J. A. SCHOUTEN
D. VAN DANTZIG

On ordinary quantities and *W*-quantities. Classification and geometrical applications

Compositio Mathematica, tome 7 (1940), p. 447-473
<http://www.numdam.org/item?id=CM_1940__7__447_0>

On ordinary quantities and W-quantities

Classification and geometrical applications

by

J. A. Schouten and D. van Dantzig

Delft

1. Interior and exterior orientation.

We consider an E_n [1]) in which an orientation (n-dimensional screw-sense) is determined by an ordered sequence of n independent directions each with a definite sense [2]). Now suppose an E_p, $0 < p < n$ to be given in E_n. This E_p in E_n determines uniquely an E_{n-p} (not lying in E_n) in the following way: all E_p's totally parallel with the given one can be considered as elements of a set, which is an $(n-p)$-dimensional plane manifold in which an affine geometry is induced, i.e. an E_{n-p}. This process of obtaining the E_{n-p} is called „Zusammenlegung" by Weyl (analogous to „stetige Zerlegung" in topology) and is also described by saying that in each E_p all points are „identified".

Now we can either define a p-dimensional orientation in the E_p or an $(n-p)$-dimensional orientation in the E_{n-p}. In the first case we say that the E_p as well as its p-direction has got an *interior orientation*, in the second case we say that it is provided with an *exterior orientation*. The notions of interior and exterior orientation were introduced by Veblen and Whitehead [3]). For $p = 0$ and $p = n$ we define the orientation as follows:

The interior orientation of an E_0 is a $+$ or a $-$-sign, the exterior one is an ordinary orientation of the E_n. The interior orientation of the E_n is just this ordinary orientation, the exterior one is a $+$ or $-$-sign.

---

1) E_n = n-dimensional space with ordinary affine geometry.

2) Comp. E. I (Einführung in die neueren Methoden der Differentialgeometrie by J. A. Schouten and D. J. Struik, Vol I [Noordhoff 1935]), p. 16.

3) O. Veblen & J. H. C. Whitehead, The foundations of differential geometry [Cambr. Tracts in Math. 29 (1932)], 55, 56.

2. Contra- and covariant p-vectors. [4])

Let $x^{\varkappa}$ ($\varkappa, \lambda, \mu, \nu, \pi, \varrho, \sigma, \tau = 1, \ldots, n$) be cartesian coordinates in an E_n. Any other system $x^{\varkappa'}$ ($\varkappa', \lambda', \ldots, \tau' = 1', \ldots, n'$) of cartesian coordinates in this E_n is connected with the first one by equations of the form:

$$x^{\varkappa'} = A^{\varkappa'}_{\varkappa} x^{\varkappa} + A^{\varkappa'}_{0}; \qquad \operatorname{Det}(A^{\varkappa'}_{\varkappa}) \neq 0 \tag{1}$$

where the $A^{\varkappa'}_{\varkappa}$ and $A^{\varkappa'}_{0}$ are constants.

A *contravariant* p-vector $v^{\varkappa_1 \cdots \varkappa_p}$ is defined by its transformation formula

$$v^{\varkappa'_1 \cdots \varkappa'_p} = A^{\varkappa'_1}_{\varkappa_1} \cdots A^{\varkappa'_p}_{\varkappa_p} v^{\varkappa_1 \cdots \varkappa_p} \tag{2}$$

and its property of being alternating with respect to all suffixes. If it is simple (i.e. the alternated product of p vectors) it can be represented by a (e.g. simply connected) part of an E_p with an *interior* orientation. Two such parts determine the same p-vector if and only if 1⁰ the E_p's are totally parallel, 2⁰ the two p-dimensional volumes are equal, 3⁰ the orientations are the same. The ($\varkappa_1, \ldots, \varkappa_p$)-component is determined by the projection of the oriented part of the E_p upon the E_p of the contravariant

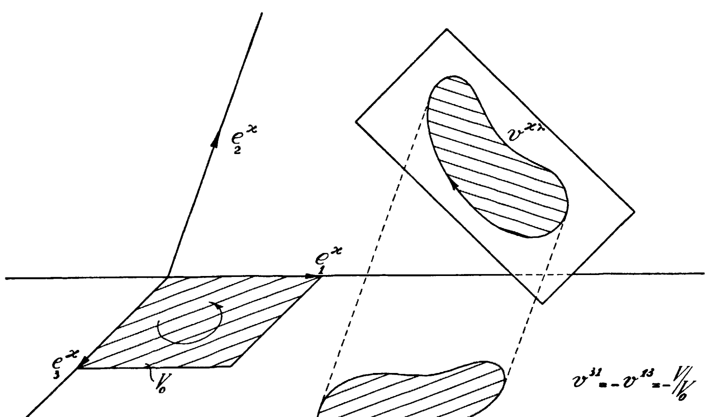


Fig. 1 [5]).

measuring vectors $\underset{\varkappa_1}{e}^{\varkappa}, \ldots, \underset{\varkappa_p}{e}^{\varkappa}$; the $(n-p)$-direction in which the

---

[4]) The results of § 2 — 5 were first published in J. A. Schouten, Über die geometrische Deutung von gewöhnlichen p-Vektoren und W-p-Vektoren und den korrespondierenden Dichten. [Proc. Amsterdam, **41** (1938), 709—716].

[5]) The orientation of the parallelogram on $\underset{3}{e}^{\varkappa}$, $\underset{1}{e}^{\varkappa}$ is the one belonging to $\underset{3}{e}^{[\varkappa}\underset{1}{e}^{\lambda]}$.

projection is performed is the $(n-p)$-direction common to the $(n-1)$-directions of the covariant measuring vectors $\overset{\varkappa_1}{e}_\lambda, \ldots, \overset{\varkappa_p}{e}_\lambda$. The value of this $(\varkappa_1, \ldots, \varkappa_p)$-component is the p-dimensional volume of the projection as measured by the p-dimensional volume of the parallelotope with edges $\underset{\varkappa_1}{e}^\varkappa, \ldots, \underset{\varkappa_p}{e}^\varkappa$ and provided with a factor $+1$ or -1 if its orientation is the same or opposite as the orientation of $\underset{\varkappa_1}{e}^\varkappa, \ldots, \underset{\varkappa_p}{e}^\varkappa$ in this order.

The projection of the orientation becomes undetermined if and only if the E_p of $v^{\varkappa_1 \cdots \varkappa_p}$ has a direction in common with the $(n-p)$-direction of the projection, in which case the volume of the projection is zero.

A *covariant* p-vector $w_{\lambda_1 \ldots \lambda_p}$ is defined by its transformation formula

$$(3) \qquad w_{\lambda_1' \ldots \lambda_p'} = A_{\lambda_1'}^{\lambda_1} \cdots A_{\lambda_p'}^{\lambda_p} w_{\lambda_1 \ldots \lambda_p}$$

and its property of being alternating. If it is simple it can be represented by a cylinder (the interior of which may be chosen simply connected) consisting of ∞^{p-1} (2 for $p=1$) totally parallel E_{n-p}'s (the generators) with an *exterior* orientation of their $(n-p)$-direction. Hence the set of ∞^{p-1} E_{n-p}'s is oriented. Two such cylinders determine the same p-vector if and only if 1^0 their $(n-p)$-directions are the same, 2^0 they intersect from

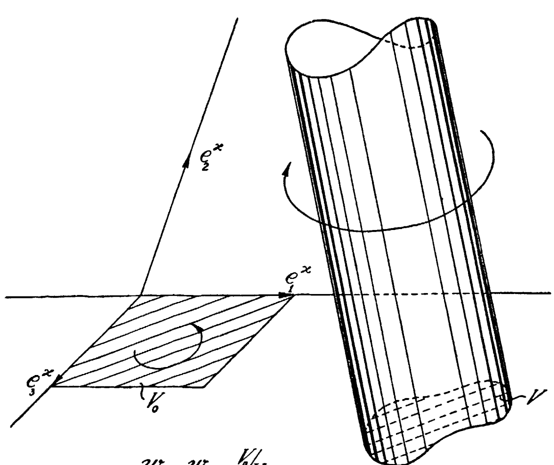


Fig. 2 [5]).

one (and then from every) E_p which has no direction in common with the $(n-p)$-direction, parts with equal p-dimensional volumes, and 3^0 the orientations are the same. The $\lambda_1 \dots \lambda_p$-component is the reciprocal value of the p-dimensional volume of the intersection of the cylinder with the E_p of $\underset{\lambda_1}{e^\varkappa}, \dots, \underset{\lambda_p}{e^\varkappa}$, as measured by the parallelotope of these vectors. It is positive or negative if the orientation of $w_{\lambda_1 \dots \lambda_p}$ has the same or the opposite sense resp. as the orientation of $\underset{\lambda_1}{e^\varkappa}, \dots, \underset{\lambda_p}{e^\varkappa}$ in this order.

3. Ordinary contra- and covariant p-vectordensities.

An ordinary contra- or covariant p-vectordensity of weight w is defined by its transformation-formula

$$(4) \qquad \mathfrak{v}^{\varkappa_1' \dots \varkappa_p'} = \Delta^{-w} A^{\varkappa_1'}_{\varkappa_1} \dots A^{\varkappa_p'}_{\varkappa_p} \mathfrak{v}^{\varkappa_1 \dots \varkappa_p}; \qquad \Delta = \operatorname{Det}(A^{\varkappa'}_{\lambda})$$

$$(5) \qquad \mathfrak{w}_{\lambda_1' \dots \lambda_p'} = \Delta^{-w} A^{\lambda_1}_{\lambda_1'} \dots A^{\lambda}_{\lambda_p'} \mathfrak{w}_{\lambda_1 \dots \lambda_p}$$

resp. and by its property of being alternating. For $p = 0$ we get ordinary scalardensities of weight w.

In an E_n three quantities are given a priori:

A. The unit affinor $A^\varkappa_\lambda$ with the components

$$(6) \qquad A^\varkappa_\lambda = \begin{cases} 1, & \varkappa = \lambda \\ 0, & \varkappa \neq \lambda \end{cases};$$

with respect to *every* system of coordinates.

B. The contravariant unit n-vectordensity $\mathfrak{E}^{\varkappa_1 \dots \varkappa_n}$ of weight $+1$ defined by

$$(7) \qquad \mathfrak{E}^{1 \dots n} = +1; \qquad \mathfrak{E}^{\varkappa_1 \dots \varkappa_n} = \mathfrak{E}^{[\varkappa_1 \dots \varkappa_n]}$$

with respect to *every* system of coordinates;

C. The covariant unit n-vectordensity $\mathfrak{e}_{\lambda_1 \dots \lambda_n}$ of weight -1 defined by

$$(8) \qquad \mathfrak{e}_{1 \dots n} = +1; \qquad \mathfrak{e}_{\lambda_1 \dots \lambda_n} = \mathfrak{e}_{[\lambda_1 \dots \lambda_n]}$$

with respect to *every* system of coordinates.

Hence a one to one correspondence exists between the set of all contravariant p-vectors and the set of all covariant $(n-p)$-vectordensities of weight -1 and also between the set of all covariant p-vectors and the set of all contravariant $(n-p)$-vectordensities of weight $+1$:

$$(9) \quad \mathfrak{v}_{\lambda_1 \dots \lambda_{n-p}} = \frac{1}{p!} \mathfrak{e}_{\lambda_1 \dots \lambda_{n-p} \varkappa_1 \dots \varkappa_p} v^{\varkappa_1 \dots \varkappa_p}; \quad v^{\varkappa_1 \dots \varkappa_p} = \frac{1}{(n-p)!} \mathfrak{v}_{\lambda_1 \dots \lambda_{n-p}} \mathfrak{E}^{\lambda_1 \dots \lambda_{n-p} \varkappa_1 \dots \varkappa_p}$$

$$(10) \quad \mathfrak{w}^{\varkappa_1 \dots \varkappa_{n-p}} = \frac{1}{p!} w_{\lambda_1 \dots \lambda_p} \mathfrak{E}^{\lambda_1 \dots \lambda_p \varkappa_1 \dots \varkappa_{n-p}}; \quad w_{\lambda_1 \dots \lambda_p} = \frac{1}{(n-p)!} \mathfrak{e}_{\lambda_1 \dots \lambda_p \varkappa_1 \dots \varkappa_{n-p}} \mathfrak{w}^{\varkappa_1 \dots \varkappa_{n-p}}.$$

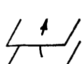
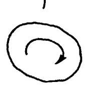
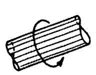
In particular with a scalar p correspond both a co- and a contravariant n-vectordensity

$$
(11)\quad \begin{cases} \mathfrak{p}'_{\lambda_1\ldots\lambda_n} = p\,\mathfrak{e}_{\lambda_1\ldots\lambda_n};\ p = \mathfrak{p}'_{1\ldots n} = \frac{1}{n!}\,\mathfrak{p}'_{\lambda_1\ldots\lambda_n}\,\mathfrak{E}^{\lambda_1\ldots\lambda_n} \\ \mathfrak{p}^{\varkappa_1\ldots\varkappa_n} = p\,\mathfrak{E}^{\varkappa_1\ldots\varkappa_n};\ p = \mathfrak{p}^{1\ldots n} = \frac{1}{n!}\,\mathfrak{p}^{\varkappa_1\ldots\varkappa_n}\,\mathfrak{e}_{\varkappa_1\ldots\varkappa_n}. \end{cases}
$$

Of course the geometrical representations of corresponding quantities are the same. For densities of other weights no such simple geometrical representations exist.

Table 1 shows the quantities considered here for $n = 3$.

Table 1.

Figure	Notation 1	Notation 2	Number of independent components	Orientation
none	p; scalar	$\mathfrak{p}'_{\mu\lambda\varkappa}$; cov. triv. dens. $w = -1$ $\mathfrak{p}^{\varkappa\lambda\mu}$; contr. triv. dens. $w = +1$	1	none
	$v^{\varkappa}$; contr. vect.	$\mathfrak{v}_{\lambda\varkappa}$; cov. biv. dens. $w = -1$	3 (proj.)	interior
	w_{λ}; cov. vect.	$\mathfrak{w}^{\varkappa\lambda}$; contr. biv. dens. $w = +1$	$3\left(\frac{1}{\text{inters.}}\right)$	exterior
	$f^{\varkappa\lambda}$; contr. biv.	$\mathfrak{f}_{\lambda}$; cov. vect. dens. $w = -1$	3 (proj.)	interior
	$h_{\lambda\varkappa}$; cov. biv.	$\mathfrak{h}^{\varkappa}$; contr. vect. dens. $w = +1$	$3\left(\frac{1}{\text{inters.}}\right)$	exterior
	$p^{\varkappa\lambda\mu}$; contr. triv.	$\mathfrak{p}$; scalardens. $w = -1$	1 (vol.)	screw
	$q_{\mu\lambda\varkappa}$; cov. triv.	$\mathfrak{q}$; scalardens. $w = +1$	$1\left(\frac{1}{\text{vol.}}\right)$	

4. W-quantities.

Aside ordinary densities we can also consider densities, in the transformation-formulae of which the *absolute value* $|\Delta|$ is taken instead of Δ itself. As they were introduced by H. Weyl [6]) we call them W-densities and distinguish them from ordinary densities by a $\smile$ above the central letter. If the weight is w, their transformation formulae are

[6]) RZM, § 13, 4th Aufl., 98.

$$(12)\qquad \tilde{\mathfrak{v}}^{\varkappa_1'\ldots\varkappa_p'} = |\Delta|^{-w} A^{\varkappa_1'}_{\varkappa_1} \cdots A^{\varkappa_p'}_{\varkappa_p} \tilde{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_p}$$

$$(13)\qquad \tilde{\mathfrak{w}}_{\lambda_1'\ldots\lambda_p'} = |\Delta|^{-w} A^{\lambda_1}_{\lambda_1'} \cdots A^{\lambda_p}_{\lambda_p'} \tilde{\mathfrak{w}}_{\lambda_1\ldots\lambda_p}.$$

For $p = 0$ we get W-scalardensities. As long as we consider only (real) transformations with $\Delta > 0$, there is no difference between W-densities and ordinary densities. But this restriction is equivalent with giving an orientation in X_n, viz. the orientation of $\underset{1}{e}^{\varkappa}, \ldots, \underset{n}{e}^{\varkappa}$ in this order [7]). Hence the geometric interpretation of a W-density can only differ from the interpretation of a corresponding ordinary density by the orientation. Hence it is to be expected that we may, at least in the cases $w = +1$ resp. $w = -1$ where we have simple geometrical representations, obtain W-densities from ordinary densities by interchanging *interior* and *exterior* orientations.

Take for instance a part of an E_p with an *exterior* orientation and fix the rules about equivalence and building of components in the same way as for an ordinary p-vector, the $(\varkappa_1 \ldots \varkappa_p)$-component being positive or negative if the projection of the *exterior* orientation has the same or the opposite sense resp. as the orientation of $\underset{\varkappa_{p+1}}{e}^{\varkappa}, \ldots, \underset{\varkappa_n}{e}^{\varkappa}$ in this order, where $\varkappa_1, \ldots, \varkappa_n$ is an even permutation of $1, \ldots, n$. Hence, if the quantity thus defined, which we denote by $\tilde{v}^{\varkappa_1\ldots\varkappa_p}$, has the same components as an ordinary contravariant p-vector with respect to one system of coordinates, this will be true also for all systems of coordinates that can be deduced from the first one by transformations with $\Delta > 0$, but the sign changes if we take a transformation with $\Delta < 0$. From this follows the transformation-formula of $\tilde{v}^{\varkappa_1\ldots\varkappa_p}$

$$(14)\qquad \tilde{v}^{\varkappa_1'\ldots\varkappa_p'} = \Delta\,|\Delta|^{-1} A^{\varkappa_1'}_{\varkappa_1} \cdots A^{\varkappa_p'}_{\varkappa_p} \tilde{v}^{\varkappa_1\ldots\varkappa_p}$$

and, if we define $\tilde{\mathfrak{v}}_{\lambda_1\ldots\lambda_{n-p}}$ by

$$(15)\qquad \tilde{\mathfrak{v}}_{\lambda_1\ldots\lambda_{n-p}} = \frac{1}{p!}\, \mathfrak{e}_{\lambda_1\ldots\lambda_{n-p}\varkappa_1\ldots\varkappa_p} \tilde{\mathfrak{v}}^{\varkappa_1\ldots\varkappa_p},$$

also:

$$(16)\qquad \mathfrak{v}_{\lambda_1'\ldots\lambda_{n-p}'} = |\Delta|\, A^{\lambda_1}_{\lambda_1'} \cdots A^{\lambda_{n-p}}_{\lambda_{n-p}'} \mathfrak{v}_{\lambda_1\ldots\lambda_{n-p}}.$$

---

[7]) Cf. Veblen & Whitehead l.c.

$\overset{\frown}{\mathfrak{v}}_{\lambda_1 \ldots \lambda_{n-p}}$ is a covariant W-$(n-p)$-vectordensity of weight -1. Accordingly the quantity $\overset{\frown}{v}^{\varkappa_1 \ldots \varkappa_p}$ will be called a contravariant W-p-vector.

In the same way we define a covariant W-p-vector $\overset{\frown}{w}_{\lambda_1 \ldots \lambda_p}$ by its transformation-formula

$$\overset{\frown}{w}_{\lambda'_1 \ldots \lambda'_p} = \Delta \,|\, \Delta \,|^{-1} A^{\lambda_1}_{\lambda'_1} \cdots A^{\lambda_p}_{\lambda'_p} \overset{\frown}{w}_{\lambda_1 \ldots \lambda_p} \tag{17}$$

and its alternating property. This quantity is geometrically equivalent with the contravariant W-$(n-p)$-vectordensity

$$\overset{\frown}{\mathfrak{w}}^{\varkappa_1 \ldots \varkappa_{n-p}} = \frac{1}{p!} \overset{\frown}{w}_{\lambda_1 \ldots \lambda_p} \mathfrak{E}^{\lambda_1 \ldots \lambda_p \varkappa_1 \ldots \varkappa_{n-p}} \tag{18}$$

of weight $+1$ and can, if it is simple, be represented by a cylinder (the interior of which may be chosen simply connected) consisting of ∞^{p-1} totally parallel E_{n-p}'s with an *interior* orientation of their $(n-p)$-direction. The rules for equivalence and the building of components are the same as in the case of the ordinary covariant p-vector; the component is positive or negative if the orientation of $\overset{\frown}{w}_{\lambda_1 \ldots \lambda_p}$ has the same or the opposite sense resp. as the orientation of $\underset{\lambda_{p+1}}{e^{\varkappa}}, \ldots, \underset{\lambda_n}{e^{\varkappa}}$ in this order, where $\lambda_1, \ldots, \lambda_n$ is an even permutation of $1, \ldots, n$.

For $p = 0$ we get W-scalars with the transformation-formula

$$\overset{(\varkappa')}{\overset{\frown}{p}} = \Delta \,|\, \Delta \,|^{-1} \overset{(\varkappa)}{\overset{\frown}{p}}. \tag{19}$$

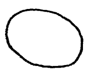
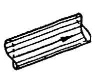
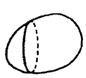
With $\overset{\frown}{p}$ correspond both a co- and a contravariant W-n-vectordensity of weight -1 and $+1$ resp.:

$$\begin{aligned} \overset{\frown}{\mathfrak{p}}'_{\lambda_1 \ldots \lambda_n} &= \overset{\frown}{p}\, \mathfrak{e}_{\lambda_1 \ldots \lambda_n}; \quad \overset{\frown}{p} = \frac{1}{n!} \overset{\frown}{\mathfrak{p}}'_{\lambda_1 \ldots \lambda_n} \mathfrak{E}^{\lambda_1 \ldots \lambda_n} \\ \overset{\frown}{\mathfrak{p}}^{\varkappa_1 \ldots \varkappa_n} &= \overset{\frown}{p}\, \mathfrak{E}^{\varkappa_1 \ldots \varkappa_n}; \quad \overset{\frown}{p} = \frac{1}{n!} \overset{\frown}{\mathfrak{p}}^{\varkappa_1 \ldots \varkappa_n} \mathfrak{e}_{\varkappa_1 \ldots \varkappa_n}. \end{aligned} \tag{20}$$

Contrary to an ordinary scalar a W-scalar has an (n-dimensional) orientation, viz. the orientation of the coordinate system with respect to which p is positive. But a W-scalardensity of weight $+1$ or -1 has no orientation because it is equivalent with a covariant or contravariant W-n-vector resp. and is represented by an n-dimensional volume without orientation.

Table 2 shows the W-quantities considered here for $n = 3$.

Table 2.

Figure	Notation 1	Notation 2	Components	Orientation
none	$\tilde{p}$; W-scalar	$\tilde{\mathfrak{p}}'_{\mu\lambda\varkappa}$; cov. W-triv.d.; $w = -1$ $\tilde{\mathfrak{p}}^{\varkappa\lambda\mu}$; contr. W-triv.d.; $w = +1$	1	screw
	$\tilde{v}^{\varkappa}$; contr. W-vect.	$\tilde{\mathfrak{v}}_{\lambda\varkappa}$; cov. W-biv.d.; $w = -1$	3 (proj.)	outside
	$\tilde{w}_{\lambda}$; cov. W-vect.	$\tilde{\mathfrak{w}}^{\varkappa\lambda}$; contr. W-biv.d.; $w = +1$	$3\left(\frac{1}{\text{inters.}}\right)$	inside
	$\tilde{f}^{\varkappa\lambda}$; contr. W-biv.	$\tilde{\mathfrak{f}}_{\lambda}$; cov. W-vect.d.; $w = -1$	3 (proj.)	outside
	$\tilde{h}_{\lambda\varkappa}$; cov. W-biv.	$\tilde{\mathfrak{h}}^{\varkappa}$; contr. W-vect.d.; $w = +1$	$3\left(\frac{1}{\text{inters.}}\right)$	inside
	$\tilde{p}^{\varkappa\lambda\mu}$; contr. W-triv. $\tilde{q}_{\mu\lambda\varkappa}$; cov. W-triv.	$\tilde{\mathfrak{p}}$; W-dens.; $w = -1$ $\tilde{\mathfrak{q}}$; W-dens.; $w = +1$	1 (vol.) $1\left(\frac{1}{\text{vol.}}\right)$	none

An example of a W-scalar is: $p = +1$ for all right-handed systems and $p = -1$ for all left-handed systems. Quantities of this kind are called sometimes „pseudoscalars'' in physics. W-vectors are occasionally used in physics but only after the introduction of a metric [8]). Mr. St. Gołab [9]) has proved by solving a functional equation, that *all* geometric objects with only one component, whose transformation depends on Δ only, can be deduced from the four objects: scalars, W-scalars, ordinary scalardensities and W-scalardensities. From this theorem follows that the classification we have used here is really exhaustive.

5. Identification of quantities.

After introducing a unit of volume, a metric or an n-dimensional orientation identifications arise between the different quantities derived. We take the case $n = 3$ as an illustration, the generalisation being obvious. (Cf. table 3)

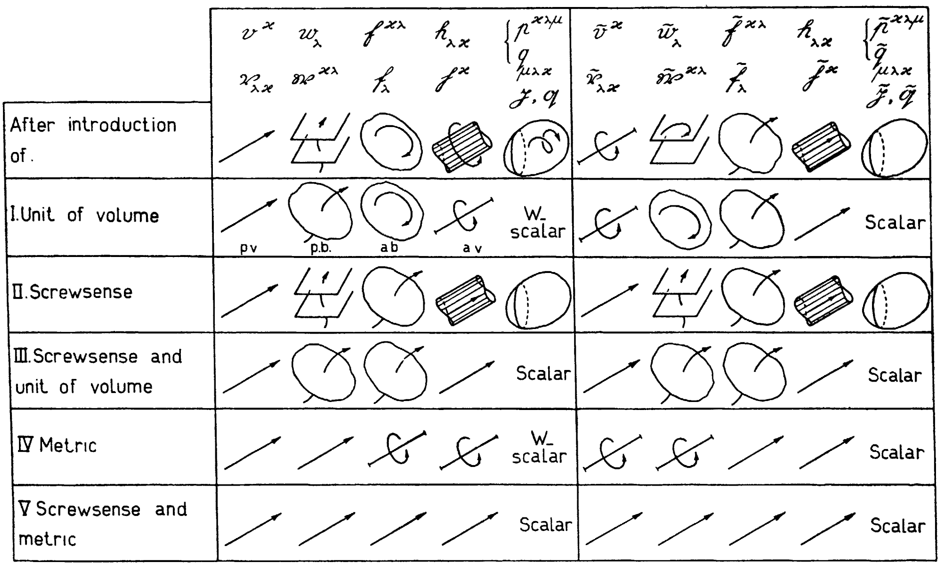
The four directed quantities occuring after introduction of

[8]) For applications without a metric comp. D. v. DANTZIG, On the phenomenological thermodynamics, [Physica **6** (1939), 673—704]; On relativistic thermodynamics, [Proc. Amsterdam, **42** (1939) 601—607]; On relativistic gas theory, [l.c., 608—625].

[9]) ST. GOLAB, Über die Klassifikation der geometrischen Objekte [Math. Zeitschr. **44** (1938) 104—114].

a unit of volume, are known in litterature (from left to right) as polar vector, polar bivector, axial bivector, axial vector. After introduction of a metric the difference between polar and

Table 3.



axial and between scalars and W-scalars remains. This is the point of view often found in publications on physics. After introduction of a screw-sense and a metric (this includes **I**, **II**, **III** and **IV**) all differences between directed quantities and the difference between scalars and W-scalars vanish.

6. Quantities in X_n.

We consider an n-dimensional differentiable manifold X_n. Let $\xi^\varkappa (\varkappa, \lambda, \mu, \nu, \pi, \varrho, \sigma, \tau = 1, \ldots, n)$ be a set of coordinates in a sufficiently small part of this X_n and $\xi^{\varkappa'} (\varkappa', \lambda', \ldots, \tau' = 1', \ldots, n')$ another set of coordinates with

$$(21) \qquad \Delta = \mathrm{Det}\,(A_\lambda^{\varkappa'}) \neq 0; \; A_\lambda^{\varkappa'} = \partial_\lambda \xi^{\varkappa'}; \; \partial_\lambda = \frac{\partial}{\partial \xi^\lambda}.$$

It is well known that we define all quantities in the local E_n of a point of X_n using in the formulae of transformation $A_\lambda^{\varkappa'} = \partial_\lambda \xi^{\varkappa'}$ in stead of the constants $A_\lambda^{\varkappa'}$ in (1). In this way we get in each local E_n ordinary quantities and W-quantities.

7. Imbedding of an X_m in an X_n.[10])

If an X_m is imbedded in an X_n the following eight cases are important. We write k for $n - m$ and suppose η^a $(a, b, \ldots, g = 1, \ldots, \mathrm{m})$ to be coordinates in the X_m. Further we denote by B_b^a the unity-affinor in X_m, by $\mathfrak{E}'^{a_1 \ldots a_m}$ and $\mathfrak{e}'_{b_1 \ldots b_m}$ the unit-m-vectordensities in X_m and by D the transformation-modulus $D = \det(B_b^{a'})$ in X_m.

Case 1. *Pure imbedding without any auxiliary assumptions.*

To every point η^a of the X_m belongs one and only one point of the X_n, given by equations of the form

$$\xi^\varkappa = \xi^\varkappa(\eta^1, \ldots, \eta^{\mathrm{m}}). \tag{22}$$

In every point of the X_m two quantities exist:

1⁰ the affinor

$$B_b^\varkappa = \frac{\partial \xi^\varkappa}{\partial \eta^b}, \tag{23}$$

a connecting quantity behaving like a system of m contravariant vectors with respect to transformations in X_n and like a system of n covariant vectors with respect to transformations in X_m;

2⁰ the simple k-vectordensity

$$\mathfrak{t}_{\lambda_1 \ldots \lambda_k} = \frac{1}{m!} \mathfrak{e}_{\lambda_1 \ldots \lambda_k \varkappa_1 \ldots \varkappa_m} B_{b_1}^{\varkappa_1} \ldots B_{b_m}^{\varkappa_m} \mathfrak{E}'^{b_1 \ldots b_m}, \tag{24}$$

a connecting quantity with the weights $+1$ and -1 with respect to transformations in X_m and X_n respectively:

$$\mathfrak{t}_{\lambda_1' \ldots \lambda_k'} = \Delta\, D^{-1} A_{\lambda_1'}^{\lambda_1} \ldots A_{\lambda_k'}^{\lambda_k} \mathfrak{t}_{\lambda_1 \ldots \lambda_k}. \tag{25}$$

$B_b^\varkappa$ as well as $\mathfrak{t}_{\lambda_1 \ldots \lambda_k}$ are geometrically represented by the tangent E_m in the local E_n. They are related by the identity

$$B_b^{\lambda_1} \mathfrak{t}_{\lambda_1 \ldots \lambda_k} = 0. \tag{26}$$

Case 1′. *Imbedding with exterior orientation.*

The orientation is an exterior orientation of the tangent E_m in every point of the X_m and can be given by any quantity ω with the absolute value $+1$ and the transformation-formula

---

[10]) The eight different cases of imbedding were first treated in J. A. Schouten, Über die Beziehungen zwischen den geometrischen Größen in einer X_n und in einer in der X_n eingebetteten X_m, [Proc. Amsterdam, **41** (1938) 568—575].

$$(27)\qquad \overset{(\varkappa',a')}{\omega} = \Delta^{-1} \,|\, \Delta \,|\, D \,|\, D \,|^{-1} \overset{(\varkappa,a)}{\omega}.$$

In fact, because of (25), (27) the quantity $\omega t_{\lambda_1 \ldots \lambda_k}$ has the transformation-formula

$$(28)\qquad \overset{(\varkappa',a')}{\omega} t_{\lambda'_1 \ldots \lambda'_k} = |\, \Delta \,|\, |\, D \,|^{-1} A^{\lambda_1}_{\lambda'_1} \cdots A^{\lambda_k}_{\lambda'_k}\ \omega\ t_{\lambda_1 \ldots \lambda_k},$$

and this is the same as that of a covariant k-vector except for an always *positive* factor. Hence $\omega\, t_{\lambda_1 \ldots \lambda_k}$ determines a k-dimensional orientation in every E_k having no direction in common with the tangent E_m. If in any point of the X_m the orientation of $\underset{1}{e}^a, \ldots, \underset{m}{e}^a$ in this order *followed* by the exterior orientation gives the orientation of $\underset{1}{e}^\varkappa, \ldots, \underset{n}{e}^\varkappa$ in this order, then we choose in that point always $\overset{(\varkappa,a)}{\omega} = +1$.

In each E_k having no direction in common with the tangent E_m in case 1′ a contravariant alternating quantity $\pi^{\varkappa_1 \ldots \varkappa_k}$ with the transformation-formula

$$(29)\qquad \pi^{\varkappa'_1 \ldots \varkappa'_k} = \Delta^{-1} \,|\, \Delta \,|\, D \,|\, D \,|^{-1} A^{\varkappa'_1}_{\varkappa_1} \cdots A^{\varkappa'_k}_{\varkappa_k} \pi^{\varkappa_1 \ldots \varkappa_k},$$

satisfying the invariant condition

$$(30)\qquad \pi^{\varkappa_1 \ldots \varkappa_k} t_{\varkappa_1 \ldots \varkappa_k} > 0,$$

is determined except for a *positive* factor. Except for a factor ± 1 this quantity transforms in the same way as a contravariant k-vector. Hence it determines such a k-vector except for the orientation. Now suppose a k-vector $p^{\varkappa_1 \ldots \varkappa_k}$ is given except for a factor ± 1. Then we can determine a quantity $\pi^{\varkappa_1 \ldots \varkappa_k}$ by the equation

$$(31)\qquad \begin{aligned} \pi^{\varkappa_1 \ldots \varkappa_k} &= \omega\, p^{\varkappa_1 \ldots \varkappa_k} \operatorname{sgn} (\omega\, p^{\varkappa_1 \ldots \varkappa_k} t_{\varkappa_1 \ldots \varkappa_k}) \\ &= p^{\varkappa_1 \ldots \varkappa_k} \operatorname{sgn} (p^{\varkappa_1 \ldots \varkappa_k} t_{\varkappa_1 \ldots \varkappa_k}), \end{aligned}$$

(where $\operatorname{sgn}(z) = \frac{z}{|z|}$ for $z \neq 0$ and $\operatorname{sgn}(0) = 0$). From this follows, that the geometric representation of a quantity with the transformation (29) satisfying the condition (30) is a part of an E_k having neither an interior nor an exterior orientation.

Case 2. *Rigged imbedding.*

An X_m in an X_n is called „rigged'' („eingespannt'') if in every point of X_m a k-direction is given, having no direction in common with the tangent E_m. This k-direction can be given by an affinor $B^{\varkappa}_{\lambda}$ with the following properties: 1⁰. its $\varkappa$-region consists of all contravariant vectors of the local E_m and its λ-region consists of all covariant vectors whose $(n-1)$-direction contains the k-direction of the rigging, 2⁰:

$$B^{\varkappa}_{\varrho} B^{\varrho}_{\lambda} = B^{\varkappa}_{\lambda}. \tag{32}$$

From $B^{\varkappa}_{b}$ and $B^{\varkappa}_{\lambda}$ an affinor B^{a}_{λ} can be uniquely determined by means of the equations

$$B^{a}_{\lambda} B^{\lambda}_{b} = B^{a}_{b} = \text{unitaffinor of the } X_m. \tag{33a}$$

$$B^{b}_{\lambda} B^{\varkappa}_{b} = B^{\varkappa}_{\lambda}. \tag{33b}$$

The λ-region of B^{a}_{λ} is the same as the λ-region of $B^{\varkappa}_{\lambda}$. $B^{\varkappa}_{b}$ and B^{a}_{λ} together determine $B^{\varkappa}_{\lambda}$ uniquely by (33b). The rigging determines also uniquely and is determined by the simple k-vectordensity

$$\mathfrak{e}^{\varkappa_1 \cdots \varkappa_k} = \frac{1}{m!} \mathfrak{E}^{\lambda_1 \cdots \lambda_m \varkappa_1 \cdots \varkappa_k} B^{a_1}_{\lambda_1} \cdots B^{a_m}_{\lambda_m} \mathfrak{e}'_{a_1 \cdots a_m}, \tag{34a}$$

a connecting quantity with the weights -1 and $+1$ with respect to transformations in X_m and X_n resp. and satisfying the relations

$$\mathfrak{e}^{\varkappa_1 \cdots \varkappa_k} B^{a}_{\varkappa_1} = 0; \quad \frac{1}{k!} \mathfrak{e}^{\varkappa_1 \cdots \varkappa_k} \mathfrak{t}_{\varkappa_1 \cdots \varkappa_k} = 1. \tag{34b}$$

In the cases 1 and 1′ B^{a}_{λ} can not be uniquely determined, as then (34) fails, and the solutions of (33) alone contain arbitrary parameters. But they determine B^{a}_{λ} partly and well enough in order that the expression $B^{a_1}_{\varkappa_1} \cdots B^{a_p}_{\varkappa_p} v^{\varkappa_1 \cdots \varkappa_p}$ be uniquely determined if $v^{\varkappa_1 \cdots \varkappa_p}$ is a simple contravariant p-vector, $p \leqq m$, parallel to the tangent E_m.

We often make use of a special system of coordinates for which in every point of the X_m $\xi^1 = \eta^1, \ldots, \xi^m = \eta^m$ [11]). If the X_m is rigged we choose the parameterlines of $\xi^{m+1}, \ldots, \xi^n$ in such a way that they have in every point of the X_m a direction lying

---

[11]) Coordinate systems of this kind are transformed into each other by transformations of a group studied by A. Kawaguchi, The foundation of the theory of displacements II [Proc. Imp. Academy **10** (1934), 45—48]. From this remark follow the relations between our paper and the investigations of Kawaguchi, S. Hokari, Über die Übertragungen, die der erweiterten Transformationsgruppe angehören [Journ. Hokkaido Imp. Univ. **3** (1935), 15—26; **4** (1935) 14—50], S. Golab, Über eine Art der Geometrie von Kawaguchi-Hokari [Ann. Soc. Polon. Math. **16** (1937) 25—30], and the investigations of T. Hosokawa, A. Wundheiler, V. Hlatatу, E. Cartan, T. Y. Thomas, J. A. Schouten and S. Golab, quoted in this latter paper.

in the k-direction of the rigging. If we use ξ^α $(\alpha, \ldots, \delta = 1, \ldots, m)$ as coordinates in the X_m we have for that system

$$B^\alpha_\beta = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases} \quad B^{m+1}_\beta = \ldots = B^n_\beta = 0, \tag{35}$$

$$\mathfrak{t}_{m+1, \ldots, n} = 1; \; \mathfrak{e}^{m+1, \ldots, n} = 1 \tag{36}$$

and all components of $\mathfrak{t}_{\lambda_1 \ldots \lambda_k}$ and $\mathfrak{e}^{\varkappa_1, \ldots, \varkappa_k}$ with a suffix $\leqq m$ vanish. $\omega = +1$ if $\underset{m+1}{e^\varkappa}, \ldots, \underset{n}{e^\varkappa}$ in this order determine the exterior orientation of the tangent E_m and -1 in the other case.

CASE 2′. ***Rigged imbedding with exterior orientation.***

This case is a combination of 1′ and 2 and requires ω as well as $B^\varkappa_\lambda$ to be given.

With respect to the special coordinate system mentioned above the condition (30) for the non oriented quantity $\pi^{\varkappa_1 \ldots \varkappa_k}$ is

$$\pi^{1+m, \ldots, n} > 0 \tag{37}$$

and the equation (31) takes the form

$$\pi^{\varkappa_1 \ldots \varkappa_k} = p^{\varkappa_1 \ldots \varkappa_k} \operatorname{sign} (p^{m+1, \ldots, n}). \tag{38}$$

CASE 3. *Imbedding with normalisation and exterior orientation.*

In every point of the X_m a simple covariant k-vector $t_{\lambda_1 \ldots \lambda_k}$ is given whose m-direction lies in the tangent E_m. It determines uniquely and is uniquely determined by the scalardensity

$$\mathfrak{z} = \frac{\mathfrak{t}_{\lambda_1 \ldots \lambda_k}}{t_{\lambda_1 \ldots \lambda_k}} \tag{39}$$

of weights -1 and $+1$ with respect to transformations in X_n and X_m respectively. Obviously

$$\omega = \frac{\mathfrak{z}}{|\mathfrak{z}|}\,. \tag{40}$$

In each E_k in the local E_n, having no direction in common with the tangent E_m, $t_{\lambda_1 \ldots \lambda_k}$ determines (by section) uniquely a contravariant k-vector $p^{\varkappa_1 \ldots \varkappa_k}$ satisfying the equation

$$\frac{1}{k!}\, t_{\lambda_1 \ldots \lambda_k}\, p^{\lambda_1 \ldots \lambda_k} = 1. \tag{41}$$

With respect to the special coordinate system mentioned above we have $t_{m+1, \ldots, n} = \mathfrak{z}^{-1}$ and all components with a suffix $\leqq m$ vanish.

CASE 3′. *Imbedding with normalisation without orientation.*

We get this case by giving $|\mathfrak{z}|$ instead of $\mathfrak{z}$. Then instead of $t_{\lambda_1 \ldots \lambda_k}$ only the quantity

(42) $$\tau_{\lambda_1\dots\lambda_k} = \omega\, t_{\lambda_1\dots\lambda_k} = |\,\mathfrak{z}\,|^{-1}\, \mathfrak{t}_{\lambda_1\dots\lambda_k}$$

is determined, and v.v. $\tau_{\lambda_1\dots\lambda_k}$ determines $|\,\mathfrak{z}\,|$. The geometric representation is a cylinder consisting of ∞^{k-1} (2 for $k=1$) totally parallel E_m's, all parallel with the tangent E_m, but having neither interior nor exterior orientation. With respect to the special system of coordinates mentioned before, $\tau_{m+1,\dots,n} = |\,\mathfrak{z}\,|^{-1}$, and all components with a suffix $\leqq m$ vanish. In each E_k, having no direction in common with the tangent E_m, $\tau_{\lambda_1\dots\lambda_k}$ determines a non oriented contravariant quantity as considered under case 1′ and determined by the equation

(43) $$\frac{1}{k!}\, \tau_{\lambda_1\dots\lambda_k}\, \pi^{\lambda_1\dots\lambda_k} = 1.$$

Case 4. *Rigged imbedding with normalisation and exterior orientation.*

This case is a combination of 2′ and 3 and requires $B^{\varkappa}_{\lambda}$ and $\mathfrak{z}$ to be given. In the k-direction of the rigging a simple contravariant k-vector $n^{\varkappa_1\dots\varkappa_k} = \mathfrak{z}\, \mathfrak{e}^{\varkappa_1\dots\varkappa_k}$ exists, which is uniquely determined by the conditions

(44) $$\frac{1}{k!}\, n^{\varkappa_1\dots\varkappa_k}\, t_{\varkappa_1\dots\varkappa_k} = 1;$$

(45) $$B^{\varkappa}_{\varkappa_1}\, n^{\varkappa_1\dots\varkappa_k} = 0,$$

and for which the equation

(46) $$t_{\lambda_1\dots\lambda_k}\, n^{\varkappa_1\dots\varkappa_k} = k!\, C^{[\varkappa_1}_{[\lambda_1}\cdots C^{\varkappa_k]}_{\lambda_k]};\quad C^{\varkappa}_{\lambda} = A^{\varkappa}_{\lambda} - B^{\varkappa}_{\lambda}$$

holds. With respect to the special system of coordinates mentioned before $n^{m+1,\dots,n} = \mathfrak{z}$ and all components with a suffix $\leqq m$ vanish.

Case 4′. *Rigged imbedding with normalisation without orientation.*

This case is a combination of 2 and 3′ and requires $B^{\varkappa}_{\lambda}$ and $|\,\mathfrak{z}\,|$ to be given. In the k-direction of the rigging a non oriented contravariant quantity $\nu^{\varkappa_1\dots\varkappa_k} = |\,\mathfrak{z}\,|\, \mathfrak{e}^{\varkappa_1\dots\varkappa_k}$ exists, which is uniquely determined by the conditions

(47) $$\frac{1}{k!}\, \nu^{\varkappa_1\dots\varkappa_k}\, \tau_{\varkappa_1\dots\varkappa_k} = 1$$

(48) $$B^{\varkappa}_{\varkappa_1}\, \nu^{\varkappa_1\dots\varkappa_k} = 0$$

and for which the equation

(49) $$\tau_{\lambda_1\dots\lambda_k}\, \nu^{\varkappa_1\dots\varkappa_k} = k!\, C^{[\varkappa_1}_{[\lambda_1}\cdot\cdot\, C^{\varkappa_k]}_{\lambda_k]}$$

holds. With respect to the special system of coordinates mentioned before $\nu^{m+1,\ldots,n} = |\mathfrak{z}|$ and all components with a suffix $\leqq m$ vanish.

The following table shows the different cases and the quantities involved [12]).

Table 4.

Imbedding:	1	1′	2	2′	3′	3	4′	4	quantities occurring besides $B_b^\varkappa$ and $\mathfrak{t}$
included cases:		1	1	1, 1′, 2	1	1, 1′, 3′	1, 2, 3′	all	
orientation:	—	+	—	+	—	+	—	+	ω
rigging:	—	—	+	+	—	—	+	+	$B_\lambda^\varkappa$, (B_λ^a), $(\mathfrak{n})$
normalisation:	—	—	—	—	+	+	+	+	$\lvert\mathfrak{z}\rvert$, $(\mathfrak{r})$
quantities occuring besides $B_b^\varkappa$	$\mathfrak{t}$	$\mathfrak{t}$, ω	$\mathfrak{t}$, $\mathfrak{n}$, $(B_\lambda^\varkappa, B_\lambda^a)$	$\mathfrak{t}$, $\mathfrak{n}$, ω, $(B_\lambda^\varkappa, B_\lambda^a)$	$\mathfrak{t}$, (τ), $\lvert\mathfrak{z}\rvert$	$\mathfrak{t}$, (τ, t), ω, $\lvert\mathfrak{z}\rvert$, $(\mathfrak{z})$	$\mathfrak{t}$, (τ), $\mathfrak{n}$, (ν), $\lvert\mathfrak{z}\rvert$, $(B_\lambda^\varkappa, B_\lambda^a)$	$\mathfrak{t}$, $(\tau, \mathfrak{t})$, $\mathfrak{n}$, (ν, n), ω, $\lvert\mathfrak{z}\rvert$, $(\mathfrak{z})$, $(B_\lambda^\varkappa, B_\lambda^a)$	

Fig. 3 shows the eight different cases for the imbedding of an X_2 in ordinary space.

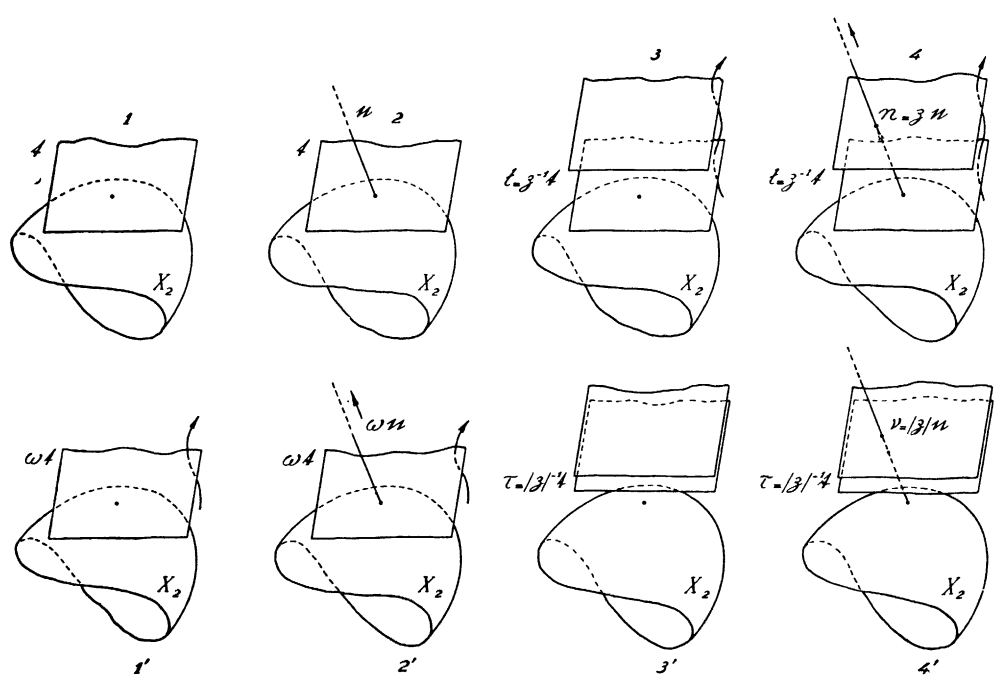


Fig. 3.

[12]) The quantities with k indices are denoted by their central letter. Quantities in brackets in the last row or column can be derived from the other quantities in

8. Relations between quantities in X_m and X_n.[13])

A. *Ordinary contravariant p-vectors.*

A contravariant p-vector $'v^{a_1 \dots a_p}$ $(p \leqq m)$ in a point of X_m determines uniquely a contravariant p-vector $'v^{\varkappa_1 \dots \varkappa_p}$ in the corresponding point of X_m according to the equation

$$(50) \qquad 'v^{\varkappa_1 \dots \varkappa_p} = B^{\varkappa_1}_{a_1} \dots B^{\varkappa_p}_{a_p}\, 'v^{a_1 \dots a_p}.$$

Because of (26) we have

$$(51) \qquad 'v^{\varkappa \varkappa_2 \dots \varkappa_p}\, \mathfrak{t}_{\varkappa \lambda_2 \dots \lambda_k} = 0.$$

At the other hand each p-vector $'v^{\varkappa_1 \dots \varkappa_p}$ $(p \leqq m)$ with respect to transformations in X_n, defined in a point of X_m and satisfying (51)[14]) determines *uniquely* a p-vector $'v^{a_1 \dots a_p}$ defined by (50) as these equations have a unique solution. It can be written in the form

$$(52) \qquad 'v^{a_1 \dots a_p} = \frac{1}{p!}\, \beta^{a_1 \dots a_p}_{\cdot \quad \cdot\, \varkappa_1 \dots \varkappa_p}\, 'v^{\varkappa_1 \dots \varkappa_p}$$

where $\beta^{a_1 \dots a_p}_{\cdot \quad \cdot\, \lambda_1 \dots \lambda_p}$ is determined except for the product of an *arbitrary* alternating quantity $\mathfrak{w}^{a_1 \dots a_p \lambda_{p+1} \dots \lambda_m}$ with $\mathfrak{t}_{\lambda_1 \dots \lambda_m}$. These quantities can all be derived from $\beta^{a_1 \dots a_m}_{\cdot \quad \cdot\, \lambda_1 \dots \lambda_m}$, viz.

$$(53) \quad \beta^{a_1 \dots a_p}_{\cdot \quad \cdot\, \lambda_1 \dots \lambda_p} = \beta^{a_1 \dots a_p b_{p+1} \dots b_m}_{\cdot \qquad \qquad \cdot\, \lambda_1 \dots \lambda_p \varkappa_{p+1} \dots \varkappa_m}\, B^{\varkappa_{p+1}}_{b_{p+1}} \dots B^{\varkappa_m}_{b_m}.$$

In case 2 (rigging; existence of B^a_λ) the solution has the form

$$(54) \qquad 'v^{a_1 \dots a_p} = B^{a_1}_{\varkappa_p} \dots B^{a_p}_{\varkappa_p}\, 'v^{\varkappa_1 \dots \varkappa_p},$$

i.e. a particular choice of $\beta^{a_1 \dots a_p}_{\cdot \quad \cdot\, \lambda_1 \dots \lambda_p}$ is $p!\, B^{[a_1}_{[\lambda_1} \dots B^{a_p]}_{\lambda_p]}$. Moreover in this case an *arbitrary* p-vector $v^{\varkappa_1 \dots \varkappa_p}$ defined in a point of X_m also determines a p-vector in X_m according to

$$(55) \qquad 'v^{a_1 \dots a_p} = B^{a_1}_{\varkappa_1} \dots B^{a_p}_{\varkappa_p}\, v^{\varkappa_1 \dots \varkappa_p},$$

but is not itself uniquely determined by $'v^{a_1 \dots a_p}$ except for

---

the same row *and* column. + stands for „given", — for „not given". All cases can be obtained by giving none, one, two or three of the three independent quantities $\mathfrak{n}$, ω and $|\mathfrak{z}|$, which determine independently the rigging the orientation and the normalisation respectively.

[13]) The relations between ordinary quantities have been treated by J. A. Schouten [l.c. in note [8])].

[14]) Such a p-vector in X_n is said to „lie in X_m".

$p=0$, where (55) simply becomes $'v=v$. $'v^{a_1\dots a_p}$ can be called the *projection* of $v^{\varkappa_1\dots\varkappa_p}$ in the direction of the rigging upon the local E_n tangent to X_m.

By means of (9) and the analogous equation in X_m the equations corresponding with (50), (54) and (55) are found. They contain a covariant $(n-p)$-vectordensity of weight -1 in X_n and a covariant $(m-p)$-vectordensity of weight $+1$ in X_m.

In case 3 (normalisation; existence of $t_{\lambda_1\dots\lambda_k}$) for $p\geqq k=n-m$ still another quantity in X_m is determined by an *arbitrary* p-vector $v^{\varkappa_1\dots\varkappa_p}$, viz.

$$(56)\qquad {}''v^{a_1\dots a_{p-k}}=\frac{1}{k!}\,B^{a_1}_{\varkappa_1}\dots B^{a_{p-k}}_{\varkappa_{p-k}}v^{\varkappa_1\dots\varkappa_p}\,t_{\varkappa_{p-k+1}\dots\varkappa_p}.$$

If $v^{\varkappa_1\dots\varkappa_p}$ is simple, the $(p-k)$-direction of $v^{\varkappa_1\dots\varkappa_p}t_{\varkappa_{p-k+1}\dots\varkappa_p}$ and of $''v^{a_1\dots a_{p-k}}$ is contained in the $(p+m-n)$-dimensional intersection of the p-direction of $v^{\varkappa_1\dots\varkappa_p}$ and the local m-direction. By (56) $v^{\varkappa_1\dots\varkappa_p}$ is not uniquely determined for a given $''v^{a_1\dots a_{p-k}}$ (except for $p=n$, $p-k=m$). In case 4 however (and in case 3 for $p=n$) each q-vector $''v^{a_1\dots a_q}$ in X_m determines a $(q+k)$-vector $''v^{\varkappa_1\dots\varkappa_{q+k}}$ in X_n, according to

$$(57)\qquad {}''v^{\varkappa_1\dots\varkappa_{q+k}}=\binom{q+k}{k}{}''v^{a_1\dots a_q}\,B^{[\varkappa_1}_{a_1}\dots B^{\varkappa_q}_{a_q}\,n^{\varkappa_{q+1}\dots\varkappa_{q+k}]},$$

which is uniquely determined by $''v^{a_1\dots a_q}$, the solution of (57) being

$$(58)\qquad {}''v^{a_1\dots a_q}=\frac{1}{k!}\,B^{a_1}_{\varkappa_1}\dots B^{a_q}_{\varkappa_q}v^{\varkappa_1\dots\varkappa_{q+k}}\,t_{\varkappa_{q+1}\dots\varkappa_{q+k}}.$$

By means of (9) and the analogous equation in X_m the equations corresponding with (56), (57) and (58) are found, containing a covariant $(n-p)$-vectordensity of weight -1 in X_n and a covariant $(n-p)$-vectordensity of weight -1 in X_m:

$$(59)\qquad {}''\mathfrak{v}_{b_1\dots b_l}=\mathfrak{z}^{-1}\,B^{\lambda_1}_{b_1}\dots B^{\lambda_l}_{b_l}\,\mathfrak{v}_{\lambda_1\dots\lambda_l};\quad\text{(case 3)}$$

$$\left.\begin{aligned}&(60)\qquad {}''\mathfrak{v}_{\lambda_1\dots\lambda_l}=\mathfrak{z}\,B^{b_1}_{\lambda_1}\dots B^{b_l}_{\lambda_l}\,{}''\mathfrak{v}_{b_1\dots b_l}\\&(61)\qquad {}''\mathfrak{v}_{b_1\dots b_l}=\mathfrak{z}^{-1}\,B^{\lambda_1}_{b_1}\dots B^{\lambda_l}_{b_l}\,{}''\mathfrak{v}_{\lambda_1\dots\lambda_l}\end{aligned}\right\}\ \begin{aligned}&\text{(case 4, for } l=0\\&\text{also case 3)}\end{aligned}$$

where $l=n-p$.

With respect to the special coordinate-system the equations (55) and (56) take the very simple form

$$(62)\qquad {}'v^{\alpha_1\dots\alpha_p}=v^{\alpha_1\dots\alpha_p};\qquad (p\leqq m,\text{ case 2, for } p=0\text{ also case 1})$$

(63) $''v^{\alpha_1 \dots \alpha_{p-k}} = \mathfrak{z}^{-1}\, v^{\alpha_1 \dots \alpha_{p-k},\ m+1, \dots, n}$ $(p \geqq k = n-m,$ case 3$)$,

e.g. for $m = n-1$, $p = 1$, $(k=1, q=0)$, $n \geqq 2$:

$$\left.\begin{array}{lll} (64) & 'v^{\alpha} = v^{\alpha}; & \text{(case 2)} \\ (65) & ''v = \mathfrak{z}^{-1}\, v^{n}; & \text{(case 3)} \end{array}\right\} \text{ together valid only in case 4.}$$

The equations of the corresponding densities are $(l = n-p$, $k = n-m)$

$$'\mathfrak{v}_{\beta_1 \dots \beta_{l-k}} = \mathfrak{v}_{\beta_1 \dots \beta_{l-k},\ m+1, \dots, n} \tag{66}$$

$$''\mathfrak{v}_{\beta_1 \dots \beta_l} = \mathfrak{z}^{-1}\, \mathfrak{v}_{\beta_1 \dots \beta_l}, \tag{67}$$

e.g. for $m = n-1$, $p = n-1$, $(k=1,\ l=1)$

$$'\mathfrak{v} = \mathfrak{v}_n \tag{68}$$

$$''\mathfrak{v} = \mathfrak{z}^{-1}\, \mathfrak{v}. \tag{69}$$

It is remarkable, that in case 4, if $m \geqq p \geqq k = n - m$ (which is only possible if $m \geqq \frac{1}{2}n$) both quantities $'v^{a_1 \dots a_p}$ and $''v^{a_1 \dots a_{p-k}}$ in X_m exist and *that these two quantities together for* $m = n-1$ *determine completely the* p*-vector* $v^{\varkappa_1 \dots \varkappa_p}$ *in* X_n as follows from (62) and (63).

B. *Ordinary covariant* p*-vectors.*

For $p \leqq m$ a covariant p-vector $w_{\lambda_1 \dots \lambda_p}$ in X_n, defined in a point of X_m, always (i.e. in case 1) determines a covariant p-vector in X_m, viz.

$$'w_{b_1 \dots b_p} = B_{b_1}^{\lambda_1} \cdots B_{b_p}^{\lambda_p}\, w_{\lambda_1 \dots \lambda_p}. \tag{70}$$

If $w_{\lambda_1 \dots \lambda_p}$ is simple this quantity is the intersection of $w_{\lambda_1 \dots \lambda_p}$ with the local E_m. By equation (70) $w_{\lambda_1 \dots \lambda_p}$ is not uniquely determined. In case 2 (rigging) however (and in case 1 for $p=0$) each covariant p-vector $'w_{b_1 \dots b_p}$ in X_m determines such a quantity in X_n according to

$$'w_{\lambda_1 \dots \lambda_p} = B_{\lambda_1}^{b_1} \cdots B_{\lambda_p}^{b_p}\, 'w_{b_1 \dots b_p}, \tag{71}$$

which is uniquely determined by $'w_{b_1 \dots b_p}$, the solution of (71) being

$$'w_{b_1 \dots b_p} = B_{b_1}^{\lambda_1} \cdots B_{b_p}^{\lambda_p}\, 'w_{\lambda_1 \dots \lambda_p}. \tag{72}$$

If $'w_{b_1 \dots b_p}$ is simple, the $(n-p)$-direction of $'w_{\lambda_1 \dots \lambda_p}$ contains and is composed of the $(m-p)$-direction of $'w_{b_1 \dots b_p}$ and the

$(n-m)$-direction of the rigging. By means of (10) and the analogous equations in X_m the equations corresponding with (70), (71) and (72) are found, containing a contravariant $(n-p)$-vector-density of weight $+1$ in X_n and a contravariant $(m-p)$-vector-density of weight $+1$ in X_m.

In case 3 (normalisation, existence of $t_{\lambda_1 \ldots \lambda_k}$) a covariant q-vector $''w_{b_1 \ldots b_q}$ in X_m determines also a covariant $(q+k)$-vector $''w_{\lambda_1 \ldots \lambda_{q+k}}$ in X_n, according to

$$(73) \quad ''w_{\lambda_1 \ldots \lambda_{q+k}} = \binom{q+k}{k} ''w_{b_1 \ldots b_q} \beta^{b_1 \ldots b_q}_{\cdot \ \cdot \ [\lambda_1 \ldots \lambda_q} t_{\lambda_{q+1} \ldots \lambda_{q+k}]} ,$$

as the alternated product of $\beta^{b_1 \ldots b_q}_{\cdot \ \cdot \ \lambda_1 \ldots \lambda_q}$ with $t_{\lambda_{q+1} \ldots \lambda_{q+k}}$ is not affected by the ambiguity of $\beta^{b_1 \ldots b_q}_{\cdot \ \cdot \ \lambda_1 \ldots \lambda_q}$.

If $''w_{b_1 \ldots b_q}$ is simple, $''w_{\lambda_1 \ldots \lambda_{q+k}}$ also is; its $(n-q-k)=(m-q)$-direction is the same as the $(m-q)$-direction of $''w_{b_1 \ldots b_q}$ and is contained in the local m-direction of X_m.

In case 4 equations (73) have the solution

$$(74) \qquad ''w_{b_1 \ldots b_q} = \frac{1}{k!} B^{\lambda_1}_{b_1} \cdots B^{\lambda_q}_{b_q} \, ''w_{\lambda_1 \ldots \lambda_{q+k}} \, n^{\lambda_{q+1} \ldots \lambda_{q+k}} .$$

This solution however is valid already in case 3 though $n^{\varkappa_1 \ldots \varkappa}$ is not uniquely determined then. In fact, as the ambiguity of $n^{\varkappa_1 \ldots \varkappa_k}$ consists in alternated products containing a factor B, and as the transvection of $''w_{\lambda_1 \ldots \lambda_{q+k}}$ with $q+1$ or more factors B vanishes, this ambiguity bears no influence upon the left side of (74). Hence in this case $''w_{\lambda_1 \ldots \lambda_{q+k}}$ and $''w_{b_1 \ldots b_q}$ can be considered to represent the same object.

Moreover in case 4 an *arbitrary* covariant $(q+k)$-vector $w_{\lambda_1 \ldots \lambda_{q+k}}$ defined in a point of X_m determines a covariant q-vector in X_m according to

$$(75) \qquad ''w_{b_1 \ldots b_q} = \frac{1}{k!} B^{\lambda_1}_{b_1} \cdots B^{\lambda_q}_{b_q} \, w_{\lambda_1 \ldots \lambda_{q+k}} \, n^{\lambda_{q+1} \ldots \lambda_{q+k}},$$

but evidently is not uniquely determined by it (except for $q=m$, $q+k=n$). $''w_{b_1 \ldots b_q}$ can be called the *projection* of $w_{\lambda_1 \ldots \lambda_{q+k}}$ in the direction of the rigging upon the local E_m tangent to X_m. If we take for $m=n-1$ the special coordinate-system mentioned above, the equation (75) takes the very simple form

$$(76) \qquad ''w_{\beta_1 \ldots \beta_q} = \mathfrak{z} \, w_{\beta_1 \ldots \beta_q n}.$$

By means of (10) and the analogous equation in X_m the equations corresponding with (73), (74) and (75) are found, containing a contravariant $(n-p)$-vectordensity of weight $+1$ in X_n and a contravariant $(n-p)$-vectordensity of weight $+1$ in X_m

$$(77)\qquad ''\mathfrak{w}^{a_1\ldots a_l} = \mathfrak{z}\, B^{a_1\ldots a_l}_{\varkappa_1\ldots\varkappa_l}\, \mathfrak{w}^{\varkappa_1\ldots\varkappa_l} \quad \text{(case 4, for } l=0 \text{ also in case 3)}$$

$$\left.\begin{aligned} (78)\qquad ''\mathfrak{w}^{\varkappa_1\ldots\varkappa_l} &= \mathfrak{z}^{-1}\, B^{\varkappa_1\ldots\varkappa_l}_{a_1\ldots a_l}\, ''\mathfrak{w}^{a_1\ldots a_l} \\ (79)\qquad ''\mathfrak{w}^{a_1\ldots a_l} &= \mathfrak{z}\quad B^{a_1\ldots a_l}_{\varkappa_1\ldots\varkappa_l}\, ''\mathfrak{w}^{\varkappa_1\ldots\varkappa_l} \end{aligned}\right\}\ \text{(case 3)}.$$

With respect to the special coordinatesystem the equations (70) and (75) get the very simple form

$$(80)\qquad 'w_{\beta_1\ldots\beta_p} = w_{\beta_1\ldots\beta_p}\,;\ (p \leqq m,\ \text{case 1})$$

$$(81)\qquad ''w_{\beta_1\ldots\beta_{p-k}} = \mathfrak{z}\, w_{\beta_1\ldots\beta_{p-k},\, m+1,\ldots,n}\,;$$
$$(p \geqq k,\ \text{case 4, for } p=n \text{ also in case 3}),$$

f.i. for $m = n-1$, $p = 1$, $n \geqq 2$

$$(82)\qquad 'w_\beta = w_\beta\,;\ \text{(case 1)}$$

$$(83)\qquad ''w = \mathfrak{z}\, w_n;\ \text{(case 4)}.$$

The equations of the corresponding densities are $(l=n-p)$

$$(84)\qquad '\mathfrak{w}^{\alpha_1\ldots\alpha_{l-k}} = \mathfrak{w}^{\alpha_1\ldots\alpha_{l-k},\, m+1,\ldots,n}$$

$$(85)\qquad ''\mathfrak{w}^{\alpha_1\ldots\alpha_l} = \mathfrak{z}\, \mathfrak{w}^{\alpha_1\ldots\alpha_l}$$

f.i. for $m = n-1$, $p = n-1$

$$(86)\qquad '\mathfrak{w} = \mathfrak{w}_n$$

$$(87)\qquad ''\mathfrak{w} = \mathfrak{z}^{-1}\, \mathfrak{w}.$$

It is remarkable, that in case 4, if $m \geqq p \geqq k$ (only possible if $m \geqq \frac{1}{2}n$) both quantities $'w_{b_1\ldots b_p}$ and $w_{b_1\ldots b_{p-k}}$ in X_m exist and *that these two quantities together determine for* $m = n-1$ *completely the p-vector* $w_{\lambda_1\ldots\lambda_p}$ *in* X_n as follows from (80) and (81).

We collect the results in the following table [15]):

Table 5.

X_n $p \leqq m$	Case 1	Case 2	X_m	X_n $p \geqq k$	Case 3	Case 4	X_m
$v^{\varkappa_1 \cdots \varkappa_p}$ $\mathfrak{v}_{\lambda_1 \cdots \lambda_l}$	$\rightleftarrows p = 0$	B^a_λ $\longrightarrow$	$'v^{a_1 \cdots a_p}$ $'\mathfrak{v}_{b_1 \cdots b_{m-p}}$	$v^{\varkappa_1 \cdots \varkappa_p}$ $\mathfrak{v}_{\lambda_1 \cdots \lambda_l}$	(B^a_λ), t $\longrightarrow$	B^a_λ, t $\longrightarrow$	$''v^{a_1 \cdots a_{p-k}}$ $''\mathfrak{v}_{b_1 \cdots b_l}$
$'v^{\varkappa_1 \cdots \varkappa_p}$ $'\mathfrak{v}_{\lambda_1 \cdots \lambda_l}$	$B^\varkappa_b$ $\leftrightarrows$ (B^a_λ)	$B^\varkappa_b$ $\leftrightarrows$ B^a_λ	$'v^{a_1 \cdots a_p}$ $'\mathfrak{v}_{b_1 \cdots b_{m-p}}$	$''v^{\varkappa_1 \cdots \varkappa_p}$ $''\mathfrak{v}_{\lambda_1 \cdots \lambda_l}$	$B^\varkappa_b$, (n) $\leftrightarrows p = n$ (B^a_λ), t	$B^\varkappa_b$, n $\leftrightarrows$ B^a_λ, t	$''v^{a_1 \cdots a_{p-k}}$ $''\mathfrak{v}_{b_1 \cdots b_l}$
$w_{\lambda_1 \cdots \lambda_p}$ $\mathfrak{w}^{\varkappa_1 \cdots \varkappa_l}$	$B^\varkappa_b$ $\longrightarrow$	$B^\varkappa_b$ $\longrightarrow$	$'w_{b_1 \cdots b_p}$ $'\mathfrak{w}^{a_1 \cdots a_{m-p}}$	$w_{\lambda_1 \cdots \lambda_p}$ $\mathfrak{w}^{\varkappa_1 \cdots \varkappa_l}$	$B^\varkappa_b$, (n) $\rightleftarrows p = n$ (B^a_λ), t	$B^\varkappa_b$, n $\longrightarrow$	$''w_{b_1 \cdots b_{p-k}}$ $''\mathfrak{w}^{a_1 \cdots a_l}$
$'w_{\lambda_1 \cdots \lambda_p}$ $'\mathfrak{w}^{\varkappa_1 \cdots \varkappa_l}$	$\leftrightarrows p = 0$	B^a_λ $\leftrightarrows$ $B^\varkappa_b$	$'w_{b_1 \cdots b_p}$ $'\mathfrak{w}^{a_1 \cdots a_{m-p}}$	$''w_{\lambda_1 \cdots \lambda_p}$ $''\mathfrak{w}^{\varkappa_1 \cdots \varkappa_l}$	(B^a_λ), t $\leftrightarrows$ $B^\varkappa_b$, (n)	B^a_λ, t $\leftrightarrows$ $B^\varkappa_b$, n	$''w_{b_1 \cdots b_{p-k}}$ $''\mathfrak{w}^{a_1 \cdots a_l}$

$k = n - m; \quad l = n - p.$

C. *W-Quantities.*

For W-p-vectors and their corresponding W-$(n-p)$-vector-densities the equations are:

Contravariant W-p-vectors, $p \leqq m$:

$$\left.\begin{aligned} &(50W) \quad '\overset{\frown}{v}^{\varkappa_1 \cdots \varkappa_p} = \omega\, B^{\varkappa_1}_{a_1} \cdots B^{\varkappa_p}_{a_p}\, '\overset{\frown}{v}^{a_1 \cdots a_p} \\ &(54W) \quad '\overset{\frown}{v}^{a_1 \cdots a_p} = \omega\, B^{a_1}_{\varkappa_1} \cdots B^{a_p}_{\varkappa_p}\, '\overset{\frown}{v}^{\varkappa_1 \cdots \varkappa_p} \end{aligned}\right\} \text{(case } 1'\text{)}.$$

$$(55W) \quad '\overset{\frown}{v}^{a_1 \cdots a_p} = \omega\, B^{a_1}_{\varkappa_1} \cdots B^{a_p}_{\varkappa_p} \overset{\frown}{v}^{\varkappa_1 \cdots \varkappa_p} \quad \text{(case } 2'\text{, for } p = 0 \text{ also case } 1'\text{)};$$

Contravariant W-p-vectors, $p \geqq k$:

$$(56W) \quad ''\overset{\frown}{v}^{a_1 \cdots a_{p-k}} = \frac{1}{k!} B^{a_1}_{\varkappa_1} \cdots B^{a_{p-k}}_{\varkappa_{p-k}} \overset{\frown}{v}^{\varkappa_1 \cdots \varkappa_p} \tau_{\varkappa_{p-k+1} \cdots \varkappa_p} \quad \text{(case } 3'\text{)};$$

[15]) The arrows in the 2nd and 3rd column show which of the quantities in the 1st and 4th column is determined by which, and analogously in the last four columns. Near the arrows the quantities are written, used for the determination. Quantities, which are not uniquely determined but can be used because their ambiguity does not affect the result, are written in brackets.

$$\left.\begin{aligned}&(57W)\quad {}''\overset{\frown}{v}{}^{\varkappa_1\ldots\varkappa_p} = \binom{p}{k}\, {}''\overset{\frown}{v}{}^{a_1\ldots a_{p-k}}\, B^{[\varkappa_1}_{a_1}\cdots B^{\varkappa_{p-k}}_{a_{p-k}}\, v^{\varkappa_{p-k}\ldots\varkappa_p]}\\ &(58W)\quad {}''\overset{\frown}{v}{}^{a_1\ldots a_{p-k}} = \frac{1}{k!} B^{a_1}_{\varkappa_1}\cdots B^{a_{p-k}}_{\varkappa_{p-k}}\, \overset{\frown}{v}{}^{\varkappa_1\ldots\varkappa_p}\, \tau_{\varkappa_{p-k+1}\ldots\varkappa_p}\end{aligned}\right\}\ \text{(case 4}', \text{ for } p=n \text{ also case 3}').$$

Covariant W-$(n-p)$-vectordensities of weight -1, $p \geqq k$:

$$(59W)\quad {}''\overset{\frown}{\mathfrak{v}}_{b_1\ldots b_l} = |\,\mathfrak{z}\,|^{-1} B^{\lambda_1}_{b_1}\cdots B^{\lambda_l}_{b_l}\, {}'\overset{\frown}{\mathfrak{v}}_{\lambda_1\ldots\lambda_l}\quad \text{(case 3}');$$

$$\left.\begin{aligned}&(60W)\quad {}''\overset{\frown}{\mathfrak{v}}_{\lambda_1\ldots\lambda_l} = |\,\mathfrak{z}\,|\, B^{b_1}_{\lambda_1}\cdots B^{b_l}_{\lambda_l}\, {}''\overset{\frown}{\mathfrak{v}}_{b_1\ldots b_l}\\ &(61W)\quad {}''\overset{\frown}{\mathfrak{v}}_{b_1\ldots b_l} = |\,\mathfrak{z}\,|^{-1} B^{\lambda_1}_{b_1}\cdots B^{\lambda_l}_{b_l}\, {}''\overset{\frown}{\mathfrak{v}}_{\lambda_1\ldots\lambda_l}\end{aligned}\right\}\ \text{(case 4}', \text{ for } l=0 \text{ also case 3}').$$

Contravariant W-p-vectors w.r. to special coordinatesystem:

$$(62W)\quad {}'\overset{\frown}{v}{}^{\alpha_1\ldots\alpha_p} = \omega\overset{\frown}{v}{}^{\alpha_1\ldots\alpha_p};\ (p \leqq m,\ \text{case 2}',\ \text{for } p=0 \text{ also case 1}')$$

$$(63W)\quad {}''\overset{\frown}{v}{}^{\alpha_1\ldots\alpha_{p-k}} = |\,\mathfrak{z}\,|^{-1}\, \overset{\frown}{v}{}^{\alpha_1\ldots\alpha_{p-k},\, m+1,\ldots,n}\quad (p \geqq k,\ \text{case 3}');$$

e.g. for $m = n-1$, $p = 1$, $n \geqq 2$:

$$\left.\begin{aligned}&(64W)\quad {}'\overset{\frown}{v}{}^{\alpha} = \omega\overset{\frown}{v}{}^{\alpha};\ \text{(case 2}')\\ &(65W)\quad {}''\overset{\frown}{v} = |\,\mathfrak{z}\,|^{-1}\overset{\frown}{v}{}^{n};\ \text{(case 3}')\end{aligned}\right\}\ \text{together valid only in case 4 (not in case 4}').$$

Covariant W-$(n-p)$-vectordensities of weight -1 w.r. to special coordinatesystem:

$$(66W)\quad {}'\overset{\frown}{\mathfrak{v}}_{\beta_1\ldots\beta_{l-k}} = \overset{\frown}{\mathfrak{v}}_{\beta_1\ldots\beta_{l-k},\, m+1,\ldots,n}\quad \left\{\ (p \leqq m,\ \text{case 2}',\ \text{for } p=0 \text{ also case 1}')\right.$$

$$(67W)\quad {}''\overset{\frown}{\mathfrak{v}}_{\beta_1\ldots\beta_l} = \mathfrak{z}^{-1}\, \overset{\frown}{\mathfrak{v}}_{\beta_1\ldots\beta_l}\quad (p \geqq k,\ \text{case 3}');$$

e.g. for $m = n-1$, $p = n-1$, $n \geqq 2$:

$$\left.\begin{aligned}&(68W)\quad {}'\overset{\frown}{\mathfrak{v}} = \omega\, \overset{\frown}{\mathfrak{v}}_n\ \text{(case 2}');\\ &(69W)\quad {}''\overset{\frown}{\mathfrak{v}}_\beta = |\,\mathfrak{z}\,|^{-1}\, \overset{\frown}{\mathfrak{v}}_\beta\ \text{(case 3}');\end{aligned}\right\}\ \text{together only valid in case 4 (not in case 4}').$$

Covariant W-p-vectors, $p \leqq m$:

$$(70W)\quad {}'\overset{\frown}{w}_{b_1\ldots b_p} = \omega\, B^{\lambda_1}_{b_1}\cdots B^{\lambda_p}_{b_p}\, \overset{\frown}{w}_{\lambda_1\ldots\lambda_p}\quad \text{(case 1}');$$

$$\left.\begin{aligned}&(71W)\quad {}'w_{\lambda_1\ldots\lambda_p} = \omega\, B^{b_1}_{\lambda_1}\cdots B^{b_p}_{\lambda_p}\, {}'\overset{\frown}{w}_{b_1\ldots b_p}\\ &(72W)\quad {}'\overset{\frown}{w}_{b_1\ldots b_p} = \omega\, B^{\lambda_1}_{b_1}\cdots B^{\lambda_p}_{b_p}\, {}'\overset{\frown}{w}_{\lambda_1\ldots\lambda_p}\end{aligned}\right\}\ \text{(case 2}', \text{ for } p=0 \text{ also case 1}')$$

Covariant W-p-vectors, $p \geqq k$:

$$(73W)\quad {}''\overset{\frown}{w}_{\lambda_1\ldots\lambda_p} = \binom{p}{k}\, {}''\overset{\frown}{w}_{b_1\ldots b_{p-k}}\, B^{b_1}_{[\lambda_1}\cdots B^{b_{p-k}}_{\lambda_{p-k}}\, \tau_{\lambda_{p-k+1}\ldots\lambda_p]}\quad \text{(case 3}')$$

$$(74W)\quad {}''\overset{\frown}{w}_{b_1\ldots b_{p-k}} = \frac{1}{k!} B^{\lambda_1}_{b_1} \cdots B^{\lambda_{p-k}}_{b_{p-k}} {}''\overset{\frown}{w}_{\lambda_1\ldots\lambda_p} v^{\lambda_{p-k+1}\ldots\lambda_p} \qquad \text{(case 3')}$$

$$(75W)\quad {}''\overset{\frown}{w}_{b_1\ldots b_{p-k}} = \frac{1}{k!} B^{\lambda_1}_{b_1} \cdots B^{\lambda_{p-k}}_{b_{p-k}} \overset{\frown}{w}_{\lambda_1\ldots\lambda_p} v^{\lambda_{p-k+1}\ldots\lambda_p} \qquad \text{(case 4', for } p=n \text{ also case 3').}$$

Contravariant W-$(n-p)$-vectordensities of weight $+1$, $p \geqq k$:

$$(77W)\quad {}''\overset{\frown}{\mathfrak{w}}^{a_1\ldots a_l} = |\mathfrak{z}|\, B^{a_1}_{\varkappa_1} \cdots B^{a_l}_{\varkappa_l} \overset{\frown}{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_l} \qquad \text{(case 4', for } l=0; \text{ also case 3')}$$

$$(78W)\quad {}''\overset{\frown}{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_l} = |\mathfrak{z}|^{-1} B^{\varkappa_1}_{a_1} \cdots B^{\varkappa_l}_{a_l} {}''\overset{\frown}{\mathfrak{w}}^{a_1\ldots a_l}$$

$$(79W)\quad {}''\overset{\frown}{w}^{a_1\ldots a_l} = |\mathfrak{z}|\, B^{a_1}_{\varkappa_1} \cdots B^{a_l}_{\varkappa_l} {}''\overset{\frown}{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_l}$$

(case 3').

Covariant W-p-vectors w.r. to special coordinatesystem:

$$(80W)\quad {}'\overset{\frown}{w}_{\beta_1\ldots\beta_p} = \omega\, \overset{\frown}{w}_{\beta_1\ldots\beta_p} \qquad (p \leqq m, \text{ case 1'});$$

$$(81W)\quad {}''\overset{\frown}{w}_{\beta_1\ldots\beta_{p-k}} = |\mathfrak{z}|\, \overset{\frown}{w}_{\beta_1\ldots\beta_{p-k}m+1,\ldots,n} \qquad (p \geqq k, \text{ case 4', for } p=n \text{ also in case 3'});$$

f.i. for $m = n-1$, $p = 1$, $n \geqq 2$

$$(82W)\quad {}'\overset{\frown}{w}_\beta = \omega\, \overset{\frown}{w}_\beta; \text{ (case 1')}$$

$$(83W)\quad {}''\overset{\frown}{w} = |\mathfrak{z}|\, \overset{\frown}{w}_n; \text{ (case 4')}$$

together valid only in case 4 (not in case 4').

Contravariant W-$(n-p)$-vectordensities of weight $+1$ w.r. to special coordinatesystem:

$$(84W)\quad {}'\overset{\frown}{\mathfrak{w}}^{\alpha_1\ldots\alpha_{l-k}} = \omega\, \overset{\frown}{\mathfrak{w}}^{\alpha_1\ldots\alpha_{l-k}\, m+1,\ldots,n}; \qquad (p \leqq m, \text{ case 1'})$$

$$(85W)\quad {}''\overset{\frown}{\mathfrak{w}}^{\alpha_1\ldots\alpha_l} = |\mathfrak{z}|\, \overset{\frown}{\mathfrak{w}}^{\alpha_1\ldots\alpha_l} \qquad (p \geqq k, \text{ case 4', for } p=n \text{ also case 3'}),$$

f.i. for $m = n-1$, $p = n-1$, $n \geqq 2$.

$$(86W)\quad {}'\overset{\frown}{\mathfrak{w}} = \omega\, \overset{\frown}{\mathfrak{w}}^n \text{ (case 1')}$$

$$(87W)\quad {}''\overset{\frown}{\mathfrak{w}}^\alpha = |\mathfrak{z}|\, \overset{\frown}{\mathfrak{w}}^\alpha \text{ (case 4')}$$

together valid only in case 4 (not case 4').

From (62W) and (63W) follows that for $m = n-1$ the W-p-vector $\overset{\frown}{v}^{\varkappa_1\ldots\varkappa_p}$ is completely determined by ${}'\overset{\frown}{v}^{a_1\ldots a_p}$ and ${}''\overset{\frown}{v}^{a_1\ldots a_{p-1}}$. The same holds for $\overset{\frown}{w}_{\lambda_1\ldots\lambda_p}$, ${}'\overset{\frown}{w}_{b_1\ldots b_p}$ and ${}''\overset{\frown}{w}_{b_1\ldots b_{p-1}}$ in (80 W) and (81 W). All formulae differ from the corresponding ones for ordinary quantities only by a factor ω at the right. From this follows that the cases 1', 2', 3', 4' play now the same role as the cases 1, 2, 3, 4 for ordinary quantities [16]). We collect the results in the following table:

Table 6.

X_n $p \leqq m$	Case 1′	Case 2′	X_m	X_n $p \geqq k$	Case 3′	Case 4′	X_n
$\overleftarrow{v}^{\varkappa_1\ldots\varkappa_p}$ $\overleftarrow{\mathfrak{v}}_{\lambda_1\ldots\lambda_l}$	$\rightleftarrows p=0$	$B^a_\lambda,\ \omega$ $\longrightarrow$	$'\overleftarrow{v}^{a_1\ldots a_p}$ $'\overleftarrow{\mathfrak{v}}_{b_1\ldots b_{m-p}}$	$\overleftarrow{v}^{\varkappa_1\ldots\varkappa_p}$ $\overleftarrow{\mathfrak{v}}_{\lambda_1\ldots\lambda_l}$	$(B^a_\lambda),\ \tau$ $\longrightarrow$	$B^a_\lambda,\ \tau$ $\longrightarrow$	$''v^{a_1\ldots a_{p-k}}$ $''\mathfrak{v}_{b_1\ldots b_l}$
$'\overleftarrow{v}^{\varkappa_1\ldots\varkappa_p}$ $'\overleftarrow{\mathfrak{v}}_{\lambda_1\ldots\lambda_l}$	$B^\varkappa_b,\ \omega$ $\leftrightarrows$ $(B^a_\lambda),\ \omega$	$B^\varkappa_b,\ \omega$ $\leftrightarrows$ $B^a_\lambda,\ \omega$	$'\overleftarrow{v}^{a_1\ldots a_p}$ $'\overleftarrow{\mathfrak{v}}_{b_1\ldots b_{m-p}}$	$''v^{\varkappa_1\ldots\varkappa_p}$ $''\overleftarrow{\mathfrak{v}}_{\lambda_1\ldots\lambda_l}$	$B^\varkappa_b,\ (\nu)$ $\leftrightarrows p=n$ $(B^a_\lambda),\ \tau$	$B^\varkappa_b,\ \nu$ $\leftrightarrows$ $B^a_\lambda,\ \tau$	$''v^{\ldots a\ -k}$ $''\mathfrak{v}_{b_1\ldots b_l}$
$\overleftarrow{w}_{\lambda_1\ldots\lambda_p}$ $\overleftarrow{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_l}$	$B^\varkappa_b,\ \omega$ $\longrightarrow$	$B^\varkappa_b,\ \omega$ $\longrightarrow$	$'\overleftarrow{w}_{b_1\ldots b_p}$ $'\overleftarrow{\mathfrak{w}}^{a_1\ldots a_{m-p}}$	$\overleftarrow{w}_{\lambda_1\ldots\lambda_p}$ $\overleftarrow{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_l}$	$B^\varkappa_b,\ (\nu)$ $\rightleftarrows p=n$ $(B^a_\lambda),\ \tau$	$B^\varkappa_b,\ \nu$ $\longrightarrow$	$''w_{b_1\ldots b_{p-k}}$ $''\mathfrak{w}^{a_1\ldots a_l}$
$'\overleftarrow{w}_{\lambda_1\ldots\lambda_p}$ $'\overleftarrow{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_l}$	$\leftrightarrows p=0$	$B^a_\lambda,\ \omega$ $\leftrightarrows$ $B^\varkappa_b,\ \omega$	$'\overleftarrow{w}_{b_1\ldots b_p}$ $'\overleftarrow{\mathfrak{w}}^{a_1\ldots a_{m-p}}$	$''\overleftarrow{w}_{\lambda_1\ldots\lambda_p}$ $''\overleftarrow{\mathfrak{w}}^{\varkappa_1\ldots\varkappa_l}$	$(B^a_\lambda),\ \tau$ $\leftrightarrows$ $B^\varkappa_b,\ (\nu)$	$B^a_\lambda,\ \tau$ $\leftrightarrows$ $B^\varkappa_b,\ \nu$	$''w_{b_1\ldots b_{p-k}}$ $''\mathfrak{w}^{a_1\ldots a_l}$

From (64, 65, 68, 69, 82, 83, 86, 87) and (64W, 65W, 68W, 69W, 82W, 83W, 86W, 87W) it is easy to deduce the quantities in X_{n-1} derived from an affinor or affinordensity in X_n, which can be written as a sum of products of vectors and vectordensities of suitable weighs. E.g. the W-affinordensity $\overleftarrow{\mathfrak{P}}^{\varkappa}_{\lambda}$ of weight $+1$ can be written as a sum of products of a covariant vector and a contravariant W-vectordensity of weight $+1$ and therefore, according to (82, 83, 86W, 87W) gives rise to the following quantities in X_{n-1}:

$$(88)\left\{\begin{array}{lll} |\mathfrak{z}|\,\overleftarrow{\mathfrak{P}}^{\alpha}_{\beta} & ; & W\text{-affinordensity of weight } +1,\ (82,\ 87W,\ \text{case } 4'). \\ \mathfrak{z}\,|\mathfrak{z}|\,\overleftarrow{\mathfrak{P}}^{\alpha}_{n} & ; & \text{contr. } W\text{-vectordensity of weight } +1,\ (83,\ 87W,\ \text{case } 4). \\ \omega\,\overleftarrow{\mathfrak{P}}^{n}_{\beta} & ; & \text{cov. } W\text{-vectordensity of weight } +1,\ (82,\ 86W,\ \text{case } 1'). \\ |\mathfrak{z}|\,\overleftarrow{\mathfrak{P}}^{n}_{n} & ; & W\text{-density of weight } +1,\ (83,\ 86W,\ \text{case } 4'). \end{array}\right.$$

[16]) But it must be remarked that 4′ does not include 1′, 2′ and 3′ but only 1, 2 and 3′.

9. Generalisation of Stokes formulae for ordinary quantities and W-quantities.

In X_n we consider an orientable and closed X_m called τ_m, bounding a simply connected part τ_{m+1} of an orientable X_{m+1}. The m-dimensional element with an interior orientation be $\mathrm{d}f^{\varkappa_1 \dots \varkappa_m}$, the $(m+1)$-dimensional element with an interior orientation of X_{m+1} be $\mathrm{d}f^{\varkappa_1 \dots \varkappa_{m+1}}$. The orientations are chosen in such a way that the direction from an interior point of τ_{m+1} towards the boundary followed by the orientation of $\mathrm{d}f^{\varkappa_1 \dots \varkappa_m}$ gives the orientation of $\mathrm{d}f^{\varkappa_1 \dots \varkappa_{m+1}}$. Be now $v_{\lambda_1 \dots \lambda_m}$ an m-vector field that satisfies the ordinary conditions of continuity. Then in a well known way we may derive [17])

$$(89) \qquad \int_{\tau_{m+1}} (\partial_\mu v_{\lambda_1 \dots \lambda_m})\, \mathrm{d}f^{\mu\lambda_1 \dots \lambda_m} = \int_{\tau_m} v_{\lambda_1 \dots \lambda_m}\, \mathrm{d}f^{\lambda_1 \dots \lambda_m}.$$

If we introduce in this formula $\mathfrak{v}^{\varkappa_{m+1} \dots \varkappa_n}$, $\mathfrak{f}_{\lambda_{m+1} \dots \lambda_n}$ and $\mathfrak{f}_{\lambda_{m+1} \dots \lambda_{n-1}}$ we get four other forms: [18])

$$(90) \qquad \frac{1}{h!} \int_{\tau_{m+1}} \mathrm{d}\, \mathfrak{f}_{\lambda_1 \dots \lambda_h}\, \partial_\mu\, \mathfrak{v}^{\lambda_1 \dots \lambda_h \mu} = \frac{1}{(h+1)!} \int_{\tau_m} \mathrm{d}\, \mathfrak{f}_{\lambda_1 \dots \lambda_{h+1}}\, \mathfrak{v}^{\lambda_1 \dots \lambda_{h+1}}; \qquad h = n - m - 1$$

$$(91) \qquad \frac{1}{h!} \int_{\tau_{m+1}} (\partial_{[\mu}\, v_{\lambda_1 \dots \lambda_m})\, \mathrm{d}\, \mathfrak{f}_{\varkappa_1 \dots \varkappa_h]} = \frac{1}{(h+1)!} \int_{\tau_m} v_{[\lambda_1 \dots \lambda_m}\, \mathrm{d}\, \mathfrak{f}_{\varkappa_1 \dots \varkappa_h \mu]}$$

$$(92) \qquad \int_{\tau_{m+1}} \mathrm{d}f^{\mu[\lambda_1 \dots \lambda_m}\, \partial_\mu\, \mathfrak{v}^{\varkappa_1 \dots \varkappa_h]} = \int_{\tau_m} \mathrm{d}f^{[\lambda_1 \dots \lambda_m}\, \mathfrak{v}^{\varkappa_1 \dots \varkappa_h]}$$

---

[17]) SCHOUTEN-STRUIK, Einführung I, p. 130; Ricci-Kalkül, p. 97, (204); earlier literature is mentioned there. $\mathrm{d}f^{\varkappa_1 \dots \varkappa_m}$ and $\mathrm{d}f^{\varkappa_1 \dots \varkappa_{m+1}}$ differ by a factor $m!$ and $(m+1)!$ resp. from $f^{\varkappa_1 \dots \varkappa_m}\, d\tau_m$ and $f^{\varkappa_1 \dots \varkappa_{m+1}}\, d\tau_{m+1}$ used in R.K.

[18]) The formulae (90), (91) an (92) correspond with (211), (210) and (208) in R.K. p. 98 where still multivectors in stead of multivectordensities were used. (90) occurs for the special but typical case of Maxwell's equations in D. VAN DANTZIG, The fundamental equations of electromagnetism independent of metrical geometry [Proc. Cambr. Phil. Soc. **30** (1934), 421—427]. The formulae (89—93) occur as (I), (II), (IV), (III′) and (III) in J. VAN WEYSSENHOFF, Duale Größen, Großrotation, Großdivergenz und die Stokes-Gaußchen Sätze in allgemeinen Räumen [Ann. de la Soc. Pol. de Math. **16** (1937), 127—144], 141, 142. In III there is a misprint, the index $\varkappa$ being not excluded from the alternation.

$$(93)\quad \binom{n}{m+1}\int\limits_{\tau_{m+1}} \mathrm{d}\,f^{[\nu\lambda_1\cdots\lambda_m}\,\partial_\mu\,\mathfrak{v}^{\varkappa_1\cdots\varkappa_h]\mu} = \binom{n}{m}\int\limits_{\tau_m}\mathrm{d}\,f^{[\lambda_1\cdots\lambda_m}\,\mathfrak{v}^{\varkappa_1\cdots\varkappa_h\nu]}.$$

The formulae (89) and (90) are valid for an X_m in X_n with an interior orientation. But they cannot be used in the case more frequently occurring in physical applications of an X_m with an exterior orientation (inducing also an exterior orientation in τ_{m+1}). In order to derive the formulae of Stokes for this case we introduce the following W-quantities

$$(94)\quad \mathrm{d}\overset{\frown}{f}^{\varkappa_1\cdots\varkappa_{m+1}} = \overset{\frown}{k}\,\mathrm{d}\,f^{\varkappa_1\cdots\varkappa_{m+1}};\quad \mathrm{d}\overset{\frown}{\mathfrak{f}}_{\lambda_1\cdots\lambda_h} = \overset{\frown}{k}\,\mathrm{d}\,\mathfrak{f}_{\lambda_1\cdots\lambda_h}$$

$$(95)\quad \mathrm{d}\overset{\frown}{f}^{\varkappa_1\cdots\varkappa_m} = \overset{\frown}{k}\,\mathrm{d}\,f^{\varkappa_1\cdots\varkappa_m};\quad \mathrm{d}\,\overset{\frown}{\mathfrak{f}}_{\lambda_1\cdots\lambda_{h+1}} = \overset{\frown}{k}\,\mathrm{d}\,\mathfrak{f}_{\lambda_1\cdots\lambda_{h+1}}$$

$$(96)\quad \overset{\frown}{v}_{\lambda_1\cdots\lambda_m} = \overset{\frown}{k}\,v_{\lambda_1\cdots\lambda_m};\quad \overset{\frown}{\mathfrak{v}}^{\varkappa_1\cdots\varkappa_{h+1}} = \overset{\frown}{k}\,\mathfrak{v}^{\varkappa_1\cdots\varkappa_{h+1}},$$

where $\overset{\frown}{k}$ is a W-scalar which has with respect to some given coordinatesystem e.g. $(\varkappa)$ the constant value $+1$ and transforms according to the formula

$$(97)\quad \overset{(\varkappa')}{\overset{\frown}{k}} = \frac{\Delta}{|\Delta|}\,\overset{(\varkappa)}{\overset{\frown}{k}}.$$

Then we get

$$(98)\quad \int\limits_{\tau_{m+1}}(\partial_{\lambda_1}\overset{\frown}{v}_{\lambda_2\cdots\lambda_{m+1}})\,\mathrm{d}\overset{\frown}{f}^{\lambda_1\cdots\lambda_{m+1}} = \int\limits_{\tau_m}\overset{\frown}{v}_{\lambda_1\cdots\lambda_m}\,\mathrm{d}\overset{\frown}{f}^{\lambda_1\cdots\lambda_m}$$

$$(99)\quad \frac{1}{h!}\int\limits_{\tau_{m+1}}\mathrm{d}\,\overset{\frown}{\mathfrak{f}}_{\lambda_1\cdots\lambda_h}\,\partial_\mu\,\overset{\frown}{\mathfrak{v}}^{\lambda_1\cdots\lambda_h\mu} = \frac{1}{(h+1)!}\int\limits_{\tau_m}\mathrm{d}\,\overset{\frown}{\mathfrak{f}}_{\lambda_1\cdots\lambda_{h+1}}\,\overset{\frown}{\mathfrak{v}}^{\lambda_1\cdots\lambda_{h+1}}$$

$$(100)\quad \frac{1}{h!}\int\limits_{\tau_{m+1}}(\partial_{[\mu}\overset{\frown}{v}_{\lambda_1\cdots\lambda_m})\,\mathrm{d}\overset{\frown}{\mathfrak{f}}_{\varkappa_1\cdots\varkappa_h]} = \frac{1}{(h+1)!}\int\limits_{\tau_m}\overset{\frown}{v}_{[\lambda_1\cdots\lambda_m}\,\mathrm{d}\overset{\frown}{\mathfrak{f}}_{\varkappa_1\cdots\varkappa_h\mu]}$$

$$(101)\quad \int\limits_{\tau_{m+1}}\mathrm{d}\overset{\frown}{f}^{\mu[\lambda_1\cdots\lambda_m}\,\partial_\mu\overset{\frown}{\mathfrak{v}}^{\varkappa_1\cdots\varkappa_h]} = \int\limits_{\tau_m}\mathrm{d}\overset{\frown}{f}^{[\lambda_1\cdots\lambda_m}\,\overset{\frown}{\mathfrak{v}}^{[\varkappa_1\cdots\varkappa_h]}$$

$$(102)\quad \binom{n}{m+1}\int\limits_{\tau_{m+1}}\mathrm{d}\overset{\frown}{f}^{[\nu\lambda_1\cdots\lambda_m}\,\partial_\mu\overset{\frown}{\mathfrak{v}}^{\varkappa_1\cdots\varkappa_h]\mu} = \binom{n}{m}\int\limits_{\tau_m}\mathrm{d}\overset{\frown}{f}^{[\lambda_1\cdots\lambda_m}\,\overset{\frown}{\mathfrak{v}}^{\varkappa_1\cdots\varkappa_h\nu]}.$$

They evidently are independent of the choice of $\overset{\frown}{k}$ and therefore invariant under arbitrary transformations of coordinates.

If we consider only changes of the coordinatesystem for which Δ in τ_{m+1} and on τ_m has everywhere the same sign, also formulae hold like

$$(103)\quad \frac{1}{h!}\int\limits_{\tau_{m+1}} \mathrm{d}\bar{\mathfrak{f}}_{\lambda_1\cdots\lambda_h}\,\partial_\mu\,\mathfrak{v}^{\lambda_1\cdots\lambda_h\mu} = \frac{1}{(h+1)!}\int\limits_{\tau_m} \mathrm{d}\bar{\mathfrak{f}}_{\lambda_1\cdots\lambda_{h+1}}\,\mathfrak{v}^{\lambda_1\cdots\lambda_{h+1}}.$$

They express equalities between two W-scalars, which in this special case can be defined for the whole region concerned.

(Received July 3rd, 1939.)