (x, y) with regard to H of two vectors x, y, as given e.g. in H. L. Hamburger and M. E. Grimshaw [4], p. 153. Such products (x, y) have also recently been used by W. Givens [2] to obtain theorems on the fields of values of a square matrix, which considerably extend the well known results due to O. Toeplitz [12] and F. Hausdorff [5].

I am very much indebted to Prof. Dr. A. Ostrowski for having most kindly allowed me to see through the manuscript of the yet unpublished book [10] from which I received many suggestions. In particular a chapter of [10] on the bounds (1) was the starting point of our investigations, which rather closely follow the disposition of this chapter.

2. *Notations and definitions.* Let $A = (a_{\mu\nu})$ $(\mu = 1, \ldots, m;$ $\nu = 1, \ldots, n)$ be an $m \times n$ matrix with real or complex elements $a_{\mu\nu}$. By $A^* = \bar{A}'$ we denote its conjugate-transpose and by $A^{(p)}$ $(p = 1, \ldots, \mathrm{Min}\,(m, n))$ its p^{th} compound matrix, i.e. the $\binom{m}{p} \times \binom{n}{p}$ matrix consisting of all minors of A of order p. The groups of p rows and columns which form the minors are supposed to be arranged in lexicographical order. We have to use the following rules concerning $A^{(p)}$:

$$(AB)^{(p)} = A^{(p)}B^{(p)}, \ (A^*)^{(p)} = (A^{(p)})^*, \tag{2}$$

if the product AB exists (cf. e.g. [1], p. 90ff). The first relation in (2) (the so called *Binet-Cauchy* theorem) is readily extended to more than two factors. Further if $m = n$ and A^{-1} exists, from (2) (with $B = A^{-1}$) it follows that

$$(A^{-1})^{(p)} = (A^{(p)})^{-1}. \tag{3}$$

tr A will denote the trace $\sum\limits_{\nu} a_{\nu\nu}$ of a square matrix A, λ_A an eigenvalue of A and $|\lambda_A|^{\max}$, $|\lambda_A|^{\min}$ respectively the maximal, minimal modulus of the eigenvalues of A.

By x, y etc. we denote *column*-vectors of a k-dimensional complex Euclidean space, by x^* the conjugate-transposed row-vector

$\bar{x}'$ and by $|x|$ the Euclidean length of x. In order to introduce an Hermitian metric we define the scalar product (x, y) of two vectors x, y by

$$(x, y) \equiv y^* H x \quad (H > 0), \tag{4}$$

where H is an Hermitian matrix of order k; the meaning of the relation $H > 0$ is that H is positive definite. In particular (x, x) is real and $\geqq 0$ with $(x, x) = 0$ only when $x = 0$. We therefore define

$$\| x \| \equiv \sqrt{(x, x)} \tag{5}$$

as the *norm of x with regard to H*. Sometimes we add the subscript H and write $\| x \|_H$ instead of $\| x \|$. By routine arguments (cf. e.g. [11], p. 5, [3], p. 90—92, or [4], p. 4—5) the following three properties of $\| x \|$ are obtained:

$$\left.\begin{array}{l} \| x \| \geqq 0 \text{ with equality if and only if } x = 0 \\ \| \gamma x \| = |\gamma| \; \| x \| \; (\gamma \text{ any complex scalar}) \\ \| x + y \| \leqq \| x \| + \| y \|. \end{array}\right\} \tag{6}$$

Now let A be an $m \times n$ matrix and $H > 0, K > 0$ be Hermitian matrices of orders m, n respectively; we then define the upper and lower bounds $\Omega_{H,K}(A)$, $\omega_{H,K}(A)$ of A by

$$\left.\begin{array}{l} \Omega_{H,K}(A) = \underset{\| x \|_K = 1}{\operatorname{Max}} \| Ax \|_H = \left(\underset{\| x \|_K = 1}{\operatorname{Max}} x^* A^* H A x \right)^{\frac{1}{2}}, \\ \omega_{H,K}(A) = \underset{\| x \|_K = 1}{\operatorname{Min}} \| Ax \|_H = \left(\underset{\| x \|_K = 1}{\operatorname{Min}} x^* A^* H A x \right)^{\frac{1}{2}}. \end{array}\right\} \tag{7}$$

If in particular $m = n$ and $H = K$ we write $\Omega_{H,H} \equiv \Omega_H$, $\omega_{H,H} \equiv \omega_H$. The definition (7) can also (partly) be expressed in terms of Euclidean lengths: Let K be transformed to a diagonal matrix by the unitary matrix U:

$$D = U^* K U = \operatorname{Diag}(k_1, \ldots, k_n), \; U^* U = I_n, \tag{8}$$

where I_n is the $n \times n$ unity matrix. Since $k_\nu > 0$ $(\nu = 1, \ldots, n)$ D can further be reduced to I_n by multiplying on the right and left by $\Delta = \operatorname{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right)$:

$$\Delta U^* K U \Delta = I_n, \; \Delta = \operatorname{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right). \tag{9}$$

If we now apply to x the substitution $x = U\Delta y$ we get

$$\| A x \|_H^2 = x^* A^* H A x = y^* \Delta U^* A^* H A U \Delta y \; (x = U \Delta y)$$

and by (9)

$$x^*Kx = y^*\Delta U^*KU\Delta y = y^*y.$$

Hence $\| x \|_K = 1$ implies $| y | = 1$ and viceversa; we therefore have

$$\underset{\| x \|_K = 1}{\mathfrak{F}} \| Ax \|_H^2 = \underset{| y | = 1}{\mathfrak{F}} y^*By, \quad B = \Delta U^*A^*HAU\Delta, \tag{10}$$

where $\underset{\| x \|_K=1}{\mathfrak{F}}$, $\underset{| y |=1}{\mathfrak{F}}$ denote the fields of values over the sets of vectors x, y with $\| x \|_K = 1$, $| y | = 1$ respectively. Since B is non-negative definite, from (10) we see that both Max, Min in (7) actually exist and

$$\Omega^2_{H,K}(A) = \lambda_B^{\max}, \ \omega^2_{H,K}(A) = \lambda_B^{\min}, \ B = \Delta U^*A^*HAU\Delta. \tag{11}$$

If in particular we take $H = I_m$, $K = I_n$, so that clearly $U = \Delta = I_n$, we obtain the bounds defined in (1).

Throughout this paper we denote respectively by $h_1, \ldots, h_m > 0$, $k_1, \ldots, k_n > 0$ the eigenvalues (not necessarily distinct and arranged in any order) of H, K and we put $h' = \underset{\mu=1,\ldots,m}{\text{Max}} h_\mu$, $h'' = \underset{\mu=1,\ldots,m}{\text{Min}} h_\mu$; $k' = \underset{\nu=1,\ldots,n}{\text{Max}} k_\nu$, $k'' = \underset{\nu=1,\ldots,n}{\text{Min}} k_\nu$.

3. *Elementary properties of $\Omega_{H,K}$, $\omega_{H,K}$*. If not otherwise stated in this section A, $H > 0$, $K > 0$ are respectively $m \times n$, $m \times m$, $n \times n$ matrices.

(i) The following properties of $\Omega_{H,K}$, $\omega_{H,K}$ are immediate consequences of (6) and (7):

$$\begin{aligned} &\Omega_{H,K}(\gamma A) = |\gamma| \Omega_{H,K}(A), \omega_{H,K}(\gamma A) = |\gamma| \omega_{H,K}(A) \ (\gamma \text{ any complex scalar}) \\ &\Omega_{H,K}(A+B) \leqq \Omega_{H,K}(A) + \Omega_{H,K}(B), \ \omega_{H,K}(A+B) \geqq \omega_{H,K}(A) - \Omega_{H,K}(B), \end{aligned} \tag{12}$$

$$\begin{aligned} &\Omega_{H,K}(A) = 0 \text{ if and only if } A = 0 \\ &\omega_{H,K}(A) = 0 \text{ if and only if the rank of } A \text{ is } < n. \end{aligned} \tag{13}$$

(ii) Obviously we can also write

$$\Omega_{H,K}(A) = \underset{x \neq 0}{\text{Max}} \frac{\| Ax \|_H}{\| x \|_K}, \ \omega_{H,K}(A) = \underset{x \neq 0}{\text{Min}} \frac{\| Ax \|_H}{\| x \|_K}, \tag{14}$$

so that for any $m \times n$ matrix C:

$\| Cx \|_H \leqq \Omega_{H,K}(C) \| x \|_K$, $\| Cx \|_H \geqq \omega_{H,K}(C) \| x \|_K$. Hence, if A, B, $L > 0$ respectively are $m \times l$, $l \times n$, $l \times l$ matrices, we have

$$\Omega_{H,K}(AB) \leqq \Omega_{H,L}(A) \Omega_{L,K}(B), \omega_{H,K}(AB) \geqq \omega_{H,L}(A) \omega_{L,K}(B). \tag{15}$$

On the other hand, for any vector x with $Bx \neq 0$

$$\frac{\| ABx \|_H}{\| x \|_K} = \frac{\| A(Bx) \|_H}{\| Bx \|_L} \frac{\| Bx \|_L}{\| x \|_K} \; (Bx \neq 0). \tag{16}$$

Suppose now that for the vector x: $\dfrac{\| Bx \|_L}{\| x \|_K} = \Omega_{L,K}(B)$. Then by (16) and (14)

$$\Omega_{H,K}(AB) \geqq \frac{\| ABx \|_H}{\| x \|_K} = \frac{\| A(Bx) \|_H}{\| Bx \|_L} \Omega_{L,K}(B) \geqq \omega_{H,L}(A) \Omega_{L,K}(B).$$

Similarly, if B is of rank n, from (16) we deduce $\omega_{H,K}(AB) \leqq \Omega_{H,L}(A)\omega_{L,K}(B)$. If B is of rank $< n$, then the same holds for AB (cf. e.g. [1], p. 96—97) and therefore $\omega_{H,K}(AB) = \omega_{L,K}(B) = 0$. Thus we can extend (15) as follows:

$$\Omega_{H,K}(AB) \geqq \omega_{H,L}(A)\Omega_{L,K}(B), \omega_{H,K}(AB) \leqq \Omega_{H,L}(A)\omega_{L,K}(B). \tag{17}$$

(iii) Suppose that $m = n$ and A^{-1} exists; then putting $x = A^{-1}y$ we see that

$$\underset{x\neq 0}{\mathfrak{F}} \frac{\| Ax \|_H}{\| x \|_K} = \underset{y\neq 0}{\mathfrak{F}} \frac{\| y \|_H}{\| A^{-1}y \|_K} = \underset{y\neq 0}{\mathfrak{F}} \left(\frac{\| A^{-1}y \|_K}{\| y \|_H}\right)^{-1} \; (x = A^{-1}y).$$

Hence in using (14) we get

$$\Omega_{H,K}(A) = \frac{1}{\omega_{K,H}(A^{-1})}, \quad \omega_{H,K}(A) = \frac{1}{\Omega_{K,H}(A^{-1})}. \tag{18}$$

(iv) *Let S, T be two nonsingular matrices of orders m, n respectively; then we have*

$$\Omega_{H,K}(A) = \Omega_{S^*HS,\, T^*KT}(S^{-1}AT), \omega_{H,K}(A) = \omega_{S^*HS,\, T^*KT}(S^{-1}AT). \tag{19}$$

If in particular $m = n$, $\Omega_{H,K}$, $\omega_{H,K}$ do not change, if a unitary transformation S is applied both to A, H and K.

Indeed, putting $x = Ty$ we see that the field of values x^*A^*HAx over the set of vectors x with $x^*Kx = 1$ coincides with the field of values

$$y^*T^*A^*HATy = y^*(T^*A^*(S^*)^{-1})(S^*HS)(S^{-1}AT)y$$

taken for all vectors y with $y^*T^*KTy = 1$. Hence (19) follows at once from the definition (7).

(v) *Suppose that both A, H and K are respectively the "direct sums" of $A_1, \ldots, A_s$, $H_1, \ldots, H_s$ and $K_1, \ldots, K_s$, i.e. that in an obvious notation*

$$A = \mathrm{Diag}(A_1, \ldots, A_s),\; H = \mathrm{Diag}(H_1, \ldots, H_s),\; K = \mathrm{Diag}(K_1, \ldots, K_s),$$

where A_σ *is an* $m_\sigma \times n_\sigma$, $H_\sigma > 0$ *an* $m_\sigma \times m_\sigma$ *and* $K_\sigma > 0$ *an* $n_\sigma \times n_\sigma$ *matrix* $(\sigma = 1, \ldots, s)$. *Then we have*

$$\Omega_{H,K}(A) = \operatorname*{Max}_{\sigma=1,\ldots,s} \Omega_{H_\sigma, K_\sigma}(A_\sigma), \omega_{H,K}(A) = \operatorname*{Min}_{\sigma=1,\ldots,s} \omega_{H_\sigma, K_\sigma}(A_\sigma). \quad (20)$$

In fact, let K_σ be transformed to a diagonal matrix by the unitary matrix U_σ $(\sigma = 1, \ldots, s)$ and put $U = \text{Diag}(U_1, \ldots, U_s)$, so that clearly (8) holds. Put $\Delta = \text{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right) = \text{Diag}(\Delta_1, \ldots, \Delta_s)$, Δ_σ being of the same order as K_σ $(\sigma = 1, \ldots, s)$. Then obviously the matrix B in (10) is the direct sum of $\Delta_\sigma U_\sigma^* A_\sigma^* H_\sigma A_\sigma U_\sigma \Delta_\sigma$ $(\sigma = 1, \ldots, s)$, whence (20) follows from (11).

(vi) *For every* $H > 0$, $K > 0$ *we have*

$$\sqrt{\frac{h''}{k'}}\, \omega(A) \leqq \omega_{H,K}(A) \leqq \Omega_{H,K}(A) \leqq \sqrt{\frac{h'}{k''}}\, \Omega(A), \qquad (21)$$

where $\omega(A)$, $\Omega(A)$ *are the bounds defined in* (1).

Indeed, (21) follows from (14) by putting $y = Ax$ in

$$h'' \mid y \mid^2 \leqq \| y \|_H^2 \leqq h' \mid y \mid^2, \; k'' \mid x \mid^2 \leqq \| x \|_K^2 \leqq k' \mid x \mid^2.$$

(vii) *We have for any eigenvalue* λ_A *of a square matrix* A:

$$\omega_H(A) \leqq \mid \lambda_A \mid \leqq \Omega_H(A). \qquad (22)$$

In fact, let x be an eigenvector corresponding to λ_A with $\| x \|_H = 1$. Then $Ax = \lambda_A x$, $\| Ax \|_H = \mid \lambda_A \mid$, whence (22) follows directly from (7).

4. For the proof of our first theorem we need the following

Lemma 1. *Let* S *be an* $n \times m$ *matrix and* T *an* $m \times n$ *matrix. Then, if* $m < n$, *we have*

$$(ST)^{(p)} = 0 \quad (p > m). \qquad (23)$$

Proof. Put $S_0 = (SO_1)$, $T_0 = \begin{pmatrix} T \\ O_2 \end{pmatrix}$, where O_1, O_2 are $n \times (n-m)$, $(n-m) \times n$ zero-matrices respectively. Obviously both S_0 and T_0 are $n \times n$ matrices and $ST = S_0 T_0$. Hence by (2) $(ST)^{(p)} = S_0^{(p)} T_0^{(p)}$, and (23) follows from $S_0^{(p)} = T_0^{(p)} = 0$ $(p > m)$.

Lemma 2. *Suppose that* $D = \text{Diag}(k_1, \ldots, k_n)$ $(k_\nu > 0)$, $\Delta = \text{Diag}\left(\frac{1}{\sqrt{k_1}}, \ldots, \frac{1}{\sqrt{k_n}}\right)$ *and* $G = \text{Diag}(h_1, \ldots, h_m)$ $(h_\mu > 0)$, $\Gamma = \text{Diag}\left(\frac{1}{\sqrt{h_1}}, \ldots, \frac{1}{\sqrt{h_m}}\right)$. *Further let* $R = (r_{\mu\nu})$ *be an* $m \times n$

matrix and put $B = \Gamma R D R^* \Gamma$, $C = \Delta^{-1} R^* G^{-1} R \Delta^{-1}$. *Then, if* $\varphi(\lambda) = |\lambda I_m - B|$, $\psi(\lambda) = |\lambda I_n - C|$ *are the characteristic polynomials of* B, C, *we have*

$$\psi(\lambda) = \lambda^{n-m}\varphi(\lambda).$$

Proof. Without loss of generality we may assume $m \leqq n$. Put $B = (b_{\mu\nu})$ $(\mu, \nu = 1, \ldots, m)$, $C = (c_{\mu\nu})$ $(\mu, \nu = 1, \ldots, n)$; by direct multiplication we get

$$b_{\mu\nu} = \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}} \sum_{\sigma=1}^{n} k_\sigma r_{\mu\sigma} \bar{r}_{\nu\sigma}$$

$$c_{\mu\nu} = \sqrt{k_\mu} \sqrt{k_\nu} \sum_{\tau=1}^{m} \frac{1}{h_\tau} \bar{r}_{\tau\mu} r_{\tau\nu}.$$

Hence

$$\operatorname{tr} B = \sum_{\mu=1}^{m} b_{\mu\mu} = \sum_{\mu=1}^{m} \sum_{\sigma=1}^{n} \frac{k_\sigma}{h_\mu} |r_{\mu\sigma}|^2 = \sum_{\nu=1}^{n} \sum_{\tau=1}^{m} \frac{k_\nu}{h_\tau} |r_{\tau\nu}|^2 = \sum_{\nu=1}^{n} c_{\nu\nu} = \operatorname{tr} C.$$

We now form the p^{th} compound matrices $B^{(p)}$, $C^{(p)}$ of B, C; from (2), (3) it follows that

$$\begin{aligned} B^{(p)} &= \Gamma^{(p)} R^{(p)} D^{(p)} (R^{(p)})^* \Gamma^{(p)} \\ C^{(p)} &= (\Delta^{(p)})^{-1} (R^{(p)})^* (G^{(p)})^{-1} R^{(p)} (\Delta^{(p)})^{-1} \end{aligned} \quad (p = 1, \ldots, m).$$

Evidently $B^{(p)}$, $C^{(p)}$ are built analogously to B, C. Therefore our first conclusion again is applicable and we get

$$\operatorname{tr} B^{(p)} = \operatorname{tr} C^{(p)} \quad (p = 1, \ldots, m). \tag{24}$$

If $m < n$ by the lemma 1 with $S = \Delta^{-1} R^*$, $T = G^{-1} R \Delta^{-1}$ we have

$$C^{(p)} = 0 \quad (p > m). \tag{25}$$

Since generally $(-1)^p \operatorname{tr} A^{(p)}$ is the coefficient of λ^{n-p} in the characteristic polynomial $|\lambda I_n - A|$ of an $n \times n$ matrix A, (cf. e.g. [1], p. 88), our assertion now follows immediately from (24) and (25).

Theorem 1. *Let* A *be an* $m \times n$ *matrix and* $H > 0$, $K > 0$ *be respectively of orders* m, n; *then we have*

$$\Omega_{K,H}(A^*) = \Omega_{H^{-1},K^{-1}}(A), \tag{26}$$

and, if $m = n$,

$$\omega_{K,H}(A^*) = \omega_{H^{-1},K^{-1}}(A). \tag{27}$$

Proof. Let

$$G = V^* H V = \operatorname{Diag}(h_1, \ldots, h_m), \; V^* V = I_m, \tag{28}$$

$$\Gamma V^* H V \Gamma = I_m, \; \Gamma = \operatorname{Diag}\left(\frac{1}{\sqrt{h_1}}, \ldots, \frac{1}{\sqrt{h_m}}\right) \tag{29}$$

be the equations corresponding to (8), (9), applied to the matrix H. Then in using (8), (28) we have

$$D = U^*KU,\ D^{-1} = U^*K^{-1}U;\ G = V^*HV,\ G^{-1} = V^*H^{-1}V, \quad (30)$$
$$K = UDU^*,\ K^{-1} = UD^{-1}U^*;\ H = VGV^*,\ H^{-1} = VG^{-1}V^*. \quad (31)$$

According to (11) and (28)—(30) we have to examine the eigenvalues of

$$B = \Gamma V^*AKA^*V\Gamma,\quad C = \Delta^{-1}U^*A^*H^{-1}AU\Delta^{-1}.$$

By means of (31) we can write

$$B = \Gamma(V^*AU)D(U^*A^*V)\Gamma = \Gamma RDR^*\Gamma$$
$$C = \Delta^{-1}(U^*A^*V)G^{-1}(V^*AU)\Delta^{-1} = \Delta^{-1}R^*G^{-1}R\Delta^{-1},$$

putting $R = V^*AU$. If we now apply the lemma 2, our assertion follows at once.

COROLLARY 1. *For any square matrix A and $H > 0$ we have*

$$\Omega_H(A^*) = \Omega_{H^{-1}}(A),\ \omega_H(A^*) = \omega_{H^{-1}}(A). \quad (32)$$

COROLLARY 2. *If A is an Hermitian matrix, then for any $H > 0$*

$$\Omega_H(A) = \Omega_{H^{-1}}(A),\ \omega_H(A) = \omega_{H^{-1}}(A). \quad (33)$$

§ 2. Examples.

For the sake of simplicity in this section we only consider square $n \times n$ matrices A and we take $H = K$. As to the selection of examples we follow very closely the arrangement given by A. Ostrowski in [10].

(i) Let $A = (a_{\mu\nu})$ be a matrix the only non-vanishing element of which is $a_{ik} \equiv a$. Put $H = (h_{\mu\nu})$, $B = (b_{\mu\nu})$, $U = (u_{\mu\nu})$, where U satisfies (8) and B is the matrix defined in (10). By direct multiplication we get

$$b_{\mu\nu} = |\, a\,|^2\, h_{ii}\, \bar{u}_{k\mu}\, u_{k\nu}\, \frac{1}{\sqrt{h_\mu}}\, \frac{1}{\sqrt{h_\nu}}.$$

If by v we denote the row-vector $\left(\frac{u_{k1}}{\sqrt{h_1}}, \ldots, \frac{u_{kn}}{\sqrt{h_n}}\right)$, B can be considered as the product $|\, a\,|^2\, h_{ii}\, v^*\, v$ and is therefore of rank 1. Hence by (11) we have

$$\Omega_H^2(A) = \overset{\max}{\lambda_B} = \operatorname{tr} B = |\, a\,|^2\, h_{ii} \sum_{\nu=1}^{n} \frac{|\, u_{k\nu}\,|^2}{h_\nu}.$$

On the other hand, by (31), $h_{ii} = \sum_{\nu=1}^{n} h_\nu \mid u_{i\nu} \mid^2$ and so

$$\Omega_H^2(A) = \mid a \mid^2 \sum_{\nu=1}^{n} h_\nu \mid u_{i\nu} \mid^2 \sum_{\nu=1}^{n} \frac{\mid u_{k\nu} \mid^2}{h_\nu}. \tag{34}$$

If in particular H is a diagonal matrix, and therefore $U = I_n$, we get

$$\Omega_H(A) = \mid a \mid \sqrt{\frac{h_i}{h_k}},\ H = \operatorname{Diag}(h_1, \ldots, h_n). \tag{35}$$

Let us in this example discuss, to what extent $\Omega_H(A)$ is determined by the eigenvalues of H. Clearly all Hermitian matrices having the *fixed* eigenvalues $h_1, \ldots, h_n > 0$ are obtained by letting U in $H = UDU^*$, $D = \operatorname{Diag}(h_1, \ldots, h_n)$, run through *all unitary* $n \times n$ matrices. In the case $i \neq k$, from (34) we can derive the following bounds, between which $\Omega_H^2(A)$ varies:

$$\frac{h''}{h'} \leqq \frac{1}{\mid a \mid^2} \Omega_H^2(A) \leqq \frac{h'}{h''},$$

where the upper and lower bounds are attained by taking in (34) for $(u_{i1}, \ldots, u_{in})$, $(u_{k1}, \ldots, u_{kn})$ suitable unit vectors. Similarly, if $i = k$, from (34) we see that $\frac{1}{\mid a \mid^2} \Omega_H^2(A)$ takes values in a certain closed interval, the left-hand end point of which by (22) is equal to 1.

If on the other hand we let H run through all diagonal matrices, (35) shows that in the case $i \neq k$ the range of $\Omega_H(A)$ is the whole interval $(0, \infty)$, while $\Omega_H(A)$ for $i = k$ is always equal to $\mid a \mid$.

Evidently in this example $\omega_H(A) = 0$ by (13).

(ii) Let $A = (a_{\mu\nu})$ be a matrix all elements of which are zero except those lying in the i^{th} row, and put $(a_{i1}, \ldots, a_{in}) = (a_1, \ldots, a_n) = \alpha$. We suppose that $H = \operatorname{Diag}(h_1, \ldots, h_n)$, i.e. $U = I_n$. Then for the matrix B in (10) we have $B = \Delta A^* H A \Delta$, $b_{\mu\nu} = h_i \bar{a}_\mu a_\nu \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}}$ and as in our example (i) the rank of B is equal to 1, so that

$$\Omega_H^2(A) = \operatorname{tr} B = h_i \sum_{\nu=1}^{n} \frac{\mid a_\nu \mid^2}{h_\nu} = \mid a_i \mid^2 + h_i \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n} \frac{\mid a_\nu \mid^2}{h_\nu}. \tag{36}$$

Clearly we have always $\omega_H(A) = 0$, and, by a suitable choice of H, $\Omega_H(A)$ can take values arbitrarily near to $\mid a_i \mid = \mid \lambda_A \mid^{\max}$.

(iii) If all elements of the matrix A are equal to $a \neq 0$ and if we take $H = \text{Diag}\,(h_1, \ldots, h_n)$, we have $b_{\mu\nu} = \frac{1}{\sqrt{h_\mu}} \frac{1}{\sqrt{h_\nu}} \mid a \mid^2 \sum_{\varkappa=1}^{n} h_\varkappa$ and therefore by the Cauchy-Schwarz inequality

$$\Omega_H^2(A) = \text{tr}\, B = \mid a \mid^2 \left(\sum_{\nu=1}^{n} h_\nu\right)\left(\sum_{\nu=1}^{n} \frac{1}{h_\nu}\right) \geqq \mid a \mid^2 n^2. \tag{37}$$

The lower bound for $\Omega_H(A)$, $\mid a \mid n = \mid \lambda_A \mid^{\max}$, is attained for $H = I_n$, while $\Omega_H(A)$ is not bounded at all from above. On the other hand $\omega_H(A) = 0$.

(iv) Let $A = \text{Diag}\,(a_1, \ldots, a_n)$, $H = \text{Diag}\,(h_1, \ldots, h_n)$. Then $B = \text{Diag}\,(\mid a_1 \mid^2, \ldots, \mid a_n \mid^2)$, so that

$$\Omega_H(A) = \operatorname*{Max}_{\nu} \mid a_\nu \mid = \mid \lambda_A \mid^{\max}, \quad \omega_H(A) = \operatorname*{Min}_{\nu} \mid a_\nu \mid = \mid \lambda_A \mid^{\min}. \tag{38}$$

(v) Let $A = (a_{\mu\nu})$ be a matrix all elements of which are zero except those lying in the i^{th} row and k^{th} column, while we have also $a_{ik} = 0$. We further assume H to be a diagonal matrix. In applying to both A and H the same permutation to the rows and columns, whereby in virtue of § 1, section 3(iv), $\Omega_H(A)$, $\omega_H(A)$ are not changed, we can make $k = 1$. Having carried through this transformation we denote by $\alpha = (a_2, \ldots, a_n)$ the i^{th} $(n - 1)$-dimensional row-vector of A (without its first element), by $\beta = (b_1, \ldots, b_n)$ the first (n-dimensional) column-vector of A (where $b_i = 0$) and we put $H = \text{Diag}\,(h_1, \ldots, h_n)$, $\Delta_1 = \text{Diag}\left(\frac{1}{\sqrt{h_2}}, \ldots, \frac{1}{\sqrt{h_n}}\right)$. For the matrix B of (10) we then obtain by direct multiplication (observing that $U = I_n$)

$$B = \begin{pmatrix} \frac{1}{h_1} \beta^* H \beta & 0 \\ 0 & h_i \Delta_1 \alpha^* \alpha \Delta_1 \end{pmatrix}.$$

Since again the $(n - 1) \times (n - 1)$ matrix in the lower right-hand corner of B is of rank 1, it follows from (11) that

$$\Omega_H(A) = \text{Max}\,(\Omega_1, \Omega_2), \text{ where } \begin{cases} \Omega_1 = \left(\frac{1}{h_1} \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n} h_\nu \mid b_\nu^2 \mid\right)^{\frac{1}{2}} \\ \Omega_2 = \left(h_i \sum_{\nu=2}^{n} \frac{1}{h_\nu} \mid a_\nu \mid^2\right)^{\frac{1}{2}} \end{cases}. \tag{39}$$

$$\omega_H(A) = 0 \quad (n > 2).$$

§ 3. A generalization of Ledermann's theorem and the determination of $\operatorname*{Inf}_{H>0} \Omega_H$, $\operatorname*{Sup}_{H>0} \omega_H$.

1. The reason we succeeded to calculate directly Ω_H, ω_H in the examples given in § 2 was that the matrices A contained a sufficiently large number of zeros. We now prove a general theorem which in similar cases always yields an upper bound for $\Omega_{H,K}(A)$ and which is a generalization of a theorem due to W. Ledermann [7]. More precisely:

THEOREM 2. *Let $A = (a_{\mu\nu})$ be an $m \times n$ matrix and denote by α_μ its μ^{th} row-vector; then, if $H = \mathrm{Diag}\,(h_1, \ldots, h_m)$ $(h_\mu > 0)$, $K = \mathrm{Diag}\,(k_1, \ldots, k_n)$ $(k_\nu > 0)$ and if every column-vector of A contains at most s non-vanishing elements, we have*

$$\Omega^2_{H,K}(A) \leqq \sum_{\sigma=1}^{s} h_{\mu_\sigma} \,||\, \alpha_{\mu_\sigma} \,||^2_{K^{-1}}, \tag{40}$$

where the sum on the right-hand side has to be taken over the s largest numbers $h_{\mu_\sigma} \,||\, \alpha_{\mu_\sigma} \,||^2_{K^{-1}}$ $(\sigma = 1, \ldots, s)$ among $h_\mu \,||\, \alpha_\mu \,||^2_{K^{-1}}$ $(\mu = 1, \ldots, m)$.

PROOF. Our proof is essentially the same as that given for the case $H = I_m$, $K = I_n$ by A. Ostrowski in [10].

Without loss of generality we may assume that

$$h_1 \,||\, \alpha_1 \,||^2_{K^{-1}} \geqq h_2 \,||\, \alpha_2 \,||^2_{K^{-1}} \geqq \cdots \geqq h_m \,||\, \alpha_m \,||^2_{K^{-1}}. \tag{41}$$

Indeed, let a permutation P be applied to the rows of A; if we further permute the rows *and the columns* of H according to P, by (19) (with $T = I_n$) $\Omega_{H,K}(A)$ does not change and the numbers $h_\mu \,||\, \alpha_\mu \,||^2_{K^{-1}}$ $(\mu = 1, \ldots, m)$ are arranged as required.

Let $\operatorname*{Max}_{||x||_{K^{-1}}} ||\, Ax \,||^2_H$ be attained for the vector $x = (x_1, \ldots, x_n)$ and put $y = Ax$, $y = (y_1, \ldots, y_m)$. For every μ $(\mu = 1, \ldots, m)$ replace the coordinates x_ν of x for which $a_{\mu\nu} = 0$ by zeros and denote the vector so obtained by $x^{(\mu)}$. Then we have

$$\Omega^2_{H,K}(A) = ||\, y \,||^2_H = \sum_{\mu=1}^{m} h_\mu \,|\, y_\mu \,|^2 = \sum_{\mu=1}^{m} h_\mu \,|\, \alpha_\mu \, x^{(\mu)} \,|^2. \tag{42}$$

We further put $x^{(\mu)} = (x^{(\mu)}_1, \ldots, x^{(\mu)}_n)$ $(\mu = 1, \ldots, m)$. Since by the Cauchy-Schwarz inequality

$$|\, \alpha_\mu x^{(\mu)} \,|^2 = \left| \sum_{\nu=1}^{n} a_{\mu\nu} x^{(\mu)}_\nu \right|^2 \leqq \left(\sum_{\nu=1}^{n} \frac{1}{\sqrt{k_\nu}} \,|\, a_{\mu\nu} \,|\, \sqrt{k_\nu} \,|\, x^{(\mu)}_\nu \,| \right)^2 \leqq$$

$$\leqq \left(\sum_{\nu=1}^{n} \frac{1}{k_\nu} \,|\, a_{\mu\nu} \,|^2 \right) \left(\sum_{\nu=1}^{n} k_\nu \,|\, x^{(\mu)}_\nu \,|^2 \right) = ||\, \alpha_\mu \,||^2_{K^{-1}} \sum_{\nu=1}^{n} k_\nu \,|\, x^{(\mu)}_\nu \,|^2,$$

from (42) we get

$$\Omega^2_{H,K}(A) \leqq \sum_{\mu=1}^{m} h_\mu \| \alpha_\mu \|^2_{K^{-1}} \sum_{\nu=1}^{n} k_\nu | x_\nu^{(\mu)} |^2 =$$

$$= \sum_{\nu=1}^{n} k_\nu \left[\sum_{\mu=1}^{m} | x_\nu^{(\mu)} |^2 h_\mu \| \alpha_\mu \|^2_{K^{-1}} \right]. \quad (43)$$

If $x_\nu^{(\mu)} \neq 0$ then

$$x_\nu^{(\mu)} = x_\nu \ (x_\nu^{(\mu)} \neq 0) \quad (44)$$

and $a_{\mu\nu} \neq 0$. From this and the hypothesis it follows that for any fixed ν at most s of the $x_\nu^{(\mu)}$ are $\neq 0$. Therefore taking the sum in brackets on the right-hand side of (43) only over the terms with $x_\nu^{(\mu)} \neq 0$ and using (44), (41) we see that

$$\sum_{\mu=1}^{m} | x_\nu^{(\mu)} |^2 h_\mu \| \alpha_\mu \|^2_{K^{-1}} \leqq | x_\nu |^2 \sum_{\sigma=1}^{s} h_\sigma \| \alpha_\sigma \|^2_{K^{-1}} \ (\nu = 1, \ldots, n),$$

whence by (43)

$$\Omega^2_{H,K}(A) \leqq \left(\sum_{\nu=1}^{n} k_\nu | x_\nu |^2 \right) \left(\sum_{\sigma=1}^{s} h_\sigma \| \alpha_\sigma \|^2_{K^{-1}} \right).$$

This proves our assertion, since $\sum_{\nu=1}^{n} k_\nu | x_\nu |^2 = \| x \|^2_K = 1$.

REMARKS. The theorem of Ledermann is obtained by taking $H = I_m$, $K = I_n$. If in particular we apply (40) with $H = K$ to our examples (i), (ii) and (iv) we obtain respectively as upper bounds $\frac{h_i}{h_k} | a |^2$, $h_i \| \alpha \|^2_{H^{-1}}$, $\operatorname{Max}_\nu | a_\nu |^2$, which all coincide with the corresponding Ω^2_H.

Even if $s = m$ the theorem 2 is often useful. Take e.g.

$$A = \begin{pmatrix} 2 & 1 & 8 \\ 7 & 0 & 5 \\ 0 & 1 & 3 \end{pmatrix},$$

where the elements of the second column are comparatively small. In order to get favourable bounds for $\Omega_H(A)$ in applying (40), we choose h_2 relatively small. With $H = \text{Diag}(4, 1, 20)$ we obtain $\Omega^2_H(A) \lneqq 63{,}3$, while $H = I$ gives $\Omega^2(A) \leqq 153$.

2. We now use our theorem 2 to give a refinement of (22):

THEOREM 3. *For any $n \times n$ matrix A we have*

$$\operatorname*{Inf}_{H>0} \Omega_H(A) = | \lambda_A |^{\max}, \ \operatorname*{Sup}_{H>0} \omega_H(A) = | \lambda_A |^{\min}. \quad (45)$$

If in particular A has only simple elementary divisors both Inf and Sup in (45) are attained for suitable matrices $H > 0$.

PROOF. Since for a nonsingular matrix $|\lambda_{A^{-1}}|^{\max} = 1/|\lambda_A|^{\min}$ and by (18) $\omega_H(A) = \dfrac{1}{\Omega_H(A^{-1})}$, it is sufficient to prove the relation concerning Inf Ω_H. Let A be transformed to Jordan's canonical form by the nonsingular matrix S:

$$S^{-1}AS = \Lambda + C,$$

where Λ is a diagonal matrix the elements of which are the eigenvalues of A, and C denotes a matrix consisting of zeros except possibly some elements $c_{\mu\nu} = 1$ with $\nu = \mu + 1$. If in (19) we take $T = S$ we have by (22), (12)

$$|\lambda_A|^{\max} \leqq \Omega_H(A) = \Omega_K(\Lambda + C) \leqq \Omega_K(\Lambda) + \Omega_K(C), \quad (46)$$

where $K = S^*HS$. It suffices to show that for a suitable choice of K the sum on the right-hand side of (46) is arbitrarily near to $|\lambda_A|^{\max}$. Take $K = \text{Diag}(k_1, \ldots, k_n)$; then by (38) $\Omega_K(\Lambda) = |\lambda_A|^{\max}$; if on the other hand γ_ν is the ν^{th} row-vector of C, then $\|\gamma_\nu\|^2_{K^{-1}} = \begin{cases} 0 (\gamma_\nu = 0) \\ 1/k_{\nu+1} (\gamma_\nu \neq 0) \end{cases}$. Hence by the theorem 2

$$\Omega^2_K(C) \leqq \operatorname*{Max}_{\nu=1,\ldots,n-1} \frac{k_\nu}{k_{\nu+1}},$$

which obviously can be made as small as we please.

If all elementary divisors of A are simple we have in (46) $C = 0$, $\Omega_K(C) = 0$, so that $\Omega_H(A)$ attains the value $|\lambda_A|^{\max}$ for a suitable matrix $H > 0$.

It is natural to ask whether we could in (45) take Inf, Sup only over the set of all *diagonal* matrices $H > 0$. This is however not true as the following example shows: Take

$$A = \begin{pmatrix} 0 & i & 1 \\ i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad (i^2 = 1),$$

where $|\lambda_A|^{\max} = 0$. If $H = \text{Diag}(h_1, h_2, h_3)$, it follows from § 2, Ex. (v), that

$$\Omega_H(A) = \text{Max}(\Omega_1, \Omega_2), \text{ where } \begin{cases} \Omega_1 = \sqrt{\dfrac{h_2 + h_3}{h_1}} \\ \Omega_2 = \sqrt{h_1\left(\dfrac{1}{h_2} + \dfrac{1}{h_3}\right)} = \sqrt{\dfrac{h_1(h_2 + h_3)}{h_2 h_3}}. \end{cases}$$

But by the inequality of the arithmetic and geometric mean

$$\Omega_1 \Omega_2 = \frac{h_2 + h_3}{\sqrt{h_2 h_3}} \geqq 2,$$

so that certainly $\Omega_H(A) \geqq \sqrt{2}$ for all diagonal matrices $H > 0$.

The second statement in theorem 3 can be made more precise by the following

THEOREM 4. *In order that for some matrix* $H > 0$

$$\Omega_H(A) = |\lambda_A|^{\max} \tag{47}$$

it is necessary and sufficient that the elementary divisors corresponding to the eigenvalues of A *with maximal modulus are simple. Similarly, if* A *is a non-singular matrix, we have*

$$\omega_H(A) = |\lambda_A|^{\min} \quad (|\lambda_A|^{\min} > 0) \tag{48}$$

for some $H > 0$, *if and only if all elementary divisors associated with the eigenvalues of* A *of minimal modulus are simple.*

PROOF. *Necessity*: let λ be an eigenvalue of A of either maximal or minimal modulus having multiple elementary divisors. It then suffices to show that, given a matrix $H > 0$, there always exists a vector x for which

$$\frac{\| Ax \|_H}{\| x \|_H} \begin{cases} < |\lambda_A|^{\min}, \text{ if } |\lambda| = |\lambda_A|^{\min} \\ > |\lambda_A|^{\max}, \text{ if } |\lambda| = |\lambda_A|^{\max}. \end{cases} \tag{49}$$

From Jordan's canonical form of A it is easily seen that under our hypothesis on λ there exist two linearly independent vectors u_1, u_2 such that $Au_1 = \lambda u_1$, $Au_2 = \lambda u_2 + u_1$. Put $v_1 = u_1$, $v_2 = \alpha u_1 + u_2$; in order to make v_1, v_2 orthogonal with respect to H, using the notation (4) we must have

$$(v_2, v_1) = (\alpha u_1 + u_2, u_1) = \alpha(u_1, u_1) + (u_2, u_1) = 0,$$
$$\alpha = -(u_2, u_1)/(u_1, u_1).$$

Clearly $Av_1 = \lambda v_1$, $Av_2 = \alpha\lambda u_1 + \lambda u_2 + u_1 = \lambda v_2 + v_1$, and so the vectors $w_1 = v_1/\| v_1 \|_H$, $w_2 = v_2/\| v_2 \|_H$ satisfy

$$\begin{aligned} Aw_1 &= \lambda w_1 \\ Aw_2 &= \lambda w_2 + \beta w_1 \end{aligned} \quad (\| w_1 \|_H = \| w_2 \|_H = 1,\ (w_2, w_1) = 0), \tag{50}$$

where $\beta = \| v_1 \|_H / \| v_2 \|_H > 0$. We now take

$$x = \gamma w_1 + w_2 \tag{51}$$

and determine the scalar γ in such a way that (49) holds. In fact, by (50)

$$Ax = \gamma\lambda w_1 + \lambda w_2 + \beta w_1 = \lambda x + \beta w_1,$$
$$\| Ax \|_H^2 = x^* A^* H A x = (\bar\lambda x^* + \beta w_1^*)(\lambda H x + \beta H w_1) =$$
$$= |\lambda|^2 \| x \|_H^2 + 2\Re(\beta\lambda w_1^* H x) + \beta^2.$$

Substituting the expression (51) for x in $w_1^* H x$ we obtain

$$\frac{\| Ax \|_H^2}{\| x \|_H^2} = |\lambda|^2 + \frac{\beta}{\| x \|_H^2} \left\{ 2\Re(\gamma\lambda) + \beta \right\} \qquad (\beta > 0).$$

Now (49) certainly holds, if in the case $|\lambda| = |\lambda_A|^{\min}$ we choose γ such that $\Re(\gamma\lambda) < \frac{-\beta}{2}$ and $\gamma = 0$ if $|\lambda| = |\lambda_A|^{\max}$.

Sufficiency: suppose that all eigenvalues with maximal modulus have simple elementary divisors. Let A be transformed to Jordan's canonical form $S^{-1} A\, S = J = \mathrm{Diag}(J_1, J_2)$, where J_1 is a *diagonal* matrix containing the eigenvalues λ with $|\lambda| = |\lambda_A|^{\max}$. By (19) we have $\Omega_H(A) = \Omega_K(J)$ $(K = S^*HS)$. To show that for a suitable matrix $K > 0$: $\Omega_K(J) = |\lambda_A|^{\max}$, take $K = \mathrm{Diag}$ (K_1, K_2), where $K_1, K_2 > 0$ are matrices of the same order as J_1, J_2 respectively and K_1 is a unity matrix. Since $\Omega_{K_1}(J_1) = |\lambda_A|^{\max}$, from (20) we get

$$\Omega_K(J) = \mathrm{Max}\,\{|\lambda_A|^{\max},\ \Omega_{K_2}(J_2)\}.$$

On the other hand $|\lambda_{J_2}|^{\max} < |\lambda_A|^{\max}$, whence, by theorem 3, K_2 can be chosen such that $\Omega_{K_2}(J_2) < |\lambda_A|^{\max}$.

A similar argument shows that (48) holds for some $H > 0$, if all eigenvalues of A with minimal modulus are simple.

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(P. 91, line 10: instead of $|y|^2$ read $\|y\|^2$
p. 91, line 15: instead of $R[\alpha\beta(x, y)]$ read $R[\alpha\bar{\beta}(x, y)]$.)

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