COMPOSITIO MATHEMATICA

J. DE GROOT
T. DEKKER

Free subgroups of the orthogonal group

Compositio Mathematica, tome 12 (1954-1956), p. 134-136
<http://www.numdam.org/item?id=CM_1954-1956__12__134_0>

Free subgroups of the orthogonal group

by

J. de Groot and T. Dekker

1. Let G^n be the group of all proper orthogonal transformations in Euclidean space E^n (therefore represented by real orthogonal n-matrices (a_{ik}) with determinant $+1$). We shall prove in this note — using the axiom of choice —, that for $n > 2$ G^n contains a free (non Abelian) subgroup with just as many free generators as the potency of G^n itself (which is the potency $\aleph$ of real numbers). The theorem is clear, if we can prove it for G^3. *Hausdorff* [1] showed how to find two rotations φ and ψ in G^3 which are independent except for the relations $\varphi^2 = \psi^3 = 1$. *Robinson* [2] showed that $\varphi\psi\varphi\psi$ and $\varphi\psi^2\varphi\psi^2$ generate a free group of rank two. Since any free group of rank two contains a subgroup of rank $\aleph_0$ (comp. Kurosch [3] f.i.), it is already clear that G^3 contains a free subgroup G_0 with an infinite, but countable number of free generators.

These results are used essentially to prove certain theorems concerning congruence relations for subsets of a sphere (comp. f.i. Hausdorff [1], Robinson [2], Dekker and de Groot [4]).

The rotationgroup G^2, being commutative, obviously does not contain a free non Abelian subgroup. Moreover *the group of all congruent mappings of E^2 on itself does not contain a free non Abelian subgroup.* Indeed, suppose the congruent mappings α and β generate a free subgroup. Then α^2 and β^2 are rotations or translations. It follows that γ and δ defined by

$$\gamma = \alpha^2\beta^2\alpha^{-2}\beta^{-2}$$
$$\delta = \alpha^4\beta^2\alpha^{-4}\beta^{-2}$$

are translations, which yields to $\gamma\delta = \delta\gamma$. Hence there exists a non-trivial relation between α and β, q.e.d.

2. Lemma. *Let $F = \{f_\alpha\}$ be a family of potency $\overline{\overline{F}} < \aleph$ of functions $f_\alpha(x_1, x_2, \ldots, x_n) \not\equiv 0$ each analytic* (in terms of power-series) *in its n real variables x_i. Then there are real values a_i $(i = 1, 2, \ldots, n)$, such that $f_\alpha(a_i) \neq 0$ for any $f_\alpha \in F$.*

PROOF. For $n = 1$ the lemma is trivial. Consider $f_\alpha(x_1, \ldots, x_n)$ for a fixed α and for $0 \leqq x_i \leqq 1$. There is only a finite number of values $x_1 = b$ such that for a fixed $b: f_\alpha\ (b, x_2, \ldots, x_n) \equiv 0$ (otherwise the analytic function of one variable $f_\alpha(x_1, c_2, c_3, \ldots, c_n)$ should vanish identically for fixed but arbitrary $x_i = c_i\ (1 < i \leqq n)$. From this follows $f_\alpha(x_i) \equiv 0$). For each α we leave out this finite number of values x_1. Because $\overline{\overline{F}} < \aleph$ there remains a number $x_1 = a_1$ such that for each α: $f_\alpha(a_1, x_2, \ldots, x_n) \not\equiv 0$.

This is for any α a function of $n - 1$ variables, satisfying the conditions of the lemma. Hence we find by induction: there are real values $a_i (i = 2, \ldots, n)$ such that $f_\alpha(a_1, a_2, \ldots, a_n) \neq 0$ for any $f_\alpha \in F$ q.e.d.

3. THEOREM. *The group G^n of all rotations of n-dimensional Euclidean space ($n > 2$) for which the origin is a fixed point contains a free (non Abelian) subgroup with $\aleph$ free generators.*

PROOF. We have to prove the theorem for G^3. Let G_0 be defined as in 1., G_0 being a free subgroup of G^3 with rank $\aleph_0$. We shall prove by transfinite induction the existence of a free subgroup of rank $\aleph$.

Suppose that for a certain limitnumber $\alpha \leqq \omega_\aleph$ (the initialnumber of $\aleph$) the groups G_β, $\beta < \alpha$ are defined, where G_β is a free rotationgroup with $\aleph_0 + \overline{\overline{\beta}}$ free generators such that

$$G_0 \subset G_1 \subset \ldots \subset G_\omega \subset \ldots \subset G_\beta \subset \ldots\ (\beta < \alpha).$$

Moreover we assume that for any $\beta < \alpha$, the $\aleph_0 + \overline{\overline{\beta}} + 1$ free generators by which $G_{\beta+1}$ is defined consist of the $\aleph_0 + \overline{\overline{\beta}}$ free generators of G_β (by which G_β is defined) to which one new generator is added.

Now it is clear, that for a limitnumber α the sum $\bigcup_{\beta<\alpha} G_\beta = G_\alpha$ is a *free* group. Indeed the generators are the union of the already defined generators of G_β, $\beta < \alpha$; a relation (between a finite number of generators) in G_α is already a relation in a certain G_β and therefore a trivial one. The theorem is therefore proved, if — given a certain G_β — we may define a rotation χ such that the $\aleph_0 + \overline{\overline{\beta}}$ free generators of G_β together with χ are free generators of a group $G_{\beta+1}$.

A non-trivial relation in $G_{\beta+1}$ may be written (after simplifications) in the form

$$g_1 \chi^{j_1} g_2 \chi^{j_2} \ldots g_r \chi^{j_r} = 1 \quad (j_l \text{ integer}, \ g_l \in G_\beta). \tag{1}$$

We must find a rotation χ for which no relation (1) is true.

Consider a fixed relation (1). The g_l may be represented by matrices with known elements:

$$(2) \qquad g_l = (g^l_{ik}).$$

The unknown χ can be expressed like any rotation under consideration as a product of three matrices:

$$(3)\chi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi_3 & -\sin\xi_3 \\ 0 & \sin\xi_3 & \cos\xi_3 \end{pmatrix} \begin{pmatrix} \cos\xi_2 & 0 & -\sin\xi_2 \\ 0 & 1 & 0 \\ \sin\xi_2 & 0 & \cos\xi_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\xi_1 & -\sin\xi_1 \\ 0 & \sin\xi_1 & \cos\xi_1 \end{pmatrix}$$

(ξ_1, ξ_2, ξ_3 are the so called angles of Euler). Using the substitutions (2) and (3) we get relation (1) in matrixform. This leads to a finite number of equations in the real variables ξ_1, ξ_2, ξ_3.

We show first that at least one of these equations does not vanish identically (for all values of ξ_1, ξ_2, ξ_3). Indeed the g_l ($l = 1, 2, \ldots, r$) of (1) may be expressed uniquely in a finite number of free generators of G_β. If we substitute in (1) for χ a free generator of G_β not occurring in one of these expressions g_l, the relation (1) is certainly not fulfilled (since G_β is a free group). At least one of the mentioned equations is therefore untrue for well chosen numbers ξ_1, ξ_2, ξ_3. We call this equation in ξ_1, ξ_2, ξ_3 an equation connected with (1). The total number of relations (1) with variables $g_i \in G_\beta$ and j_i has clearly a potency less than $\aleph$. The number of connected equations $f_\alpha(\xi_1, \xi_2, \xi_3) = 0$ has therefore a cardinal less than $\aleph$. From (3) it follows that the f_α are analytic in the real variables ξ_1, ξ_2, ξ_3. Therefore we can apply the preceding lemma. This gives real values a_1, a_2, a_3 with $f_\alpha(a_1, a_2, a_3) \neq 0$ for any α. The corresponding χ (substituting $a_i = \xi_i$ in (3)) therefore does not satisfy any relation of the form (1), which we had to prove.

REFERENCES

Hausdorff, F.

[1] Grundzüge der Mengenlehre (1914), p. 469—472.

Robinson, R. M.

[2] Fund. Math. 34 (1947), p. 246—260.

Kurosch, A. G.

[3] Gruppentheorie, Berlin (1954), p. 274.

Dekker, T. and de Groot, J.

[4] Proc. Int. Congr. Math. (1954), II, p. 209.

Institute of Mathematics
University of Amsterdam.

(Oblatum 18-9-54)