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On the projective geometry of K-spreads

by

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1. An N-dimensional space is called by J. Douglas [1] a space of K-spreads if there is given a system of K-dimensional manifolds such that there exists one and only one member of the system passing through any $K+1$ points given in general position. He shows that such a system of K-dimensional manifolds, that is, a system of K-spreads is represented by a system of completely integrable partial differential equations of the form

$$(1.1)\qquad \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} + H^i_{\beta\gamma}(x, p) = 0, \quad \left(p^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}\right)$$

where (x^i) $(i, j, k, \ldots = 1, 2, \ldots, N)$ are coordinates of a point of the space, and (u^α) $(\alpha, \beta, \gamma, \ldots = \dot{1}, \dot{2}, \ldots, \dot{K})$ parameters of a point on a K-spread. The functions $H^i_{\beta\gamma}$ $(= H^i_{\gamma\beta})$ form a homogeneous function system of p^i_α with respect to the lower indices, and consequently satisfy the relations

$$(1.2)\qquad p^a_\lambda(H^i_{\beta\gamma}|^\alpha_a) = \delta^\alpha_\beta H^i_{\lambda\gamma} + \delta^\alpha_\gamma H^i_{\beta\lambda},$$

where $|^\alpha_a$ denotes partial differentiation with respect to p^a_α.

The components of affine connection and those of projective connection are given respectively by

$$(1.3)\qquad \Gamma^i_{jk} = \frac{1}{K(K+1)} H^i_{\beta\gamma}|^\beta_j|^\gamma_k$$

and

$$(1.4)\qquad {}^*\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{N+1}(\delta^i_j \Gamma^a_{ak} + \delta^i_k \Gamma^a_{aj}) -$$
$$- \frac{1}{N-K} p^i_\lambda \{\Gamma^a_{jk}|^\lambda_a - \frac{1}{N+1}(\Gamma^a_{aj}|^\lambda_k + \Gamma^a_{ak}|^\lambda_j)\}.$$

J. Douglas states, among other things, that the quantities defined by

$$(1.5)\qquad W^i_{jkl} = {}^*B^i_{jkl} - \frac{1}{N-1}({}^*B^a_{jka}\delta^i_l - {}^*B^a_{jla}\delta^i_k)$$

are components of a projective tensor, where

$$(1.6) \quad {}^*B^i_{jkl} = ({}^*\Pi^i_{jk,l} - {}^*\Pi^i_{jk}\big|^{\alpha}_{a}\, {}^*\Pi^a_{bl}\, p^b_\alpha) - \\ - ({}^*\Pi^i_{jl,k} - {}^*\Pi^i_{jl}\big|^{\alpha}_{a}\, {}^*\Pi^a_{bk}\, p^b_\alpha) + {}^*\Pi^a_{jk}\, {}^*\Pi^i_{al} - {}^*\Pi^a_{jl}\, {}^*\Pi^i_{ak},$$

the comma followed by an index denoting partial differentiation with respect to x^l. But, it is pointed out by R. S. Clark [2] that the above defined W^i_{jkl} are not components of a tensor in the general case $1 < K < N$.

J. Douglas states also that necessary and sufficient conditions that equations (1.1) can be reduced to $\partial^2 x^i/\partial u^\beta \partial u^\gamma = 0$ by suitable coordinate and parameter transformations, in other words, those that the space be projectively flat are

$$(1.7) \qquad {}^*\Pi^i_{jk}\big|^{\alpha}_{l} = 0 \quad \text{and} \quad W^i_{jkl} = 0.$$

But, it is pointed out recently by Chih-Ta Yen [3] and Hsien-Chung Wang [5] that the conditions (1.7) are not independent and in fact the latter is a consequence of the former.

To study the projective geometry of K-spreads, it will be certainly helpful to construct a space of K linear elements with projective connection whose K-dimensional flat subspaces coincide with the K-spreads. This was actually done by S. S. Chern [6] and Chih-Ta Yen [3], [4].

But, all the authors mentioned above use the so-called natural frame of reference. If we use the natural frame of reference, Douglas' ${}^*\Pi^i_{jk}$ appear in the coefficients of the connection, but because of the complexity of its law of transformation, it is very difficult to deduce any tensor from it. It seems to the present authors that it is rather desirable to use the so-called semi-natural frame of reference instead of natural one. The purpose of the present Note is to show that if we take up the semi-natural frame of reference, we can find out curvature tensors and their laws of transformation with respect to a projective change of Γ^i_{jk} in a very natural way and to discuss the above topics by the use of these results.

2. We consider a space with projective connection whose elements are points (x^i) and K linearly independent contravariant vectors (p^i_α) and whose projective connection is expressed by

$$(2.1) \qquad dA_0 = \omega^0_0 A_0 + \omega^i_0 A_i, \qquad dA_j = \omega^0_j A_0 + \omega^i_j A_i,$$

where A_0 is a point coinciding with (x^i), A_j N points taken in the tangent projective space in such a way that $[A_0, A_j]$ forms a frame of reference and ω's are Pfaffian forms with respect to x^i and p^i_α. If $dx^i = 0$, then the point A_0 having the same position, we must have $\omega^i_0 = 0$, from which we can see that ω^i_0 do not contain the differentials dp^i_α. Hence, by a suitable change of frames of reference, we can put (2.1) in the form

$$(2.2)\quad \begin{cases} dA_0 = (\varphi_i dx^i + \varphi^\alpha_i dp^i_\alpha) A_0 + dx^i A_i, \\ dA_j = (\omega^0_{jk} dx^k + \omega^{0\alpha}_{jk} dp^k_\alpha) A_0 + (\omega^i_{jk} dx^k + \omega^{i\alpha}_{jk} dp^k_\alpha) A_i, \end{cases}$$

where φ_i, ω^0_{jk}, ω^i_{jk}, φ^α_i, $\omega^{0\alpha}_{jk}$, $\omega^{i\alpha}_{jk}$ are homogeneous function systems of p^i_α with respect to Greek index and satisfy

$$\varphi^\alpha_i p^i_\beta = 0, \quad \omega^{0\alpha}_{jk} p^k_\beta = 0, \quad \omega^{i\alpha}_{jk} p^k_\beta = 0.$$

Since the projective connection is completely determined by ω^0_j and $\omega^i_j - \delta^i_j \omega^0_0$, we put

$$(2.3)\quad \omega^0_j = \Pi^0_{jk} dx^k + \Pi^{0\alpha}_{jk} dp^k_\alpha, \quad \omega^i_j - \delta^i_j \omega^0_0 = \Pi^i_{jk} dx^k + \Pi^{i\alpha}_{jk} dp^k_\alpha,$$

and call Π's coefficients of the projective connection. The Π's have the same properties as ω's, and satisfy

$$(2.4)\qquad \Pi^{0\alpha}_{jk} p^k_\beta = 0, \qquad \Pi^{i\alpha}_{jk} p^k_\beta = 0.$$

If we calculate $\underset{21}{dd}A_0 - \underset{12}{dd}A_0 = \Omega^0_0 A_0 + \Omega^i_0 A_i$, then the torsion of the space is given by Ω^i_0. In what follows, we assume that our space has no torsion, from which we have

$$(2.5)\qquad \Pi^i_{jk} = \Pi^i_{kj}, \qquad \Pi^{i\alpha}_{jk} = 0.$$

To express the projective connection analytically, we have adopted the so-called semi-natural frame of reference. It will be easily seen that the transformation between two semi-natural frames of reference must be of the form

$$(2.6)\qquad \bar{A}_0 = A_0, \quad \bar{A}_j = \lambda_j A_0 + A_j,$$

where λ_j are generalized homogeneous functions of degree zero of the set p^i_α. We call such a transformation of semi-natural frame of reference a transformation of hyperplane at infinity. During a transformation of hyperplane at infinity, the coefficients of the projective connection will be transformed into

$$(2.7)\quad \begin{cases} \overline{\Pi}^0_{jk} = \Pi^0_{jk} + \lambda_{j.k} - \lambda_i \Pi^i_{jk} - \lambda_j \lambda_k, \quad \overline{\Pi}^{0\alpha}_{jk} = \Pi^{0\alpha}_{jk} + \lambda_j \big|^\alpha_k, \\ \overline{\Pi}^i_{jk} = \Pi^i_{jk} + \delta^i_j \lambda_k + \delta^i_k \lambda_j. \end{cases}$$

If we change the coordinate system from (x^i) to $(\bar{x}^i)$, the semi-natural frame of reference will be transformed into

$$(2.8) \qquad \bar{A}_o = A_o, \qquad \bar{A}_i = \frac{\partial x^a}{\partial \bar{x}^i} A_a,$$

the hyperplane at infinity being assumed to be invariant. During this transformation, the coefficients of the projective connection will be transformed into

$$(2.9) \begin{cases} \overline{\Pi}^o_{jk} = \dfrac{\partial x^b}{\partial \bar{x}^j}\dfrac{\partial x^c}{\partial \bar{x}^k}\Pi^o_{bc} + \dfrac{\partial x^b}{\partial \bar{x}^j}\dfrac{\partial^2 x^c}{\partial \bar{x}^k \partial \bar{x}^l}\dfrac{\partial \bar{x}^l}{\partial x^a} p^a_\alpha \Pi^{o\alpha}_{bc}, \quad \overline{\Pi}^{o\alpha}_{jk} = \dfrac{\partial x^b}{\partial \bar{x}^j}\dfrac{\partial x^c}{\partial \bar{x}^k}\Pi^{o\alpha}_{bc}, \\ \overline{\Pi}^i_{jk} = \dfrac{\partial \bar{x}^i}{\partial x^a}\left(\dfrac{\partial x^b}{\partial \bar{x}^j}\dfrac{\partial x^c}{\partial \bar{x}^k}\Pi^a_{bc} + \dfrac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k}\right). \end{cases}$$

Thus we can see that $\Pi^{o\alpha}_{jk}$ are components of a tensor and Π^i_{jk} are those of an affine connection under the transformation of coordinates.

3. Now let us consider a K-dimensional subspace defined by $x^i = x^i(u)$, $p^i_\alpha = \partial x^i/\partial u^\alpha$, and put

$$(3.1) \quad A_{\dot{o}} = A_o, \quad A_\alpha = p^o_\alpha A_o + p^i_\alpha A_i, \quad A_B = p^o_B A_o + p^i_B A_i,$$

where the matrix (p^i_α, p^i_A) $(A, B, C, \ldots = K+1, \ldots, N)$ is of rank N, p^o_α and p^o_B being arbitrary, then the points A_α are on the K-dimensional plane tangent to the subspace and the points $A_{\dot{o}}$, A_α, A_B are linearly independent. Equation (3.1) can be solved with respect to A_o and A_i and gives the equations of the form

$$(3.2) \qquad A_o = A_{\dot{o}}, \quad A_i = p^{\dot{o}}_i A_{\dot{o}} + p^\alpha_i A_\alpha + p^B_i A_B,$$

the matrix (p^i_α, p^i_B) and (p^α_i, p^B_i) being inverse to each other.

Now, the fact that the subspace $x^i = x^i(u)$ is flat is expressed by the equations

$$(3.3) \qquad dA_{\dot{o}} = \omega^{\dot{o}}_{\dot{o}} A_{\dot{o}} + du^\alpha A_\alpha, \quad dA_\beta = \omega^{\dot{o}}_\beta A_{\dot{o}} + \omega^\alpha_\beta A_\alpha,$$

from which we have

$$(3.4) \qquad \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} + \Pi^i_{jk} p^j_\beta p^k_\gamma = p^i_\alpha G^\alpha_{\beta\gamma},$$

as equations of the flat subspace, where

$$(3.5) \qquad G^\alpha_{\beta\gamma} = p^\alpha_i \left(\frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} + \Pi^i_{jk} p^j_\beta p^k_\gamma\right).$$

It will be easily seen that the form of the equations (3.4) is invariant under the change of hyperplane at infinity, change of coordinates and change of parameters and is independent of the choice of p^i_B.

Now, we shall consider the problem to determine a projective connection with respect to which the system of K-spreads (1.1) or

$$\frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} + \Gamma^i_{jk} p^j_\beta p^k_\gamma = 0 \tag{3.6}$$

will be exactly the system of K-dimensional flat subspaces. It was shown by J. Douglas that the so-called projective change is given by

$$\overline{H}^i_{\beta\gamma} = H^i_{\beta\gamma} + p^i_\alpha C^\alpha_{\beta\gamma} \tag{3.7}$$

or

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \lambda_k + \delta^i_k \lambda_j + p^i_\alpha B^\alpha_{jk}, \tag{3.8}$$

where

$$\lambda_k = \frac{1}{K(K+1)} C^\alpha_{\alpha\gamma}\Big|^\gamma_k, \quad B^\alpha_{jk} = \frac{1}{K(K+1)} C^\alpha_{\beta\gamma}\Big|^\beta_j\Big|^\gamma_k,$$

and, $C^\alpha_{\beta\gamma}$ being a homogeneous function system, these quantities satisfy the relations

$$p^k_\alpha (\lambda_k|^\alpha_l) = 0, \qquad p^j_\alpha B^\alpha_{jk} = 0. \tag{3.9}$$

For an ordinary space with projective connection and an ordinary space of paths, the change of hyperplane at infinity and the projective change of affine connection correspond to each other [7]. While, in our case, the transformation law of Π^i_{jk} and that of Γ^i_{jk} are not the same. But as we can obtain, from (3.8),

$$\frac{1}{N-K}\Big[\overline{\Gamma}^a_{jk}\Big|^\alpha_a - \frac{1}{N+1}\big(\overline{\Gamma}^a_{aj}\Big|^\alpha_k + \overline{\Gamma}^a_{ak}\Big|^\alpha_j\big)\Big] =$$

$$\frac{1}{N-K}\Big[\Gamma^a_{jk}\Big|^\alpha_a - \frac{1}{N+1}\big(\Gamma^a_{aj}\Big|^\alpha_k + \Gamma^a_{ak}\Big|^\alpha_j\big)\Big] + B^\alpha_{jk},$$

we can see that the transformation law of Π^i_{jk} coincides with that of

$$\Gamma^i_{jk} - \frac{1}{N-K} p^i_\alpha \Big[\Gamma^a_{jk}\Big|^\alpha_a - \frac{1}{N+1}\big(\Gamma^a_{aj}\Big|^\alpha_k + \Gamma^a_{ak}\Big|^\alpha_j\big)\Big].$$

Consequently we put

$$\Pi^i_{jk} = \Gamma^i_{jk} - \frac{1}{N-K} p^i_\alpha \Big[\Gamma^a_{jk}\Big|^\alpha_a - \frac{1}{N+1}\big(\Gamma^a_{aj}\Big|^\alpha_k + \Gamma^a_{ak}\Big|^\alpha_j\big)\Big]. \tag{3.10}$$

To determine the other coefficients Π^o_{jk} and $\Pi^{o\alpha}_{jk}$, we must calculate the curvature tensors.

4. If we calculate $\underset{21}{dd}A_j - \underset{12}{dd}A_j = \Omega^o_j A_o + \Omega^i_j A_i$, the curvatures will be given by

$$(4.1)\ \Omega^o_j = P^o_{jkl}\underset{1}{dx^k}\underset{2}{dx^l} + P^o_{jkl}{}^{\alpha}(\underset{1}{dx^k}\underset{2}{\delta p^l_\alpha} - \underset{2}{dx^k}\underset{1}{\delta p^l_\alpha}) + P^o_{jkl}{}^{\alpha\beta}\underset{1}{\delta p^k_\alpha}\underset{2}{\delta p^l_\beta},$$

$$(4.2)\ \Omega^i_j - \delta^i_j\Omega^o_o = P^i_{jkl}\underset{1}{dx^k}\underset{2}{dx^l} + P^i_{jkl}{}^{\alpha}(\underset{1}{dx^k}\underset{2}{\delta p^l_\alpha} - \underset{2}{dx^k}\underset{1}{\delta p^l_\alpha}),$$

where δp^i_α defined by

$$\delta p^i_\alpha = dp^i_\alpha + \Pi^i_{jk}p^j_\alpha dx^k$$

are contravariant vectors and

$$\begin{aligned}(4.3)\ P^o_{jkl} = {} & (\Pi^o_{jk,\,l} - \Pi^o_{jk}\big|^{\alpha}_{a}\Pi^a_{bl}p^b_\alpha) - (\Pi^o_{jl,\,k} - \Pi^o_{jl}\big|^{\alpha}_{a}\Pi^a_{bk}p^b_\alpha) \\ & + \Pi^a_{jk}(\Pi^o_{al} - \Pi^{o\alpha}_{ab}\Pi^b_{cl}p^c_\alpha) - \Pi^a_{jl}(\Pi^o_{ak} - \Pi^{o\alpha}_{ab}\Pi^b_{ck}p^c_\alpha) \\ & + (\Pi^{o\alpha}_{ja,\,k} - \Pi^{o\alpha}_{ja}\big|^{\beta}_{b}\Pi^b_{ck}p^c_\beta)\Pi^a_{dl}p^d_\alpha - (\Pi^{o\alpha}_{ja,\,l} - \Pi^{o\alpha}_{ja}\big|^{\beta}_{b}\Pi^b_{cl}p^c_\beta)\Pi^a_{dk}p^d_\alpha,\end{aligned}$$

$$(4.4)\ P^o_{jkl}{}^{\alpha} = \Pi^o_{jk}\big|^{\alpha}_{l} - (\Pi^{o\alpha}_{jl,\,k} - \Pi^{o\alpha}_{jl}\big|^{\beta}_{a}\Pi^a_{bk}p^b_\beta) - \Pi^{o\beta}_{ja}\big|^{\alpha}_{l}\Pi^a_{bk}p^b_\beta + \Pi^a_{jk}\Pi^{o\alpha}_{al},$$

$$(4.5)\ P^o_{jkl}{}^{\alpha\beta} = \Pi^{o\alpha}_{jk}\big|^{\beta}_{l} - \Pi^{o\beta}_{jl}\big|^{\alpha}_{k},$$

$$\begin{aligned}(4.6)\ P^i_{jkl} = {} & (\Pi^i_{jk,\,l} - \Pi^i_{jk}\big|^{\alpha}_{a}\Pi^a_{bl}p^b_\alpha) - (\Pi^i_{jl,\,k} - \Pi^i_{jl}\big|^{\alpha}_{a}\Pi^a_{bk}p^b_\alpha) + \Pi^a_{jk}\Pi^i_{al} - \Pi^a_{jl}\Pi^i_{ak} \\ & + (\Pi^o_{jk} - \Pi^{o\alpha}_{ja}\Pi^a_{bk}p^b_\alpha)\delta^i_l - (\Pi^o_{jl} - \Pi^{o\alpha}_{ja}\Pi^a_{bl}p^b_\alpha)\delta^i_k \\ & - \delta^i_j\{(\Pi^o_{kl} - \Pi^{o\alpha}_{ka}\Pi^a_{bl}p^b_\alpha) - (\Pi^o_{lk} - \Pi^{o\alpha}_{la}\Pi^a_{bk}p^b_\alpha)\},\end{aligned}$$

$$(4.7)\ P^i_{jkl}{}^{\alpha} = \Pi^i_{jk}\big|^{\alpha}_{l} - \delta^i_j\Pi^{o\alpha}_{kl} - \delta^i_k\Pi^{o\alpha}_{jl}.$$

The P^o_{jkl} and P^i_{jkl} are generalized homogeneous functions of degree zero of the set p^i_α and $P^o_{jkl}{}^{\alpha}$, $P^o_{jkl}{}^{\alpha\beta}$ and $P^i_{jkl}{}^{\alpha}$ are homogeneous function systems with respect to the upper Greek indices and satisfy

$$(4.8)\quad P^o_{jkl}{}^{\alpha}p^l_\lambda = 0,\ P^o_{jkl}{}^{\alpha\beta}p^k_\lambda = 0,\ P^o_{jkl}{}^{\alpha\beta}p^l_\lambda = 0,\ P^i_{jkl}{}^{\alpha}p^l_\lambda = 0.$$

During a change of hyperplane at infinity, the P's are transformed respectively into $\overline{P}$'s by the following transformation laws:

$$\begin{aligned}(4.9)\ \overline{P}^o_{jkl} = {} & P^o_{jkl} - (P^o_{jkl}{}^{\alpha} - P^o_{jlk}{}^{\alpha})\lambda_\alpha + P^o_{jkl}{}^{\alpha\beta}\lambda_\alpha\lambda_\beta \\ & - \lambda_i\{P^i_{jkl} - (P^i_{jkl}{}^{\alpha} - P^i_{jlk}{}^{\alpha})\lambda_\alpha\},\end{aligned}$$

$$(4.10)\ \overline{P}^o_{jkl}{}^{\alpha} = P^o_{jkl}{}^{\alpha} - \lambda_i P^i_{jkl}{}^{\alpha} - P^o_{jkl}{}^{\beta\alpha}\lambda_\beta,$$

$$(4.11)\ \overline{P}^o_{jkl}{}^{\alpha\beta} = P^o_{jkl}{}^{\alpha\beta},$$

$$(4.12)\ \overline{P}^i_{jkl} = P^i_{jkl} - (P^i_{jkl}{}^{\alpha} - P^i_{jlk}{}^{\alpha})\lambda_\alpha,$$

$$(4.13)\qquad \overline{P}^{i}_{jkl}{}^{\alpha} = P^{i}_{jkl}{}^{\alpha},$$

where we have put $\lambda_\alpha = \lambda_i p^i_\alpha$.

During a change of coordinates, the P's, as is easily seen, behave as components of tensors.

To determine $\Pi^{o\alpha}_{jk}$, we put the condition $P^{a}_{akl}{}^{\alpha} = 0$, which is invariant one as we can see from (4.13). Thus we obtain

$$(4.14)\qquad \Pi^{o\alpha}_{kl} = \frac{1}{N+1}\Pi^{a}_{ak}\Big|^{\alpha}_{l} = \frac{1}{N+1}\Gamma^{a}_{ak}\Big|^{\alpha}_{l},$$

from which $P^{o}_{jkl}{}^{\alpha\beta} = 0$ and

$$(4.15)\quad P^{i}_{jkl}{}^{\alpha} = \Big[\Gamma^{i}_{jk} - \frac{1}{N+1}(\delta^{i}_{j}\Gamma^{a}_{ak} + \delta^{i}_{k}\Gamma^{a}_{aj}) - \\ - \frac{1}{N-K}p^{i}_{\lambda}\{\Gamma^{a}_{jk}\Big|^{\lambda}_{a} - \frac{1}{N+1}(\Gamma^{a}_{aj}\Big|^{\lambda}_{k} + \Gamma^{a}_{ak}\Big|^{\lambda}_{j})\}\Big]\Big|^{\alpha}_{l}.$$

Thus we can see that the $P^{i}_{jkl}{}^{\alpha}$ satisfy

$$(4.16)\qquad P^{a}_{akl}{}^{\alpha} = P^{a}_{jal}{}^{\alpha} = P^{a}_{jka}{}^{\alpha} = 0.$$

To determine the Π^{o}_{jk}, we put the condition $P^{a}_{jka} = 0$, which is invariant one as we can see from (4.12) and (4.16). Thus we obtain

$$(4.17)\qquad \Pi^{o}_{jk} - \Pi^{o\alpha}_{ja}\Pi^{a}_{bk}p^{b}_{\alpha} = -\frac{1}{N^2-1}(N\Pi^{a}_{jka} + \Pi^{a}_{kja})$$

and consequently

$$(4.18)\quad P^{i}_{jkl} = \Pi^{i}_{jkl} - \frac{1}{N^2-1}(N\Pi^{a}_{jka} + \Pi^{a}_{kja})\delta^{i}_{l} + \\ + \frac{1}{N^2-1}(N\Pi^{a}_{jla} + \Pi^{a}_{lja})\delta^{i}_{k} + \frac{1}{N+1}\delta^{i}_{j}(\Pi^{a}_{kla} - \Pi^{a}_{lka}),$$

where

$$(4.19)\quad \Pi^{i}_{jkl} = (\Pi^{i}_{jk,l} - \Pi^{i}_{jk}\Big|^{\alpha}_{a}\Pi^{a}_{bl}p^{b}_{\alpha}) - \\ - (\Pi^{i}_{jl,k} - \Pi^{i}_{jl}\Big|^{\alpha}_{a}\Pi^{a}_{bk}p^{b}_{\alpha}) + \Pi^{a}_{jk}\Pi^{i}_{al} - \Pi^{a}_{jl}\Pi^{i}_{ak}.$$

Thus all the coefficients of the projective connection are determined from Γ^{i}_{jk} by projectively invariant conditions.

5. Among the semi-natural frames of reference, we can select a special one for which $*\Pi^{a}_{ak} = 0$. We shall call natural frame of reference the semi-natural frame of reference for which these relations hold good. As we can see from (2.7), to obtain the natural frame of reference from an arbitrary semi-natural frame of reference, we must choose λ_j as

$$\lambda_j = -\frac{1}{N+1}\Pi^a_{aj} = -\frac{1}{N+1}\Gamma^a_{aj}$$

and effect the transformation

$$*A_o = A_o, \quad *A_j = -\frac{1}{N+1}\Gamma^a_{aj}A_o + A_j.$$

Thus, as we can see from (2.7), the coefficients of the projective connection with respect to the natural frame of reference are given by

$$(5.1)\quad \begin{cases} *\Pi^o_{jk} = \Pi^o_{jk} - \dfrac{1}{N+1}(\Gamma^a_{aj,\,k} - \Gamma^a_{ab}\Pi^b_{jk} + \dfrac{1}{N+1}\Gamma^a_{aj}\Gamma^b_{bk}), \quad *\Pi^{o\alpha}_{jk} = 0, \\ *\Pi^i_{jk} = \Gamma^i_{jk} - \dfrac{1}{N+1}(\delta^i_j\Gamma^a_{ak} + \delta^i_k\Gamma^a_{aj}) - \\ \qquad - \dfrac{1}{N-K}p^i_\lambda\{\Gamma^a_{jk}|^\lambda_a - \dfrac{1}{N+1}(\Gamma^a_{aj}|^\lambda_k + \Gamma^a_{ak}|^\lambda_j)\}. \end{cases}$$

The coefficients $*\Pi^i_{jk}$ coincide with those used by the other writers.

If we denote by $*P$ the components of the curvatures with respect to the natural frame of reference, then we have

$$(5.2)\quad *P^o_{jkl} = P^o_{jkl} + \frac{1}{N+1}(P^o_{jkl}{}^\alpha - P^o_{jlk}{}^\alpha)\Gamma^a_{ab}p^b_\alpha + \\ + \frac{1}{N+1}\Gamma^a_{ai}\{P^i_{jkl} + \frac{1}{N+1}(P^i_{jkl}{}^\alpha - P^i_{jlk}{}^\alpha)\Gamma^b_{bc}p^c_\alpha\},$$

$$(5.3)\quad *P^o_{jkl}{}^\alpha = P^o_{jkl}{}^\alpha + \frac{1}{N+1}\Gamma^a_{ai}P^i_{jkl}{}^\alpha,$$

$$(5.4)\quad *P^i_{jkl} = P^i_{jkl} + \frac{1}{N+1}(P^i_{jkl}{}^\alpha - P^i_{jlk}{}^\alpha)\Gamma^a_{ab}p^b_\alpha,$$

$$(5.5)\quad *P^i_{jkl}{}^\alpha = P^i_{jkl}{}^\alpha = *\Pi^i_{jk}|^\alpha_l.$$

Thus, $*P^o_{jkl}$, $*P^o_{jkl}{}^\alpha$ and $*P^i_{jkl}$ are not components of tensors while $*P^i_{jkl}{}^\alpha$ are components of a projective tensor. The $*P^i_{jkl}$ coincide with the W^i_{jkl} of J. Douglas. But, its transformation law is, as is easily seen from (5.4),

$$(5.6)\quad *\overline{P}^i_{jkl} = \frac{\partial\bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial\bar{x}^j}\frac{\partial x^c}{\partial\bar{x}^k}\frac{\partial x^d}{\partial\bar{x}^l}*P^a_{bcd} + *\overline{\Pi}^i_{jk}|^\alpha_l\,\bar{\theta}_m\bar{p}^m_\alpha - *\overline{\Pi}^i_{jl}|^\alpha_k\,\bar{\theta}_m\bar{p}^m_\alpha,$$

where

$$\bar{\theta}_m = \frac{1}{N+1}\frac{\partial \log \Delta}{\partial \bar{x}^m}, \quad \bar{p}^m_\alpha = \frac{\partial \bar{x}^m}{\partial u^\alpha} \qquad \left(\Delta = \left|\frac{\partial x}{\partial \bar{x}}\right|\right).$$

This is the transformation law of W^i_{jkl} obtained by R. S. Clark [2].

It will be remarked here that, when $K = 1$, the above discussion will be reduced to that given already by one of the present authors [8].

6. Now, returning to the semi-natural frame of reference, we have (3.3) along a K-spread. If we put

$$\text{(6.1)} \quad \underset{21}{dd}A_{\dot{o}} - \underset{12}{dd}A_{\dot{o}} = \Omega^{\dot{o}}_{\dot{o}} A_{\dot{o}}, \quad \underset{21}{dd}A_\beta - \underset{12}{dd}A_\beta = \Omega^{\dot{o}}_\beta A_{\dot{o}} + \Omega^\alpha_\beta A_\alpha.$$

the curvature tensors of the projective connection induced on a K-spread are given by

$$\text{(6.2)} \qquad \Omega^{\dot{o}}_\beta = P^{\dot{o}}_{\beta\gamma\delta} \underset{1}{du^\gamma} \underset{2}{du^\delta}, \quad \Omega^\alpha_\beta - \delta^\alpha_\beta \Omega^{\dot{o}}_{\dot{o}} = P^\alpha_{\beta\gamma\delta} \underset{1}{du^\gamma} \underset{2}{du^\delta}.$$

On the other hand, as we have

$$A_{\dot{o}} = A_o, \qquad A_\beta = p^o_\beta A_o + p^i_\beta A_i,$$

we find

$$\begin{cases} \underset{21}{dd}A_{\dot{o}} - \underset{12}{dd}A_{\dot{o}} = \underset{21}{dd}A_o - \underset{12}{dd}A_o, \\ \underset{21}{dd}A_\beta - \underset{12}{dd}A_\beta = p^o_\beta(\underset{21}{dd}A_o - \underset{12}{dd}A_o) + p^j_\beta(\underset{21}{dd}A_j - \underset{12}{dd}A_j), \end{cases}$$

from which

$$\text{(6.3)} \quad \begin{cases} \Omega^{\dot{o}}_{\dot{o}} A_{\dot{o}} = \Omega^o_o A_o, \\ \Omega^{\dot{o}}_\beta A_{\dot{o}} + \Omega^\alpha_\beta A_\alpha = p^o_\beta \Omega^o_o A_o + p^j_\beta (\Omega^o_j A_o + \Omega^i_j A_i). \end{cases}$$

Thus, remembering that $A_\alpha = p^o_\alpha A_o + p^i_\alpha A_i$, we find

$$p^i_\alpha \Omega^\alpha_\beta = \Omega^i_j p^j_\beta,$$

or

$$p^i_\alpha(\Omega^\alpha_\beta - \delta^\alpha_\beta \Omega^{\dot{o}}_{\dot{o}}) = (\Omega^i_j - \delta^i_j \Omega^o_o) p^j_\beta,$$

from which, δp^i_α being linear combinations of p^i_α along a K-spread, we find

$$\text{(6.4)} \qquad p^i_\alpha P^\alpha_{\beta\gamma\delta} = P^i_{jkl} p^j_\beta p^k_\gamma p^l_\delta,$$

where $P^\alpha_{\beta\gamma\delta}$ are components of the projective curvature tensor induced on the K-spreads. In a space of K-spreads, the equations of K-spreads being completely integrable, the above equations must be satisfied identically for any K-spread.

Now, we shall seek for the condition that the equations of the K-spreads can be reduced to

$$\frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} = 0$$

by suitable coordinate and parameter transformations.

If the equations are reduced to the above form, then we must have

$$P^i_{jkl}{}^\alpha \equiv {}^*\Pi^i_{jk}\Big|^\alpha_l = 0. \tag{6.5}$$

As $P^i_{jkl}{}^\alpha$ is a projective tensor, the vanishing of $P^i_{jkl}{}^\alpha$ is a condition invariant under coordinate and parameter transformations.

Conversely, if $P^i_{jkl}{}^\alpha \equiv {}^*\Pi^i_{jk}\Big|^\alpha_l = 0$, then ${}^*\Pi^i_{jk}$ and consequently

$${}^*P^i_{jkl} = P^i_{jkl}$$

are the functions of the point only. Thus applying the operator $\Big|^\lambda_i$ to (6.4), we obtain

$$(N-K)\, P^\lambda_{\beta\gamma\delta} = 0,$$

from which

$$P^i_{jkl} p^j_\beta p^k_\gamma p^l_\delta = 0 \quad \text{and consequently} \quad P^i_{jkl} = 0.$$

Thus it is proved that the necessary and sufficient condition that the space of K-spreads be projectively flat is that $P^i_{jkl}{}^\alpha = 0$, that is to say, ${}^*\Pi^i_{jk}$ be independent of p^i_α.

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