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On ordinary quantities and \( W \)-quantities. Classification and geometrical applications

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On ordinary quantities and $W$-quantities

Classification and geometrical applications

by

J. A. Schouten and D. van Dantzig

Delft

1. Interior and exterior orientation.

We consider an $E_n$ 1) in which an orientation ($n$-dimensional screw-sense) is determined by an ordered sequence of $n$ independent directions each with a definite sense 2). Now suppose an $E_p$, $0 < p < n$ to be given in $E_n$. This $E_p$ in $E_n$ determines uniquely an $E_{n-p}$ (not lying in $E_n$) in the following way: all $E_p$'s totally parallel with the given one can be considered as elements of a set, which is an $(n-p)$-dimensional plane manifold in which an affine geometry is induced, i.e. an $E_{n-p}$. This process of obtaining the $E_{n-p}$ is called „Zusammenlegung” by Weyl (analogous to „stetige Zerlegung” in topology) and is also described by saying that in each $E_p$ all points are „identified”.

Now we can either define a $p$-dimensional orientation in the $E_p$ or an $(n-p)$-dimensional orientation in the $E_{n-p}$. In the first case we say that the $E_p$ as well as its $p$-direction has got an interior orientation, in the second case we say that it is provided with an exterior orientation. The notions of interior and exterior orientation were introduced by Veblen and Whitehead 3). For $p = 0$ and $p = n$ we define the orientation as follows:

The interior orientation of an $E_0$ is a + or a --sign, the exterior one is an ordinary orientation of the $E_n$. The interior orientation of the $E_n$ is just this ordinary orientation, the exterior one is a + or --sign.

1) $E_n = n$-dimensional space with ordinary affine geometry.
2. Contra- and covariant $p$-vectors. 4)

Let $x^\kappa (\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau = 1, \ldots, n)$ be cartesian coordinates in an $E_n$. Any other system $\tilde{x}^{\kappa'} (\kappa', \lambda', \ldots, \tau' = 1', \ldots, n')$ of cartesian coordinates in this $E_n$ is connected with the first one by equations of the form:

\[(1) \quad \tilde{x}^{\kappa'} = A^{\kappa'}_\kappa x^\kappa + A_0^{\kappa'}; \quad \text{Det} (A^{\kappa'}_\kappa) \neq 0\]

where the $A^{\kappa'}_\kappa$ and $A_0^{\kappa'}$ are constants.

A contravariant $p$-vector $v^{x_1 \ldots x_p}$ is defined by its transformation formula

\[(2) \quad v^{x_1 \ldots x_p}_1 = A^{x_1'}_{x_1} \ldots A^{x_p'}_{x_p} v^{x_1 \ldots x_p}_1\]

and its property of being alternating with respect to all suffixes. If it is simple (i.e. the alternated product of $p$ vectors) it can be represented by a (e.g. simply connected) part of an $E_p$ with an interior orientation. Two such parts determine the same $p$-vector if and only if 1° the $E_p$'s are totally parallel, 2° the two $p$-dimensional volumes are equal, 3° the orientations are the same.

The $(\kappa_1, \ldots, \kappa_p)$-component is determined by the projection of the oriented part of the $E_p$ upon the $E_p$ of the contravariant measuring vectors $e^{x_1}, \ldots, e^{x_p}$; the $(n-p)$-direction in which the

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4) The results of § 2—5 were first published in J. A. Schouten, Über die geometrische Deutung von gewöhnlichen $p$-Vektoren und $W-p$-Vektoren und den korrespondierenden Dichten. [Proc. Amsterdam, 41 (1938), 709—716].

5) The orientation of the parallelogram on $v^{x_1}, e^{x_1}$ is the one belonging to $e^{(x_1)}_x$. 
projection is performed is the \((n-p)\)-direction common to the 
\((n-1)\)-directions of the covariant measuring vectors \(e^1_\lambda, \ldots, e^p_\lambda\).
The value of this \((x_1, \ldots, x_p)\)-component is the \(p\)-dimensional 
volume of the projection as measured by the \(p\)-dimensional 
volume of the parallelotope with edges \(e^1_\xi, \ldots, e^p_\xi\) and provided
with a factor \(+1\) or \(-1\) if its orientation is the same or op-
posite as the orientation of \(e^1_\lambda, \ldots, e^p_\lambda\) in this order.

The projection of the orientation becomes undetermined if
and only if the \(E_p\) of \(\xi^1 \cdot \xi^p\) has a direction in common with
the \((n-p)\)-direction of the projection, in which case the volume
of the projection is zero.

A covariant \(p\)-vector \(w_{\lambda_1 \ldots \lambda_p}\) is defined by its transformation
formula
\[
(3) \quad w'_{\lambda_1 \ldots \lambda_p} = A^1_{\lambda_1} \cdots A^p_{\lambda_p} w_{\lambda_1 \ldots \lambda_p}
\]
and its property of being alternating. If it is simple it can be
represented by a cylinder (the interior of which may be chosen
simply connected) consisting of \(\infty^{p-1}\) (2 for \(p=1\)) totally parallel
\(E_{n-p}\)'s (the generators) with an exterior orientation of their
\((n-p)\)-direction. Hence the set of \(\infty^{p-1}\) \(E_{n-p}\)'s is oriented.
Two such cylinders determine the same \(p\)-vector if and only if
1° their \((n-p)\)-directions are the same, 2° they intersect from

![Diagram](Fig. 2)

one (and then from every) \(E_p\) which has no direction in common
with the \((n-p)\)-direction, parts with equal \(p\)-dimensional volumes,
and the orientations are the same. The $\lambda_1 \ldots \lambda_p$-component is the reciprocal value of the $p$-dimensional volume of the intersection of the cylinder with the $E_p$ of $e^x, \ldots, e^x$, as measured by the parallelopotope of these vectors. It is positive or negative if the orientation of $w_{\lambda_1 \ldots \lambda_p}$ has the same or the opposite sense resp. as the orientation of $e^x, \ldots, e^x$ in this order.

3. Ordinary contra- and covariant $p$-vectordensities.

An ordinary contra- or covariant $p$-vectordensity of weight $w$ is defined by its transformation-formula

\begin{equation}
\psi^{x_1 \ldots x_p} = A^{-w} A_{x_1}^{x_1} \ldots A_{x_p}^{x_p} \psi^{x_1 \ldots x_p}; \quad A = \text{Det} (A_{x}^{x})
\end{equation}

(4)

\begin{equation}
w_{\lambda_1 \ldots \lambda_p} = A^{-w} A_{\lambda_1}^{\lambda_1} \ldots A_{\lambda_p}^{\lambda_p} w_{\lambda_1 \ldots \lambda_p}
\end{equation}

(5)

resp. and by its property of being alternating. For $p = 0$ we get ordinary scalardensities of weight $w$.

In an $E_n$ three quantities are given a priori:

A. The unit affinor $A_{\lambda}^{x}$ with the components

\begin{equation}
A_{\lambda}^{x} = \begin{cases} 1, & x = \lambda \\ 0, & x \neq \lambda \end{cases}
\end{equation}

(6)

with respect to every system of coordinates.

B. The contravariant unit $n$-vectordensity $\xi^{x_1 \ldots x_n}$ of weight $+1$ defined by

\begin{equation}
\xi^{1 \ldots n} = +1; \quad \xi^{x_1 \ldots x_n} = \xi^{[x_1 \ldots x_n]}
\end{equation}

(7)

with respect to every system of coordinates;

C. The covariant unit $n$-vectordensity $e_{\lambda_1 \ldots \lambda_n}$ of weight $-1$ defined by

\begin{equation}
e_{1 \ldots n} = +1; \quad e_{\lambda_1 \ldots \lambda_n} = e_{[\lambda_1 \ldots \lambda_n]}
\end{equation}

(8)

with respect to every system of coordinates.

Hence a one to one correspondence exists between the set of all contravariant $p$-vectors and the set of all covariant $(n-p)$-vectordensities of weight $-1$ and also between the set of all covariant $p$-vectors and the set of all contravariant $(n-p)$-vectordensities of weight $+1$:

\begin{equation}
\psi_{\lambda_1 \ldots \lambda_{n-p}} = \frac{1}{p!} \xi_{\lambda_1 \ldots \lambda_{n-p} x_1 \ldots x_p} \psi^{x_1 \ldots x_p}; \quad \psi_{\lambda_1 \ldots \lambda_{n-p} x_1 \ldots x_p} = \frac{1}{(n-p)!} \psi_{\lambda_1 \ldots \lambda_{n-p}} \xi^{x_1 \ldots x_p}
\end{equation}

(9)

\begin{equation}
w_{\lambda_1 \ldots \lambda_{n-p}} = \frac{1}{p!} w_{\lambda_1 \ldots \lambda_p} \xi_{\lambda_1 \ldots \lambda_p x_1 \ldots x_{n-p}}; \quad w_{\lambda_1 \ldots \lambda_p} = \frac{1}{(n-p)!} w_{\lambda_1 \ldots \lambda_{n-p}} \xi^{x_1 \ldots x_{n-p}}
\end{equation}

(10)
In particular with a scalar $p$ correspond both a co- and a contra-
variant $n$-vector density

\[
\begin{align*}
\left\{
\begin{array}{l}
p'_{\lambda_1\ldots\lambda_n} = p e_{\lambda_1\ldots\lambda_n}; \\
p'_{\mu_1\ldots\mu_n} = \frac{1}{n!} p'_{\lambda_1\ldots\lambda_n} \xi_{\lambda_1}\ldots\lambda_n
\end{array}
\right.
\end{align*}
\]

(11) \[
\begin{align*}
p'_{\mu_1\ldots\mu_n} = p e_{\mu_1\ldots\mu_n}; \\
p'_{\lambda_1\ldots\lambda_n} = \frac{1}{n!} p'_{\mu_1\ldots\mu_n} \xi_{\mu_1}\ldots\mu_n
\end{align*}
\]

Of course the geometrical representations of corresponding
quantities are the same. For densities of other weights no such
simple geometrical representations exist.

Table 1 shows the quantities considered here for $n = 3$.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Notation 1</th>
<th>Notation 2</th>
<th>Number of independent components</th>
<th>Orientation</th>
</tr>
</thead>
</table>
| none   | $p$; scalar | $\{p_{\mu\lambda}; \text{cov. triv. dens. } w = -1 \\
|        |            | $p_{\mu\lambda}; \text{cov. biv. dens. } w = -1 \\|        |            | $w_{\lambda}; \text{cov. biv. dens. } w = -1 \\
|        |            | $f_{\lambda}; \text{cov. vect. dens. } w = -1 \\
|        |            | $h_{\lambda}; \text{cov. biv. dens. } w = -1 \\
|        | $p_{\mu\lambda}; \text{cov. triv. } w = -1 \\
|        | $q_{\mu\lambda}; \text{cov. triv. } w = +1 \\


Aside ordinary densities we can also consider densities, in the
transformation-formulae of which the absolute value $|\Delta|$ is taken
instead of $\Delta$ itself. As they were introduced by H. Weyl \footnote{RZM, § 13, 4th Aufl., 98.} we
call them $W$-densities and distinguish them from ordinary
densities by a $\sim$ above the central letter. If the weight is $w$,
their transformation formulae are

\[
\begin{align*}
\end{align*}
\]
For $p = 0$ we get $W$-scalar densities. As long as we consider only (real) transformations with $\Delta > 0$, there is no difference between $W$-densities and ordinary densities. But this restriction is equivalent with giving an orientation in $X_n$, viz. the orientation of $e^\varepsilon_1, \ldots, e^\varepsilon_n$ in this order \(^7\). Hence the geometric interpretation of a $W$-density can only differ from the interpretation of a corresponding ordinary density by the orientation. Hence it is to be expected that we may, at least in the cases $w = +1$ resp. $w = -1$ where we have simple geometrical representations, obtain $W$-densities from ordinary densities by interchanging interior and exterior orientations.

Take for instance a part of an $E_p$ with an exterior orientation and fix the rules about equivalence and building of components in the same way as for an ordinary $p$-vector, the $(\varepsilon_1 \ldots \varepsilon_p)$-component being positive or negative if the projection of the exterior orientation has the same or the opposite sense resp. as the orientation of $e^{\varepsilon_1}, \ldots, e^{\varepsilon_p}$ in this order, where $\varepsilon_1, \ldots, \varepsilon_n$ is an even permutation of $I, \ldots, n$. Hence, if the quantity thus defined, which we denote by $\tilde{\nu}^{\varepsilon_1 \ldots \varepsilon_p}$, has the same components as an ordinary contravariant $p$-vector with respect to one system of coordinates, this will be true also for all systems of coordinates that can be deduced from the first one by transformations with $\Delta > 0$, but the sign changes if we take a transformation with $\Delta < 0$. From this follows the transformation-formula of $\tilde{\nu}^{\varepsilon_1 \ldots \varepsilon_p}$

\begin{equation}
\tilde{\nu}^{\varepsilon_1 \ldots \varepsilon_p} = \Delta \left| \Delta \right|^{-1} A_{\varepsilon_1}^{\varepsilon'_1} \ldots A_{\varepsilon_p}^{\varepsilon'_p} \tilde{\nu}^{\varepsilon'_1 \ldots \varepsilon'_p}
\end{equation}

and, if we define $\tilde{\nu}_{\lambda_1 \ldots \lambda_{n-p}}$ by

\begin{equation}
\tilde{\nu}_{\lambda_1 \ldots \lambda_{n-p}} = \frac{1}{p!} \varepsilon_{\lambda_1 \ldots \lambda_{n-p} \varepsilon_1 \ldots \varepsilon_p} \tilde{\nu}^{\varepsilon_1 \ldots \varepsilon_p},
\end{equation}

also:

\begin{equation}
\nu_{\lambda'_1 \ldots \lambda'_{n-p}} = \left| \Delta \right| A_{\lambda'_1}^{\lambda_1} \ldots A_{\lambda'_{n-p}}^{\lambda_{n-p}} \nu_{\lambda_1 \ldots \lambda_{n-p}}.
\end{equation}

\(^7\) Cf. VEBLEN & WHITEHEAD l.c.
\( \mathbf{v}_{\lambda_1 \ldots \lambda_{n-p}} \) is a covariant \( W-(n-p) \)-vectordensity of weight \(-1\). Accordingly the quantity \( \mathbf{v}_{\kappa_1 \ldots \kappa_p} \) will be called a contravariant \( W-p \)-vector.

In the same way we define a covariant \( W-p \)-vector \( \mathbf{w}_{\lambda_1 \ldots \lambda_p} \) by its transformation-formula

\[
(17) \quad \mathbf{w}_{\lambda_1 \ldots \lambda_p} = A |A|^{-1} A_{\lambda_1}^{\lambda_p} \ldots A_{\lambda_p}^{\lambda_1} \mathbf{w}_{\lambda_1 \ldots \lambda_p}
\]

and its alternating property. This quantity is geometrically equivalent with the contravariant \( W-(n-p) \)-vectordensity

\[
(18) \quad \mathbf{w}_{\kappa_1 \ldots \kappa_{n-p}} = \frac{1}{p!} \mathbf{w}_{\lambda_1 \ldots \lambda_p} e_{\lambda_1} \ldots e_{\lambda_p}
\]

of weight \(+1\) and can, if it is simple, be represented by a cylinder (the interior of which may be chosen simply connected) consisting of \( \infty^{p-1} \) totally parallel \( E_{n-p} \)'s with an interior orientation of their \((n-p)\)-direction. The rules for equivalence and the building of components are the same as in the case of the ordinary covariant \( p \)-vector; the component is positive or negative if the orientation of \( \mathbf{w}_{\lambda_1 \ldots \lambda_p} \) has the same or the opposite sense resp. as the orientation of \( e_{\lambda_1}, \ldots, e_{\lambda_n} \) in this order, where \( \lambda_1, \ldots, \lambda_n \) is an even permutation of \( 1, \ldots, n \).

For \( p = 0 \) we get \( W \)-scalars with the transformation-formula

\[
(19) \quad \mathbf{p}^{(\kappa)} = A |A|^{-1} \mathbf{p}^{(\kappa)}.
\]

With \( \mathbf{p} \) correspond both a co- and a contravariant \( W-n \)-vectordensity of weight \(-1\) and \(+1\) resp.:

\[
(20) \quad \mathbf{p}_{\lambda_1 \ldots \lambda_n} = \mathbf{p}_{\kappa_1 \ldots \kappa_n} e_{\lambda_1} \ldots e_{\lambda_n}; \quad \mathbf{p} = \frac{1}{n!} \mathbf{p}_{\lambda_1 \ldots \lambda_n} e_{\lambda_1} \ldots e_{\lambda_n}
\]

Contrary to an ordinary scalar a \( W \)-scalar has an \((n\)-dimensional\) orientation, viz. the orientation of the coordinate system with respect to which \( p \) is positive. But a \( W \)-scaldensity of weight \(+1\) or \(-1\) has no orientation because it is equivalent with a covariant or contravariant \( W-n \)-vector resp. and is represented by an \( n \)-dimensional volume without orientation.

Table 2 shows the \( W \)-quantities considered here for \( n = 3 \).
Table 2.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Notation 1</th>
<th>Notation 2</th>
<th>Components</th>
<th>Orientation</th>
</tr>
</thead>
<tbody>
<tr>
<td>none</td>
<td>$p; W$-scalar</td>
<td>$\tilde{p}^{\mu}_{\lambda};$  cov. $W$-triv.d.; $w = -1$</td>
<td>1</td>
<td>screw</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\tilde{p}^{\nu\lambda\mu};$ contr. $W$-triv.d.; $w = +1$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td>$\tilde{w};$ contr. $W$-vect.</td>
<td>$\tilde{v}^\lambda_{\lambda};$ cov. $W$-biv.d.; $w = -1$</td>
<td>3 (proj.)</td>
<td>outside</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td></td>
<td>$\tilde{w}^\nu\lambda_{\lambda};$ contr. $W$-biv.d.; $w = +1$</td>
<td>3 (proj.)</td>
<td>inside</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td>$\tilde{f}^\nu\lambda_{\lambda};$ contr. $W$-biv.</td>
<td>$\tilde{f}^\lambda_{\lambda};$ cov. $W$-vect.d.; $w = -1$</td>
<td>3 (proj.)</td>
<td>outside</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td></td>
<td>$\tilde{h}^\mu_{\lambda};$ cov. $W$-biv.</td>
<td>3 (proj.)</td>
<td>inside</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td></td>
<td>$\tilde{p}^{\nu\lambda\mu};$ contr. $W$-triv.</td>
<td>$\tilde{p};$ $W$-dens.; $w = -1$</td>
<td>1 (vol.)</td>
</tr>
<tr>
<td>$\rightarrow$</td>
<td></td>
<td>$\tilde{q}^\mu_{\lambda\nu};$ cov. $W$-triv.</td>
<td>$\tilde{q};$ $W$-dens.; $w = +1$</td>
<td>1 (vol.)</td>
</tr>
</tbody>
</table>

An example of a $W$-scalar is: $p = +1$ for all right-handed systems and $p = -1$ for all left-handed systems. Quantities of this kind are called sometimes „pseudoscalars” in physics. $W$-vectors are occasionally used in physics but only after the introduction of a metric \(^8\). Mr. St. Golab\(^9\) has proved by solving a functional equation, that all geometric objects with only one component, whose transformation depends on $\Delta$ only, can be deduced from the four objects: scalars, $W$-scalars, ordinary scalardensities and $W$-scalardensities. From this theorem follows that the classification we have used here is really exhaustive.

5. Identification of quantities.

After introducing a unit of volume, a metric or an $n$-dimensional orientation identifications arise between the different quantities derived. We take the case $n = 3$ as an illustration, the generalisation being obvious. (Cf. table 3)

The four directed quantities occurring after introduction of

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a unit of volume, are known in literature (from left to right) as polar vector, polar bivector, axial bivector, axial vector. After introduction of a metric the difference between polar and axial and between scalars and $W$-scalars remains. This is the point of view often found in publications on physics. After introduction of a screw-sense and a metric (this includes I, II, III and IV) all differences between directed quantities and the difference between scalars and $W$-scalars vanish.

### 6. Quantities in $X_n$.

We consider an $n$-dimensional differentiable manifold $X_n$. Let $\xi^\nu(\zeta, \chi, \mu, \nu, \pi, \sigma, \tau = 1, \ldots, n)$ be a set of coordinates in a sufficiently small part of this $X_n$ and $\xi'^\nu(\chi', \lambda', \ldots, \tau' = 1', \ldots, n')$ another set of coordinates with

$$
(21) \quad \Delta = \text{Det} (A^\nu_\lambda) \neq 0; \quad A^\nu_\lambda = \partial_\lambda \xi'^\nu; \quad \partial_\lambda = \frac{\partial}{\partial \xi^\nu}.
$$

It is well known that we define all quantities in the local $E_n$ of a point of $X_n$ using in the formulae of transformation $A^\nu_\lambda = \partial_\lambda \xi'^\nu$ in stead of the constants $A^\nu_\lambda$ in (1). In this way we get in each local $E_n$ ordinary quantities and $W$-quantities.
7. Imbedding of an $X_m$ in an $X_n$.  \(^{10}\)

If an $X_m$ is imbedded in an $X_n$ the following eight cases are important. We write $k$ for $n - m$ and suppose $\eta^a \ (a, b, \ldots, g = 1, \ldots, m)$ to be coordinates in the $X_m$. Further we denote by $B^a_b$ the unity-affinor in $X_m$, by $\xi^{a_1 \ldots a_m}$ and $\varepsilon_{b_1 \ldots b_m}$ the unit-m-vectordensities in $X_m$ and by $D$ the transformation-modulus $D = \det (B^a_b)$ in $X_m$.

**Case 1. Pure imbedding without any auxiliary assumptions.**

To every point $\eta^a$ of the $X_m$ belongs one and only one point of the $X_n$, given by equations of the form

\[(22) \quad \xi^\alpha = \xi^\alpha (\eta^1, \ldots, \eta^m).\]

In every point of the $X_m$ two quantities exist:
\[1^0 \quad \text{the affinor } \quad B^\alpha_b = \frac{\partial \xi^\alpha}{\partial \eta^b};\]
\[2^0 \quad \text{the simple } k\text{-vectordensity } \quad t_{\lambda_1 \ldots \lambda_k}.\]

A connecting quantity behaving like a system of $m$ contravariant vectors with respect to transformations in $X_n$ and like a system of $n$ covariant vectors with respect to transformations in $X_m$;

\[(24) \quad t_{\lambda_1 \ldots \lambda_k} = \frac{1}{m!} \xi_{\lambda_1 \ldots \lambda_k} x_1 \ldots x_m B^\alpha_{b_1} \cdots B^\alpha_{b_m} \xi^{a_1 \ldots a_m};\]

A connecting quantity with the weights $+1$ and $-1$ with respect to transformations in $X_m$ and $X_n$ respectively:

\[(25) \quad t_{\lambda_1 \ldots \lambda_k} = \Delta D^{-1} A_{\lambda_1}^{\lambda'_1} \cdots A_{\lambda_k}^{\lambda'_k} t_{\lambda_1 \ldots \lambda_k}.\]

$B^\alpha_b$ as well as $t_{\lambda_1 \ldots \lambda_k}$ are geometrically represented by the tangent $E_m$ in the local $E_n$. They are related by the identity

\[(26) \quad B^\alpha_b t_{\lambda_1 \ldots \lambda_k} = 0.\]

**Case 1'. Imbedding with exterior orientation.**

The orientation is an exterior orientation of the tangent $E_m$ in every point of the $X_m$ and can be given by any quantity $\omega$ with the absolute value $+1$ and the transformation-formula

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\(^{10}\) The eight different cases of imbedding were first treated in J. A. Schouten, "Über die Beziehungen zwischen den geometrischen Größen in einer $X_n$ und in einer in der $X_n$ eingebetteten $X_m$, [Proc. Amsterdam, 41 (1938) 568—575]."
In fact, because of (25), (27) the quantity $\omega t_{\lambda_1 \ldots \lambda_k}$ has the transformation-formula

$$\begin{align*}
^{(x',a')}(\omega) & = A^{-1} \left| A \right| D \left| D^{-1} \right| \omega \begin{pmatrix} \lambda_1 \lambda_2 \ldots \lambda_k \end{pmatrix}.
\end{align*}$$

and this is the same as that of a covariant $k$-vector except for an always positive factor. Hence $\omega t_{\lambda_1 \ldots \lambda_k}$ determines a $k$-dimensional orientation in every $E_k$ having no direction in common with the tangent $E_m$. If in any point of the $X_m$ the orientation of $e^a, \ldots, e^a$ in this order followed by the exterior orientation gives the orientation of $e^x, \ldots, e^x$ in this order, then we choose in that point always $\begin{pmatrix} (x,a) \end{pmatrix} = \pm 1$.

In each $E_k$ having no direction in common with the tangent $E_m$ in case $1'$ a contravariant alternating quantity $\pi^{x_1 \ldots x_k}$ with the transformation-formula

$$\begin{align*}
\pi^{x_1 \ldots x_k} & = A^{-1} \left| A \right| D \left| D^{-1} \right| \begin{pmatrix} \lambda_1 \lambda_2 \ldots \lambda_k \end{pmatrix} \pi^{x_1 \ldots x_k},
\end{align*}$$

satisfying the invariant condition

$$\begin{align*}
\pi^{x_1 \ldots x_k} t_{x_1 \ldots x_k} > 0,
\end{align*}$$

is determined except for a positive factor. Except for a factor $\pm 1$ this quantity transforms in the same way as a contravariant $k$-vector. Hence it determines such a $k$-vector except for the orientation. Now suppose a $k$-vector $p^{x_1 \ldots x_k}$ is given except for a factor $\pm 1$. Then we can determine a quantity $\pi^{x_1 \ldots x_k}$ by the equation

$$\begin{align*}
\pi^{x_1 \ldots x_k} & = \omega p^{x_1 \ldots x_k} \text{sgn} \left( \omega p^{x_1 \ldots x_k} t_{x_1 \ldots x_k} \right) \\
& = p^{x_1 \ldots x_k} \text{sgn} \left( p^{x_1 \ldots x_k} t_{x_1 \ldots x_k} \right),
\end{align*}$$

(where $\text{sgn} (z) = \frac{z}{|z|}$ for $z \neq 0$ and $\text{sgn} (0) = 0$). From this follows, that the geometric representation of a quantity with the transformation (29) satisfying the condition (30) is a part of an $E_k$ having neither an interior nor an exterior orientation.

**Case 2. Rigged imbedding.**

An $X_m$ in an $X_n$ is called "rigged" ("eingespannt") if in every point of $X_m$ a $k$-direction is given, having no direction in common
with the tangent $E_m$. This $k$-direction can be given by an affinor $B_\lambda^\xi$ with the following properties: 1°. its $\kappa$-region consists of all contravariant vectors of the local $E_m$ and its $\lambda$-region consists of all covariant vectors whose $(n-1)$-direction contains the $k$-direction of the rigging, 2°:

$$B_0^\xi B_\lambda^\omega = B_\lambda^\xi.$$  

From $B_0^\xi$ and $B_\lambda^\xi$ an affinor $B_\lambda^\alpha$ can be uniquely determined by means of the equations

$$B_\lambda^\alpha B_\beta^\mu = B_\beta^\alpha = \text{unitaffinor of the } X_m.$$  

$$B_\alpha^\beta B_\beta^\xi = B_\lambda^\xi.$$  

The $\lambda$-region of $B_\lambda^\alpha$ is the same as the $\lambda$-region of $B_\lambda^\xi$. $B_\alpha^\beta$ and $B_\lambda^\alpha$ together determine $B_\lambda^\xi$ uniquely by (33b). The rigging determines also uniquely and is determined by the simple $k$-vector-density

$$e^{\kappa_1 \cdots \kappa_k} = \frac{1}{m!} e^{\lambda_1 \cdots \lambda_m} e^{\kappa_1 \cdots \kappa_k} B_\lambda^\kappa_1 \cdots B_\lambda^\kappa_k e^{\alpha_1 \cdots a_m},$$

a connecting quantity with the weights $-1$ and $+1$ with respect to transformations in $X_m$ and $X_n$ resp. and satisfying the relations

$$e^{\kappa_1 \cdots \kappa_k} B_\alpha^a = 0; \quad \frac{1}{k!} e^{\kappa_1 \cdots \kappa_k} t^{\alpha_1 \cdots \alpha_k} = 1.$$  

In the cases 1 and 1' $B_\lambda^\alpha$ can not be uniquely determined, as then (34) fails, and the solutions of (33) alone contain arbitrary parameters. But they determine $B_\lambda^\alpha$ partly and well enough in order that the expression $B_\alpha^\kappa_1 \cdots B_\alpha^\kappa_p e^{\kappa_1 \cdots \kappa_p}$ be uniquely determined if $e^{\kappa_1 \cdots \kappa_p}$ is a simple contravariant $p$-vector, $p \leq m$, parallel to the tangent $E_m$.

We often make use of a special system of coordinates for which in every point of the $X_m$ $\xi^1 = \eta^1, \ldots, \xi^m = \eta^m$. If the $X_m$ is rigged we choose the parameterlines of $\xi^{m+1}, \ldots, \xi^n$ in such a way that they have in every point of the $X_m$ a direction lying

in the \( k \)-direction of the rigging. If we use \( \xi \) \((\alpha, \ldots, \delta = 1, \ldots, m)\) as coordinates in the \( X_m \) we have for that system

\[
B_{\beta}^{\alpha} = \begin{cases} 
1, \alpha = \beta \\
0, \alpha \neq \beta
\end{cases} \quad B_{\beta}^{m+1} = \ldots = B_{\beta}^{n} = 0,
\]

and all components of \( t_{\alpha_1 \ldots \alpha_k} \) and \( \varepsilon^{\alpha_1 \ldots \alpha_k} \) with a suffix \( \leq m \) vanish. \( \omega = +1 \) if \( \varepsilon^{1 \ldots n} \) in this order determine the exterior orientation of the tangent \( E_m \) and \(-1\) in the other case.

**CASE 2'. Rigged imbedding with exterior orientation.**

This case is a combination of 1' and 2 and requires \( \omega \) as well as \( B_{\alpha}^{\beta} \) to be given.

With respect to the special coordinate system mentioned above the condition (30) for the non oriented quantity \( \pi^{\alpha_1 \ldots \alpha_k} \) is

\[
\pi^{1+m, \ldots, n} > 0
\]

and the equation (31) takes the form

\[
\pi^{\alpha_1 \ldots \alpha_k} = p^{\alpha_1 \ldots \alpha_k} \text{ sign } (p^{m+1, \ldots, n}).
\]

**CASE 3. Imbedding with normalisation and exterior orientation.**

In every point of the \( X_m \) a simple covariant \( k \)-vector \( t_{\alpha_1 \ldots \alpha_k} \) is given whose \( m \)-direction lies in the tangent \( E_m \). It determines uniquely and is uniquely determined by the scalar-density

\[
\delta = \frac{t_{\alpha_1 \ldots \alpha_k}}{|t_{\alpha_1 \ldots \alpha_k}|}
\]

of weights \(-1\) and \(+1\) with respect to transformations in \( X_n \) and \( X_m \) respectively. Obviously

\[
\omega = \frac{1}{|\delta|}.
\]

In each \( E_k \) in the local \( E_n \), having no direction in common with the tangent \( E_m \), \( t_{\alpha_1 \ldots \alpha_k} \) determines (by section) uniquely a contravariant \( k \)-vector \( p^{\alpha_1 \ldots \alpha_k} \) satisfying the equation

\[
\frac{1}{k!} t_{\alpha_1 \ldots \alpha_k} p^{\alpha_1 \ldots \alpha_k} = 1.
\]

With respect to the special coordinate system mentioned above we have \( t_{m+1, \ldots, n} = \delta^{-1} \) and all components with a suffix \( \leq m \) vanish.

**CASE 3'. Imbedding with normalisation without orientation.**

We get this case by giving \( |\delta| \) instead of \( \delta \). Then instead of \( t_{\alpha_1 \ldots \alpha_k} \) only the quantity
is determined, and v.v. $\tau_{\lambda_1 \ldots \lambda_k}$ determines $|\delta|$. The geometric representation is a cylinder consisting of $\infty^{k-1} \cdot (2$ for $k=1)$ totally parallel $E_m$'s, all parallel with the tangent $E_m$, but having neither interior nor exterior orientation. With respect to the special system of coordinates mentioned before, $\tau_{m+1, \ldots, n} = |\delta|^{-1}$, and all components with a suffix $\leq m$ vanish. In each $E_k$, having no direction in common with the tangent $E_m$, $\tau_{\lambda_1 \ldots \lambda_k}$ determines a non-oriented contravariant quantity as considered under case 1' and determined by the equation

$$\frac{1}{k!} \tau_{\lambda_1 \ldots \lambda_k} \tau_{\lambda_1 \ldots \lambda_k} = 1.$$  

**Case 4. Rigged imbedding with normalisation and exterior orientation.**

This case is a combination of 2' and 3 and requires $B_{\lambda}^\times$ and $\delta$ to be given. In the $k$-direction of the rigging a simple contravariant $k$-vector $n^{x_1 \ldots x_k} = \delta e^{x_1 \ldots x_k}$ exists, which is uniquely determined by the conditions

$$\frac{1}{k!} n^{x_1 \ldots x_k} t_{x_1 \ldots x_k} = 1;$$  

$$B_{x_1}^\times n^{x_1 \ldots x_k} = 0,$$

and for which the equation

$$t_{\lambda_1 \ldots \lambda_k} n^{x_1 \ldots x_k} = k! C_{[\lambda_1}^{x_1} \ldots C_{\lambda_k]}^{x_k}; \quad C_{\lambda}^\times = A_{\lambda}^\times - B_{\lambda}^\times$$

holds. With respect to the special system of coordinates mentioned before $n^{m+1, \ldots, n} = \delta$ and all components with a suffix $\leq m$ vanish.

**Case 4'. Rigged imbedding with normalisation without orientation.**

This case is a combination of 2 and 3' and requires $B_{\lambda}^\times$ and $|\delta|$ to be given. In the $k$-direction of the rigging a non-oriented contravariant quantity $v^{x_1 \ldots x_k} = |\delta| e^{x_1 \ldots x_k}$ exists, which is uniquely determined by the conditions

$$\frac{1}{k!} v^{x_1 \ldots x_k} \tau_{x_1 \ldots x_k} = 1$$  

$$B_{x_1}^\times v^{x_1 \ldots x_k} = 0$$

and for which the equation

$$t_{\lambda_1 \ldots \lambda_k} v^{x_1 \ldots x_k} = k! C_{[\lambda_1}^{x_1} \ldots C_{\lambda_k]}^{x_k}$$

holds. With respect to the special system of coordinates men-
tioned before \( v^{m+1}, \ldots, n = | \delta | \) and all components with a suffix \( \leq m \) vanish.

The following table shows the different cases and the quantities involved 12).

<table>
<thead>
<tr>
<th>Imbedding:</th>
<th>1</th>
<th>1'</th>
<th>2</th>
<th>2'</th>
<th>3'</th>
<th>3</th>
<th>4'</th>
<th>4</th>
<th>quantities occurring besides ( B_b^x ) and ( t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>included cases:</td>
<td>1</td>
<td>1</td>
<td>1, 1', 2</td>
<td>1</td>
<td>1, 1', 3'</td>
<td>1, 2, 3'</td>
<td>all</td>
<td></td>
<td></td>
</tr>
<tr>
<td>orientation:</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>( \omega )</td>
</tr>
<tr>
<td>rigging:</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>( B_{x}^x, (B_{y}^x), (n) )</td>
</tr>
<tr>
<td>normalisation:</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>(</td>
</tr>
<tr>
<td>quantities occurring besides ( B_b^x )</td>
<td>( t )</td>
<td>( t )</td>
<td>( t )</td>
<td>( t, (\tau) )</td>
<td>( t, (\tau, t) )</td>
<td>( t, (\tau, t) )</td>
<td>( t, (\tau, t) )</td>
<td>( t, (\tau, t) )</td>
<td>( B_{x}^x, (B_{y}^x), (n) )</td>
</tr>
</tbody>
</table>

Fig. 3 shows the eight different cases for the imbedding of an \( X_2 \) in ordinary space.

---

12) The quantities with \( k \) indices are denoted by their central letter. Quantities in brackets in the last row or column can be derived from the other quantities in
8. Relations between quantities in $X_m$ and $X_n$.\(^{13}\)

A. Ordinary contravariant $p$-vectors.

A contravariant $p$-vector $\tau v^{a_1 \cdots a_p}$ ($p \leq m$) in a point of $X_m$ determines uniquely a contravariant $p$-vector $\tau v^{x_1 \cdots x_p}$ in the corresponding point of $X_m$ according to the equation

$$\tau v^{x_1 \cdots x_p} = B_{a_1}^{x_1} \cdots B_{a_p}^{x_p} \tau v^{a_1 \cdots a_p}. \quad (50)$$

Because of (26) we have

$$\tau v^{x_1 \cdots x_p} \lambda_1 \cdots \lambda_k = 0. \quad (51)$$

At the other hand each $p$-vector $\tau v^{x_1 \cdots x_p}$ ($p \leq m$) with respect to transformations in $X_n$, defined in a point of $X_m$ and satisfying (51)\(^{14}\) determines uniquely a $p$-vector $\tau v^{a_1 \cdots a_p}$ defined by (50) as these equations have a unique solution. It can be written in the form

$$\tau v^{a_1 \cdots a_p} = \frac{1}{p!} \beta^{a_1 \cdots a_p} \tau v^{x_1 \cdots x_p} \quad (52)$$

where $\beta^{a_1 \cdots a_p} \lambda_1 \cdots \lambda_p$ is determined except for the product of an arbitrary alternating quantity $w^{a_1 \cdots a_p} \lambda_{p+1} \cdots \lambda_m$ with $\lambda_1 \cdots \lambda_m$. These quantities can all be derived from $\beta^{a_1 \cdots a_m} \lambda_1 \cdots \lambda_m$, viz.

$$\beta^{a_1 \cdots a_p} \lambda_1 \cdots \lambda_p = \beta^{a_1 \cdots a_p} b_{p+1} \cdots b_m \lambda_{p+1} \cdots \lambda_m \quad (53)$$

In case 2 (rigging; existence of $B_{\lambda}^a$) the solution has the form

$$\tau v^{a_1 \cdots a_p} = B_{\lambda_1}^{a_1} \cdots B_{\lambda_p}^{a_p} \tau v^{x_1 \cdots x_p}, \quad (54)$$

i.e. a particular choice of $\beta^{a_1 \cdots a_p} \lambda_1 \cdots \lambda_p$ is $p! B_{[\lambda_1}^{a_1} \cdots B_{\lambda_p]}^{a_p}$. Moreover in this case an arbitrary $p$-vector $\tau v^{x_1 \cdots x_p}$ defined in a point of $X_m$ also determines a $p$-vector in $X_m$ according to

$$\tau v^{a_1 \cdots a_p} = B_{x_1}^{a_1} \cdots B_{x_p}^{a_p} \tau v^{x_1 \cdots x_p}, \quad (55)$$

but is not itself uniquely determined by $\tau v^{a_1 \cdots a_p}$ except for

the same row and column. + stands for „given”, — for „not given”. All cases can be obtained by giving none, one, two or three of the three independent quantities $\pi$, $\omega$ and $|\lambda|$, which determine independently the rigging the orientation and the normalisation respectively.

\(^{13}\) The relations between ordinary quantities have been treated by J. A. Schouten [I.c. in note 8].

\(^{14}\) Such a $p$-vector in $X_n$ is said to „lie in $X_m$".
\[ p = 0, \text{ where (55) simply becomes } \nu = \nu, \nu^{a_1 \ldots a_p} \text{ can be called the projection of } \nu^{x_1 \ldots x_p} \text{ in the direction of the rigging upon the local } E_n \text{ tangent to } X_m. \]

By means of (9) and the analogous equation in \( X_m \) the equations corresponding with (50), (54) and (55) are found. They contain a covariant \((n-p)\)-vectordensity of weight \(-1\) in \( X_n \) and a covariant \((m-p)\)-vectordensity of weight \(+1\) in \( X_m \).

In case 3 (normalisation; existence of \( t_{\lambda_1 \ldots \lambda_k} \)) for \( p \geq k = n - m \)

still another quantity in \( X_m \) is determined by an arbitrary \( p\)-vector \( \nu^{x_1 \ldots x_p} \), viz.

\[
(56) \quad \nu^{a_1 \ldots a_{p-k}} = \frac{1}{k!} B^a_{x_1} \cdots B^a_{x_{p-k}} \nu^{x_1 \ldots x_p} t_{x_{p-k+1} \ldots x_p}.
\]

If \( \nu^{x_1 \ldots x_p} \) is simple, the \((p-k)\)-direction of \( \nu^{x_1 \ldots x_p} t_{x_{p-k+1} \ldots x_p} \)

and of \( \nu^{a_1 \ldots a_{p-k}} \) is contained in the \((p+m-n)\)-dimensional intersection of the \( p\)-direction of \( \nu^{x_1 \ldots x_p} \) and the local \( m\)-direction. By (56) \( \nu^{x_1 \ldots x_p} \) is not uniquely determined for a given \( \nu^{a_1 \ldots a_{p-k}} \) (except for \( p=n, p-k=m \)). In case 4 however (and in case 3 for \( p = n \)) each \( q\)-vector \( \nu^{a_1 \ldots a_q} \) in \( X_m \) determines a \((q+k)\)-vector \( \nu^{x_1 \ldots x_{q+k}} \) in \( X_n \), according to

\[
(57) \quad \nu^{x_1 \ldots x_{q+k}} = \binom{q+k}{k} \nu^{a_1 \ldots a_q} B^{[x_1 \ldots x_q} \nu^{x_{q+1} \ldots x_{q+k}]},
\]

which is uniquely determined by \( \nu^{a_1 \ldots a_q} \), the solution of (57) being

\[
(58) \quad \nu^{a_1 \ldots a_q} = \frac{1}{k!} B^a_{x_1} \cdots B^a_{x_q} \nu^{x_1 \ldots x_{q+k}} t_{x_{q+1} \ldots x_{q+k}}.
\]

By means of (9) and the analogous equation in \( X_m \) the equations corresponding with (56), (57) and (58) are found, containing a covariant \((n-p)\)-vectordensity of weight \(-1\) in \( X_n \) and a covariant \((n-p)\)-vectordensity of weight \(-1\) in \( X_m \):

\[
(59) \quad \nu^{b_1 \ldots b_l} = \delta^{-1} B_{b_1}^{b_1} \cdots B_{b_l}^{b_l} \nu^{a_1 \ldots a_l}; \quad \text{(case 3)}
\]

\[
(60) \quad \nu^{b_1 \ldots b_l} = \delta B_{b_1}^{b_1} \cdots B_{b_l}^{b_l} \nu^{a_1 \ldots a_l} \quad \text{(case 4, for } l = 0)
\]

\[
(61) \quad \nu^{b_1 \ldots b_l} = \delta^{-1} B_{b_1}^{b_1} \cdots B_{b_l}^{b_l} \nu^{a_1 \ldots a_l} \quad \text{also case 3)}
\]

where \( l = n - p \).

With respect to the special coordinate-system the equations (55) and (56) take the very simple form

\[
(62) \quad \nu^{a_1 \ldots a_p} = \nu^{a_1 \ldots a_p}; \quad \text{(} p \leq m, \text{ case 2, for } p = 0 \text{ also case 1)}
\]
The equations of the corresponding densities are \((l = n - p, k = n - m)\)

\[
(63) \quad \varphi^{x_1 \ldots x_{p-k}} = \delta^{-1} \varphi^{x_1 \ldots x_{p-k}}, \quad m+1, \ldots, n \quad (p \geq k = n - m, \text{ case 3}),
\]
e.g. for \(m = n - 1, \ p = 1, \ (k = 1, q = 0), \ n \geq 2:

\[
(64) \quad \varphi^x = \varphi^x; \quad \text{(case 2)}
\]

\[
(65) \quad \varphi^x = \delta^{-1} \varphi^x; \quad \text{(case 3)}
\]

together valid only in case 4.

The equations of the corresponding densities are \((l = n - p, k = n - m)\)

\[
(66) \quad \varphi_{\beta_1 \ldots \beta_{t-k}} = \varphi_{\beta_1 \ldots \beta_{t-k}}, \ m+1, \ldots, n
\]

\[
(67) \quad \varphi_{\beta_1 \ldots \beta_t} = \delta^{-1} \varphi_{\beta_1 \ldots \beta_t},
\]
e.g. for \(m = n - 1, \ p = n - 1, \quad (k = 1, \ l = 1)

\[
(68) \quad \varphi = \varphi_n
\]

\[
(69) \quad \varphi = \delta^{-1} \varphi.
\]

It is remarkable, that in case 4, if \(m \geq p \geq k = n - m\) (which is only possible if \(m \geq \frac{1}{2}n\)) both quantities \(\varphi^{a_1 \ldots a_p}\) and \(\varphi^{a_1 \ldots a_{p-k}}\) in \(X_m\) exist and that these two quantities together for \(m = n - 1\) determine completely the \(p\)-vector \(\varphi^{x_1 \ldots x_p}\) in \(X_n\) as follows from (62) and (63).

B. Ordinary covariant \(p\)-vectors.

For \(p \leq m\) a covariant \(p\)-vector \(w_{\lambda_1 \ldots \lambda_p}\) in \(X_n\), defined in a point of \(X_m\), always (i.e. in case 1) determines a covariant \(p\)-vector in \(X_m\), viz.

\[
(70) \quad \varphi_{w_{b_1 \ldots b_p}} = B_{b_1}^{\lambda_1} \cdots B_{b_p}^{\lambda_p} \varphi_{w_{\lambda_1 \ldots \lambda_p}}.
\]

If \(w_{\lambda_1 \ldots \lambda_p}\) is simple this quantity is the intersection of \(w_{\lambda_1 \ldots \lambda_p}\) with the local \(E_m\). By equation (70) \(w_{\lambda_1 \ldots \lambda_p}\) is not uniquely determined. In case 2 (rigging) however (and in case 1 for \(p = 0\)) each covariant \(p\)-vector \(\varphi_{w_{b_1 \ldots b_p}}\) in \(X_m\) determines such a quantity in \(X_n\) according to

\[
(71) \quad \varphi_{w_{\lambda_1 \ldots \lambda_p}} = B_{\lambda_1}^{b_1} \cdots B_{\lambda_p}^{b_p} \varphi_{w_{b_1 \ldots b_p}},
\]

which is uniquely determined by \(\varphi_{w_{b_1 \ldots b_p}}\), the solution of (71) being

\[
(72) \quad \varphi_{w_{b_1 \ldots b_p}} = B_{b_1}^{\lambda_1} \cdots B_{b_p}^{\lambda_p} \varphi_{w_{\lambda_1 \ldots \lambda_p}}.
\]

If \(\varphi_{w_{b_1 \ldots b_p}}\) is simple, the \((n-p)\)-direction of \(\varphi_{w_{\lambda_1 \ldots \lambda_p}}\) contains and is composed of the \((m-p)\)-direction of \(\varphi_{w_{b_1 \ldots b_p}}\) and the
(n−m)-direction of the rigging. By means of (10) and the analogous equations in $X_m$ the equations corresponding with (70), (71) and (72) are found, containing a contravariant $(n−p)$-vector-density of weight +1 in $X_n$ and a contravariant $(m−p)$-vector-density of weight +1 in $X_m$.

In case 3 (normalisation, existence of $t_{\lambda_1\ldots\lambda_k}$) a covariant $q$-vector $w_{b_1\ldots b_q}$ in $X_m$ determines also a covariant $(q+k)$-vector $w_{\lambda_1\ldots\lambda_{q+k}}$ in $X_n$, according to

$$w_{\lambda_1\ldots\lambda_{q+k}} = \left(\begin{array}{c} q+k \\ k \end{array}\right) w_{b_1\ldots b_q} \beta_{b_1}^{\lambda_1} \ldots \lambda_{q+k} t_{\lambda_{q+1}\ldots\lambda_{q+k}},$$

as the alternated product of $\beta_{b_1}^{\lambda_1} \ldots \lambda_{q+k}$ with $t_{\lambda_{q+1}\ldots\lambda_{q+k}}$ is not affected by the ambiguity of $\beta_{b_1}^{\lambda_1} \ldots \lambda_{q+k}$.

If $w_{b_1\ldots b_q}$ is simple, $w_{\lambda_1\ldots\lambda_{q+k}}$ also is; its $(n−q−k)=(m−q)$-direction is the same as the $(m−q)$-direction of $w_{b_1\ldots b_q}$ and is contained in the local $m$-direction of $X_m$.

In case 4 equations (73) have the solution

$$w_{b_1\ldots b_q} = \frac{1}{k!} B_{b_1}^{\lambda_1} \ldots B_{b_q}^{\lambda_q} w_{\lambda_1\ldots\lambda_{q+k}} n_{\lambda_{q+1}\ldots\lambda_{q+k}}.$$

This solution however is valid already in case 3 though $n_{x_1\ldots x_k}$ is not uniquely determined then. In fact, as the ambiguity of $n_{x_1\ldots x_k}$ consists in alternated products containing a factor $B$, and as the transvection of $w_{\lambda_1\ldots\lambda_{q+k}}$ with $q+1$ or more factors $B$ vanishes, this ambiguity bears no influence upon the left side of (74). Hence in this case $w_{\lambda_1\ldots\lambda_{q+k}}$ and $w_{b_1\ldots b_q}$ can be considered to represent the same object.

Moreover in case 4 an arbitrary covariant $(q+k)$-vector $w_{\lambda_1\ldots\lambda_{q+k}}$ defined in a point of $X_m$ determines a covariant $q$-vector in $X_m$ according to

$$w_{b_1\ldots b_q} = \frac{1}{k!} B_{b_1}^{\lambda_1} \ldots B_{b_q}^{\lambda_q} w_{\lambda_1\ldots\lambda_{q+k}} n_{\lambda_{q+1}\ldots\lambda_{q+k}},$$

but evidently is not uniquely determined by it (except for $q=m$, $q+k=n$). $w_{b_1\ldots b_q}$ can be called the projection of $w_{\lambda_1\ldots\lambda_{q+k}}$ in the direction of the rigging upon the local $E_m$ tangent to $X_m$. If we take for $m=n−1$ the special coordinate-system mentioned above, the equation (75) takes the very simple form

$$w_{\beta_1\ldots\beta_q} = \delta_{\beta_1\ldots\beta_q} n.$$
By means of (10) and the analogous equation in $X_m$ the equations corresponding with (73), (74) and (75) are found, containing a contravariant $(n-p)$-vectordensity of weight $+1$ in $X_n$ and a contravariant $(n-p)$-vectordensity of weight $+1$ in $X_m$.

(77) $\omega^{a_1\ldots a_l} = \delta B_{a_1\ldots a_l}^\alpha \omega^{\alpha}_{\ldots \alpha_{l}}$ (case 4, for $l = 0$ also in case 3)

(78) $\omega^{a_1\ldots a_l} = \delta^{-1} B^{a_1\ldots a_l}_{a_1\ldots a_l} \omega^{a_1\ldots a_l}$

(79) $\omega^{a_1\ldots a_l} = \delta B_{a_1\ldots a_l}^\alpha \omega^{\alpha}_{\ldots \alpha_{l}}$ (case 3).

With respect to the special coordinatesystem the equations (70) and (75) get the very simple form

(80) $\omega^{\beta_1\ldots \beta_p} = \omega^{\beta_1\ldots \beta_p}$; $(p \leq m$, case 1 $)$

(81) $\omega^{\beta_1\ldots \beta_p} = \delta \omega^{\beta_1\ldots \beta_p}_{\ldots m+1\ldots n}$

$(p \geq k$, case 4, for $p = n$ also in case 3 $)$, f.i. for $m = n - 1$, $p = 1$, $n \geq 2$

(82) $\omega^{\beta} = \omega^{\beta}$; (case 1)

(83) $\omega = \delta \omega$; (case 4).

The equations of the corresponding densities are $(l=n-p)$

(84) $\omega^{\alpha_1\ldots \alpha_{l-k}} = \omega^{\alpha_1\ldots \alpha_{l-k}, m+1\ldots n}$

(85) $\omega^{\alpha_1\ldots \alpha_{l}} = \delta \omega^{\alpha_1\ldots \alpha_{l}}$

f.i. for $m = n - 1$, $p = n - 1$

(86) $\omega = \omega_n$

(87) $\omega = \delta^{-1} \omega$.

It is remarkable, that in case 4, if $m \geq p \geq k$ (only possible if $m \geq \frac{1}{2} n$) both quantities $\omega_{b_1\ldots b_p}$ and $\omega_{b_1\ldots b_{p-k}}$ in $X_m$ exist and that these two quantities together determine for $m = n - 1$ completely the $p$-vector $\omega_{\lambda_1\ldots \lambda_p}$ in $X_n$ as follows from (80) and (81).
We collect the results in the following table\textsuperscript{15}):

\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
\textbf{Case} & \textbf{Case} & \textbf{Case} & \textbf{Case} & \\
\hline
\textbf{$X_n$} & \textbf{$X_m$} & \textbf{$X_n$} & \textbf{$X_m$} & \\
\hline
\textbf{$p \leq m$} & \textbf{$p \geq k$} & \textbf{$p \leq m$} & \textbf{$p \geq k$} & \\
\hline
\textbf{1} & \textbf{2} & \textbf{3} & \textbf{4} & \\
\hline
\hline
\end{tabular}
\end{table}

\texttt{15}) The arrows in the 2nd and 3rd column show which of the quantities in the 1st and 4th column is determined by which, and analogously in the last four columns. Near the arrows the quantities are written, used for the determination. Quantities, which are not uniquely determined but can be used because their ambiguity does not affect the result, are written in brackets.

C. \textit{W-Quantities}.

For \textit{W-p-vectors} and their corresponding \textit{W-(n-p)-vector-densities} the equations are:

\begin{align*}
\text{Contravariant W-p-vectors, } p \leq m: \\
(50W) \quad & \overrightarrow{\nu}^{a_1 \ldots a_p} = \omega \overrightarrow{B}_{a_1}^{b_1} \ldots \overrightarrow{B}_{a_p}^{b_p} \overrightarrow{\nu}^{b_1 \ldots b_m} \\
(54W) \quad & \overrightarrow{\nu}^{a_1 \ldots a_p} = \omega \overrightarrow{B}_{a_1}^{b_1} \ldots \overrightarrow{B}_{a_p}^{b_p} \overrightarrow{\nu}^{b_1 \ldots b_m} \\
(55W) \quad & \overrightarrow{\nu}^{a_1 \ldots a_p} = \omega \overrightarrow{B}_{a_1}^{b_1} \ldots \overrightarrow{B}_{a_p}^{b_p} \overrightarrow{\nu}^{b_1 \ldots b_m} \\
(56W) \quad & \overrightarrow{\nu}^{a_1 \ldots a_{p-k}} = \frac{1}{k!} \overrightarrow{B}_{a_1}^{b_1} \ldots \overrightarrow{B}_{a_{p-k}}^{b_{p-k}} \overrightarrow{\nu}^{b_1 \ldots b_{p-k}} \\
\end{align*}

\textsuperscript{15})
Covariant $W-(n-p)$-vector densities of weight $-1$, $p \geq k$:

\begin{align*}
(57W) & \quad \tilde{v}_{\alpha_1 \cdots \alpha_p} = \left( \frac{p}{k} \right) \tilde{v}_{\alpha_1 \cdots \alpha_{p-k}} B_{\alpha_1}^{\alpha_{p-k}} \cdots B_{\alpha_{p-k}}^{\alpha_p} \quad (\text{case } 4', \text{ for } p = n \text{ also case } 3'). \\
(58W) & \quad \tilde{v}_{\alpha_1 \cdots \alpha_{p-k}} = \frac{1}{k!} B_{\alpha_1}^{\alpha_{p-k}} \cdots B_{\alpha_{p-k}}^{\alpha_p} \tilde{v}_{\alpha_{p+1} \cdots \alpha_p} \quad (\text{case } 3').
\end{align*}

Contravariant $W-p$-vectors w.r. to special coordinatesystem:

\begin{align*}
(59W) & \quad \tilde{\nu}_{\beta_1 \cdots \beta_i} = \delta \tilde{\nu}_{\beta_1 \cdots \beta_i} (\text{case } 3'); \\
(60W) & \quad \tilde{\nu}_{\lambda_1 \cdots \lambda_i} = \delta \tilde{\nu}_{\lambda_1 \cdots \lambda_i} (\text{case } 4', \text{ for } i = 0) \\
(61W) & \quad \tilde{\nu}_{\beta_1 \cdots \beta_i} = \delta \tilde{\nu}_{\beta_1 \cdots \beta_i} (\text{also case } 3').
\end{align*}

Covariant $W-(n-p)$-vector densities of weight $-1$ w.r. to special coordinatesystem:

\begin{align*}
(62W) & \quad \tilde{\nu}_{\alpha_1 \cdots \alpha_p} = \omega \tilde{\nu}_{\alpha_1 \cdots \alpha_p}; \quad (p \leq m, \text{ case } 2', \text{ for } p = 0 \text{ also case } 1') \\
(63W) & \quad \tilde{\nu}_{\alpha_1 \cdots \alpha_{p-k}} = \delta \tilde{\nu}_{\alpha_1 \cdots \alpha_{p-k}, m+1, \ldots, n} \quad (p \geq k, \text{ case } 3'); \\
\text{e.g. for } m = n - 1, \quad p = 1, \quad n \geq 2:
\end{align*}

\begin{align*}
(64W) & \quad \tilde{\nu} = \omega \tilde{\nu}; \quad (\text{case } 2') \quad \text{together valid only in case} \\
(65W) & \quad \tilde{\nu} = \delta \tilde{\nu}; \quad (\text{case } 3') \quad \text{not in case } 4'.
\end{align*}

Covariant $W-p$-vectors, $p \leq m$:

\begin{align*}
(66W) & \quad \tilde{\nu}_{\beta_1 \cdots \beta_{p-k}} = \tilde{\nu}_{\beta_1 \cdots \beta_{p-k}, m+1, \ldots, n} \quad (p \leq m, \text{ case } 2', \text{ for } p = 0 \text{ also case } 1') \\
(67W) & \quad \tilde{\nu}_{\beta_1 \cdots \beta_i} = \delta \tilde{\nu}_{\beta_1 \cdots \beta_i} \quad (p \geq k, \text{ case } 3'); \\
\text{e.g. for } m = n - 1, \quad p = n - 1, \quad n \geq 2:
\end{align*}

\begin{align*}
(68W) & \quad \tilde{\nu} = \omega \tilde{\nu}_n \quad (\text{case } 2'); \\
(69W) & \quad \tilde{\nu}_\beta = \delta \tilde{\nu}_\beta \quad (\text{case } 3'); \quad \text{together only valid in case } 4 \text{ (not in case } 4').
\end{align*}

Covariant $W-p$-vectors, $p \geq k$:

\begin{align*}
(70W) & \quad \tilde{w}_{\beta_1 \cdots \beta_p} = \omega B_{\beta_1}^{\beta_1} \cdots B_{\beta_p}^{\beta_p} \tilde{w}_{\lambda_1 \cdots \lambda_p} \quad (\text{case } 1'); \\
(71W) & \quad \tilde{w}_{\lambda_1 \cdots \lambda_p} = \omega B_{\lambda_1}^{\lambda_1} \cdots B_{\lambda_p}^{\lambda_p} \tilde{w}_{\beta_1 \cdots \beta_p} \quad (\text{case } 2', \text{ for } p = 0) \\
(72W) & \quad \tilde{w}_{\beta_1 \cdots \beta_p} = \omega B_{\beta_1}^{\beta_1} \cdots B_{\beta_p}^{\beta_p} \tilde{w}_{\lambda_1 \cdots \lambda_p} \quad \text{also case } 1')
\end{align*}

Covariant $W-p$-vectors, $p \geq k$:

\begin{align*}
(73W) & \quad \tilde{w}_{\lambda_1 \cdots \lambda_p} = \left( \frac{p}{k} \right) \tilde{w}_{\lambda_1 \cdots \lambda_{p-k}} B_{\lambda_1}^{\lambda_1} \cdots B_{\lambda_{p-k}}^{\lambda_{p-k}} \tilde{w}_{\lambda_{p-k+1} \cdots \lambda_p} \quad (\text{case } 3').
\end{align*}
Contravariant $W-(n-p)$-vector/densities of weight $+1$, $p \geq k$:

\[(77W)\] \(\frac{\delta^{a_1 \cdots a_l}}{\delta^{a_1 \cdots a_l}} = \frac{1}{b_1 \cdots b_{k-1}} \frac{B_{a_1}^{b_1}}{B_{a_1}^{b_1}} \cdots \frac{B_{a_l}^{b_l}}{B_{a_l}^{b_l}} \frac{\delta^{a_1 \cdots a_l}}{\delta^{a_1 \cdots a_l}} \quad (\text{case } 3')\)

\[(78W)\] \(\frac{\delta^{x_1 \cdots x_l}}{\delta^{x_1 \cdots x_l}} = \frac{1}{b_1 \cdots b_{k-1}} \frac{B_{a_1}^{b_1}}{B_{a_1}^{b_1}} \cdots \frac{B_{a_l}^{b_l}}{B_{a_l}^{b_l}} \frac{\delta^{a_1 \cdots a_l}}{\delta^{a_1 \cdots a_l}} \quad (\text{case } 3')\)

Covariant $W-p$-vectors w.r. to special coordinate system:

\[(80W)\] \(\frac{\bar{\omega}_{\beta_1 \cdots \beta_p}}{\bar{\omega}_{\beta_1 \cdots \beta_p}} = \omega \bar{\omega}_{\beta_1 \cdots \beta_p} \quad (p \leq m, \text{ case } 1');\)

\[(81W)\] \(\frac{\bar{\omega}_{\beta_1 \cdots \beta_p \cdots \beta_{p+m-1}, \cdots, n}}{\bar{\omega}_{\beta_1 \cdots \beta_p \cdots \beta_{p+m+1}, \cdots, n}} = \omega \bar{\omega}_{\beta_1 \cdots \beta_p \cdots \beta_{p+m+1}, \cdots, n} \quad (p \geq k, \text{ case } 4', \text{ for } p = n \text{ also in case } 3');\)

f.i. for $m = n - 1$, $p = 1$, $n \geq 2$

\[(82W)\] \(\frac{\bar{\omega}_{\beta}}{\bar{\omega}_{\beta}}\) (case 1') \quad (\text{not in case } 4').

\[(83W)\] \(\frac{\bar{\omega}}{\bar{\omega}}\) (case 4') \quad (\text{not in case } 4').

Contravariant $W-(n-p)$-vector/densities of weight $+1$ w.r. to special coordinate system:

\[(84W)\] \(\frac{\delta^{a_1 \cdots a_{k-1}}}{\delta^{a_1 \cdots a_{k-1}}} = \omega \frac{\delta^{a_1 \cdots a_{k-1}}}{\delta^{a_1 \cdots a_{k-1}}} \quad (p \leq m, \text{ case } 1');\)

\[(85W)\] \(\frac{\delta^{a_1 \cdots a_{k}}}{\delta^{a_1 \cdots a_{k}}} = \omega \frac{\delta^{a_1 \cdots a_{k}}}{\delta^{a_1 \cdots a_{k}}} \quad (p \geq k, \text{ case } 4', \text{ for } p = n \text{ also in case } 3');\)

f.i. for $m = n - 1$, $p = n - 1$, $n \geq 2$

\[(86W)\] \(\frac{\bar{\omega}}{\bar{\omega}}\) (case 1') \quad (\text{not case } 4').

\[(87W)\] \(\frac{\bar{\omega}}{\bar{\omega}}\) (case 4') \quad (\text{not case } 4').

From (62W) and (63W) follows that for $m = n - 1$ the $W-p$-vector $\bar{v}^{x_1 \cdots x_p}$ is completely determined by $\bar{\omega}^{a_1 \cdots a_p}$ and $\bar{\omega}^{a_1 \cdots a_{p-1}}$. The same holds for $\bar{\omega}_{\lambda_1 \cdots \lambda_p}$, $\bar{\omega}_{b_1 \cdots b_p}$ and $\bar{\omega}_{b_1 \cdots b_{p-1}}$ in (80W) and (81W). All formulae differ from the corresponding ones for ordinary quantities only by a factor $\omega$ at the right. From this follows that the cases 1', 2', 3', 4' play now the same role.
as the cases 1, 2, 3, 4 for ordinary quantities\(^{16}\). We collect the results in the following table:

| \(X_n\) \(p \leq m\) | Case | \(X_m\) | \(X_n\) \(p \geq k\) | Case | \(X_n\) \\
<table>
<thead>
<tr>
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<tbody>
<tr>
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<td>(\overrightarrow{\alpha_1 \cdots \alpha_p})</td>
</tr>
<tr>
<td>(\overrightarrow{\beta_1 \cdots \beta_q})</td>
<td>(\overrightarrow{\beta_1 \cdots \beta_q})</td>
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<tr>
<td>(\overrightarrow{\gamma_1 \cdots \gamma_q})</td>
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<td>(\overrightarrow{\delta_1 \cdots \delta_q})</td>
<td>(\overrightarrow{\delta_1 \cdots \delta_q})</td>
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</tbody>
</table>

\(^{16}\) But it must be remarked that 4' does not include 1', 2' and 3' but only 1, 2 and 3'.
9. Generalisation of Stokes formulae for ordinary quantities and \( W \)-quantities.

In \( X_n \) we consider an orientable and closed \( X_m \) called \( \tau_m \), bounding a simply connected part \( \tau_{m+1} \) of an orientable \( X_{m+1} \). The \( m \)-dimensional element with an interior orientation be \( df^{x_1 \cdots x_m} \), the \( (m+1) \)-dimensional element with an interior orientation of \( X_{m+1} \) be \( df^{x_1 \cdots x_{m+1}} \). The orientations are chosen in such a way that the direction from an interior point of \( \tau_{m+1} \) towards the boundary followed by the orientation of \( df^{x_1 \cdots x_m} \) gives the orientation of \( df^{x_1 \cdots x_{m+1}} \). Be now \( v_{\lambda_1} \cdots \lambda_m \) an \( m \)-vector field that satisfies the ordinary conditions of continuity. Then in a well known way we may derive \( ^{17} \)

\[
\int_{\tau_{m+1}} (\partial_{\mu} v_{\lambda_1} \cdots \lambda_m) df^{\mu \lambda_1 \cdots \lambda_m} = \int_{\tau_m} v_{\lambda_1} \cdots \lambda_m df^{\lambda_1 \cdots \lambda_m}.
\]

If we introduce in this formula \( \psi_{m+1} \cdots \psi_n \), \( \dot{f}_{\lambda_{m+1}} \cdots \lambda_n \) and \( \dot{f}_{\lambda_{m+1} \cdots \lambda_{n-1}} \) we get four other forms: \( ^{18} \)

\[
\int_{\tau_{m+1}} \frac{1}{h!} \left( \partial_{[\mu} v_{\lambda_1} \cdots \lambda_h \right) \dot{f}_{\lambda_1 \cdots \lambda_h \mu} = \frac{1}{(h+1)!} \int_{\tau_m} \dot{f}_{\lambda_1} \cdots \lambda_{h+1} v^{\lambda_1} \cdots \lambda_{h+1} ;
\]

\[
\int_{\tau_{m+1}} (\partial_{[\mu} v_{\lambda_1} \cdots \lambda_m) \dot{f}_{\lambda_1 \cdots \lambda_m \alpha}] = \frac{1}{(h+1)!} \int_{\tau_m} v_{[\lambda_1 \cdots \lambda_m} \dot{f}_{\lambda_1 \cdots \lambda_m \alpha]} \]

\[
\int_{\tau_{m+1}} df^{[\lambda_1 \cdots \lambda_m} \partial_{\mu} v^{\lambda_1 \cdots \lambda_n]} \dot{f}_{\lambda_1 \cdots \lambda_m} = \int_{\tau_m} df^{[\lambda_1 \cdots \lambda_m} v^{\lambda_1 \cdots \lambda_n]} \]

\( ^{17} \) Schouten-Struik, Einführung I, p. 130; Ricci-Kalkül, p. 97, (204); earlier literature is mentioned there. \( df^{x_1 \cdots x_m} \) and \( df^{x_1 \cdots x_{m+1}} \) differ by a factor \( m! \) and \( (m+1)! \) resp. from \( f^{x_1 \cdots x_m} d\tau_m \) and \( f^{x_1 \cdots x_{m+1}} d\tau_{m+1} \) used in R.K.

\( ^{18} \) The formulae (90), (91) an (92) correspond with (211), (210) and (208) in R.K. p. 98 where still multivectors in stead of multivector densities were used. (90) occurs for the special but typical case of Maxwell's equations in D. Van Dantzig, The fundamental equations of electromagnetism independent of metrical geometry [Proc. Cambr. Phil. Soc. 30 (1934), 421–427]. The formulae (89–93) occur as (I), (II), (IV), (III') and (III) in J. Van Weyssenhoff, Duale Größen, Größrotatation, Größdivergenz und die Stokes-Gaußchen Sätze in allgemeinen Räumen [Ann. de la Soc. Pol. de Math. 16 (1937), 127–144], 141, 142. In III there is a misprint, the index \( \varkappa \) being not excluded from the alteration.
The formulae (89) and (90) are valid for an $X_m$ in $X_n$ with an interior orientation. But they cannot be used in the case more frequently occurring in physical applications of an $X_m$ with an exterior orientation (inducing also an exterior orientation in $\tau_{m+1}$). In order to derive the formulae of Stokes for this case we introduce the following $W$-quantities

\[ \frac{\varphi}{k} = \frac{A}{\varphi} \frac{1}{k}. \]

Then we get

\[ \left( \begin{array}{c} (v) \\ \tau_{m+1} \end{array} \right) \int d\overrightarrow{f}_{\lambda_1 \ldots \lambda_{m+1}} = \frac{1}{(h+1)!} \int d\overrightarrow{\tau}_{\lambda_1 \ldots \lambda_{h+1}} \overrightarrow{v}_{\lambda_1 \ldots \lambda_{h+1}} \]

\[ \left( \begin{array}{c} \frac{\varphi}{v} \\ \tau_{m+1} \end{array} \right) \int d\overrightarrow{\tau}_{\lambda_1 \ldots \lambda_{h+1}} = \frac{1}{(h+1)!} \int d\overrightarrow{\tau}_{\lambda_1 \ldots \lambda_{h+1}} \overrightarrow{v}_{\lambda_1 \ldots \lambda_{h+1}} \]

They evidently are independent of the choice of $\overrightarrow{k}$ and therefore invariant under arbitrary transformations of coordinates.
If we consider only changes of the coordinatesystem for which \( \Delta \) in \( \tau_{m+1} \) and on \( \tau_m \) has everywhere the same sign, also formulæ hold like

\[
(103) \quad \frac{1}{h!} \int_{\tau_{m+1}} d\bar{T}_{\lambda_1 \ldots \lambda_h} \, \partial_{\mu} v^{\lambda_1 \ldots \lambda_h \mu} = \frac{1}{(h+1)!} \int_{\tau_m} d\bar{T}_{\lambda_1 \ldots \lambda_{h+1}} \, v^{\lambda_1 \ldots \lambda_{h+1}}.
\]

They express equalities between two \( W \)-scalars, which in this special case can be defined for the whole region concerned.

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