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The simultaneous theory of two linear connections in a generalized geometry with Banach coordinates

by

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Introduction.

The study of a tensor calculus for a Hausdorf space H with coordinates in a Banach space E was begun in a paper [1] by A. D. Michal. Professor Michal points out in this paper that further generalisations are possible, and it is this suggested generalisation that I have endeavored to carry out. Throughout the paper I shall use the notation and terminology introduced in the papers Michal [1]—[7], and Michal-Hyers [1]—[2].

I would like to express my thanks to Professor Michal for not only suggesting this problem to me, but also for the help that he has given me in connection with it.

§ 1.

The first section of the paper deals mainly with formal tensor theorems. I shall not give the proofs of these theorems, as they are exact analogues of those given in Michal [1].

Let us assume we have a Hausdorf space H, with allowable $K^{(m)}$ ordinate systems [2], and which also contains a symmetric linear connection $\Gamma(x, \xi_1, \xi_2)$ [3]. Let us consider in addition to E, another Banach space E_1.

DEF. 1.1. A function $V(x)$ on E to E_1 is said to be a non-holonomic contravariant vector field, if under a transformation $\bar{x} = \bar{x}(x)$ it transforms according to

$$\overline{V}(\bar{x}) = M(x, V(x)) \tag{1.1}$$

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[1]) Michal [1].

[2]) For definition of allowable $K^{(m)}$ coordinate systems see Michal-Hyers [2], 5.

[3]) For definition of contravariant vector field $\xi(x)$ and linear connection $\Gamma(x, \xi_1, \xi_2)$ see Michal [1], 396.

where $M(x, y)$ is a solvable linear function [4]) of y on EE_1 to E_1. We shall refer to $V(x)$ as the vector coordinate of E_1.

Def. 1.2. A change of representation shall mean

(I) a transformation of coordinates $\bar{x} = \bar{x}(x)$,

(II) a transformation of vector coordinates $\bar{V}(\bar{x}) = M(x, V(x))$.

Def. 1.3. Let V be an arbitrary non-holonomic contravariant vector field, and ξ a contravariant vector. A bilinear function $K(x, V, \xi)$ of V, ξ on EE_1E to E_1 is called a non-holonomic contravariant linear connection if it transforms according to

$$(1.2) \qquad \bar{K}(\bar{x}, \bar{V}, \bar{\xi}) = M(x, K(x, V, \xi)) - M(x, V; \xi)\ ^{5)}.$$

Theorem 1.1. Let $K(x, V, \xi)$ be a bilinear function of V, ξ on EE_1E to E_1. A necessary and sufficient condition that

$$(1.3) \qquad \delta V(x) + K(x, V, \delta x)$$

be a non-holonomic contravariant vector field for every Fréchet differentiable non-holonomic contravariant vector field $V(x)$ is that $K(x, V, \delta x)$ be a non-holonomic contravariant linear connection.

The expression (1.3) shal be written $V(x \mid \delta x)$ and is called the covariant differential of $V(x)$.

Theorem 1.2. Let $K(x, V, \delta x)$ be a non-holonomic contravariant linear connection, and $F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_s)$ be a non-holonomic contravariant vector field valued multilinear form in the arbitrary contravariant vectors $\xi_1, \ldots, \xi_n$, and the arbitrary non-holonomic contravariant vectors $V_1, \ldots, V_s$. Further let us assume $F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_s; \delta x)$ exists continuous in x. Then

$$\begin{aligned} F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_s \mid \delta x) &= F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_s; \delta x) \\ &- \sum_{i=1}^{n} F(x, \xi_1, \ldots, \xi_{i-1}, \Gamma(x, \xi_i, \delta x), \xi_{i+1}, \ldots, \xi_n, V_1, \ldots, V_s) \\ (1.4) \qquad &- \sum_{i=1}^{s} F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_{i-1}, K(x, V_i, \delta x), V_{i+1}, \ldots, V_s) \\ &+ K(x, F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_s), \delta x) \end{aligned}$$

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[4]) By a linear function $G(x)$ we mean an additive and continuous function. $G(x)$ is said to be solvable if there exists an inverse function $G^{-1}(x)$ such that $G(G^{-1}(x)) = G^{-1}(G(x)) = x$.

[5]) $M(x, V; \xi)$ means the partial Fréchet differential of $M(x, V)$. We shall write the Fréchet differential of $V(x)$ in the alternative forms $V(x; \delta x)$ or $\delta V(x)$. We shall assume $M(x, y)$ possesses partial Fréchet differentials of order $r + 1$. With this restriction it is easy to verify that all differentiability conditions placed on $V(x)$, or on $K(x, V, \xi)$ of order $\leqq r$, are preserved under a change of representation for $r \geqq m$.

is also a non-holonomic contravariant vector field valued multilinear form in $\xi_1, \xi_2, \ldots, \xi_n, \delta x, V_1, \ldots, V_s$.

We shall call $F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_s \mid \delta x)$ the covariant differential of F.

Let $V(x)$ be a non-holonomic contravariant vector field possessing a continuous second partial Fréchet differential, and further let us assume $K(x, V, \delta x)$ possesses a continuous first partial Fréchet differential. By use of theorems 1.1 and 1.2 we find

$$V(x \mid \delta_1 x \mid \delta_2 x) - V(x \mid \delta_2 x \mid \delta_1 x) = H(x, V, \delta_1 x, \delta_2 x) \tag{1.5}$$

where

$$\begin{aligned} H(x, V, \delta_1 x, \delta_2 x) &= K(x, V, \delta_1 x; \delta_2 x) - K(x, V, \delta_2 x; \delta_1 x) \\ &+ K(x, K(x, V, \delta_1 x), \delta_2 x) - K(x, K(x, V, \delta_2 x), \delta_1 x). \end{aligned} \tag{1.6}$$

By (1.5) and (1.6) we see that $H(x, V, \delta_1 x, \delta_2 x)$ is a skew-symmetric non-holonomic contravariant vector field valued trillinear form in $V, \delta_1 x, \delta_2 x$. This we shall call the non-holonomic curvature form.

It is possible at this stage to prove the generalised Bianchi identity

$$H(x, V, \xi_1, \xi_2 \mid \xi_3) + H(x, V, \xi_2, \xi_3 \mid \xi_1) + H(x, V, \xi_3, \xi_1 \mid \xi_2) = 0,$$

but a simpler proof can be given by means of normal representation theory.

§ 2. *Normal representation theory.*

In the following we shall assume the linear connection $K(x, V, \xi)$ possesses continuous partial Frechet differentials of order r $(r > 1)$, and shall base our normal representation theory on the solution of the differential system

$$\frac{dX(x)}{ds} + K\left(x, X(x), \frac{dx}{ds}\right) = 0, \quad X(0) = X_0 \text{ [6]} \tag{2.1}$$

along some curve $x = x(s)$. We shall consider curves $x = x(s)$ such that the inverse $s = s(x)$ exists continuously in x, and such that $x(s)$ possesses continuous derivatives of at least order r. Along such a curve 2.1 takes the form

[6]) These equations were first discussed in Michal [7], 212, and are taken to be the defining equations of the parallel displacement of a non-holonomic contravariant vector field $X(x)$, parallel to an initial value X_0 along a curve $x = x(s)$.

$$\frac{dX}{ds} = F(s, X), \quad X(0) = X_0, \tag{2.2}$$

where $F(s, X)$ is linear in X. Under these hypotheses Kerner [7]) has shown there exists a unique solution of 2.2 $X = R(s, X_0)$ with the properties

(I) $R(0, X_0) = X_0$,

(II) $R(s, X_0)$ is continuous in both its arguments.

Thus we get a unique solution of (2.1) of the form $X(x) = R(s(x), X_0) = U(x, X_0)$ which is also continuous in both its arguments. We can in addition prove the following properties of $U(x, X_0)$.

(III) There exists a constant $c > 0$, such that, for $0 \leqq s \leqq c$, $U(x, X_0)$ is a solvable linear function of X_0.

(IV) $U(x, X)$ possesses continuous partial Frechet differentials of order $r + 1$.

This latter property is obvious, by the assumption on $K(x, V, \xi)$. To prove (III) let

$$A(s) = R(s, X_0+X_1) - R(s, X_1) - R(s, X_0).$$

Obviously

$$\frac{dA}{ds} = F(s, A), \text{ and } A(0) = 0. \tag{2.3}$$

By the uniqueness of the solution of (2.3) we must have $A(s) \equiv 0$. Hence R and consequently $U(x, X_0)$ is linear in X_0. The solvability of $U(x, X_0)$ follows directly from a theorem of Banach's. [8])

Under suitable restrictions [9]) on H, and the linear connection $\Gamma(x, \xi_1, \xi_2)$, Michal-Hyers define equations of paths to be the unique solution of the differential system

$$\frac{d^2x}{ds^2} + \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) = 0, \quad x(0) = p, \quad \left(\frac{dx}{ds}\right)_0 = \xi. \tag{2.4}$$

They then show the existence of normal coordinate systems $y(p)$ with center P_0 (with coordinate $y = 0$), such that the equations of paths through P_0 have the form $y = s\xi$. If we take the curve $x = x(s)$ of (2.2) to be the equation of a path [10]), and make the

[7]) Kerner [1], 14—19.

[8]) Banach [1], 145.

[9]) Michal-Hyers [1], 8—11.

[10]) That these are suitable curves is easy to verify. Michal-Hyers has shown that solutions of 2.4 are of the form $x = f(p, s\xi)$ where $s\xi = h(p, x)$. Thus $s = \dfrac{\|h(p, x)\|}{\|\xi\|}$.

transformation $x = f(p, y)$ [11]) to normal coordinates, the solution of (2.2) takes the form

$$X = U(x(y), X_0) = G(y, X_0).$$

Def. 2.1. A normal representation with center P_0 shall mean

(I) a normal coordinate system $y(p)$ with center P_0,

(II) the vector coordinate ${}^+V(y)$ defined implicitly by

$$V(x) = G(y, {}^+V(y)). \tag{2.5}$$

It should be noticed that ${}^+V(y)$ is not the vector coordinate corresponding to the change in representation $x = f(p, y)$, $V(x) = M(y, V(y))$.

Theorem 2.1. Under a change of representation $\bar{x} = \bar{x}(x)$, $\bar{V}(\bar{x}) = M(x, V(x))$ the change of normal representation with center P_0 is

$$\begin{aligned} &\text{(I)} \quad \bar{y} = \bar{x}(p; y), \\ &\text{(II)} \quad {}^+\bar{V}(\bar{y}) = M(p, {}^+V(y)). \end{aligned} \tag{2.6}$$

Proof. The first of these relations was proven in Michal-Hyers [1]. To prove the second we let $Z(x)$ be a non-holonomic contravariant vector field parallel [12]), to an arbitrarily chosen initial value Z_0. The normal representation of $Z(x)$ is given by

$$Z(x) = G(y, {}^+Z(y)). \tag{2.7}$$

Since $Z(x)$ is parallel to Z_0 we have

$$\frac{dZ(x)}{ds} + K\left(x, Z, \frac{dx}{ds}\right) = 0, \quad Z(p) = Z, \tag{2.8}$$

where $p = x(P_0)$. But the solution of (2.8) is $Z(x) = G(y, Z_0)$ and since $G(y, Z_0)$ is solvable we must have ${}^+Z(y) = Z_0$. Since $Z(x)$ is a non-holonomic contravariant vector field we have

$$\bar{Z}(\bar{x}) = M(x, Z(x)), \frac{d\bar{Z}}{ds} + \bar{K}\left(\bar{x}, \bar{Z}, \frac{d\bar{x}}{ds}\right) = 0, \quad \bar{Z}_0 = M(p, Z_0). \tag{2.9}$$

As before we obtain ${}^+\bar{Z}(\bar{y}) = \bar{Z}_0$, and by (2.9) we find

$${}^+\bar{Z}(\bar{y}) = M(p, {}^+Z(y)). \tag{2.10}$$

Now let $V(x)$ be any non-holonomic contravariant vector field. Then the normal representations of $V(x)$ and $\bar{V}(\bar{x})$ are given by

$$V(x) = G(y, {}^+V(y)), \quad \bar{V}(\bar{x}) = \bar{G}(\bar{y}, {}^+\bar{V}(\bar{y})). \tag{2.11}$$

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[11]) See Michal-Hyers [1], 9—10.

[12]) See footnote 6.

Solving (2.11) for ${}^{+}\overline{V}(\bar{y})$ we obtain

$$(2.12) \qquad {}^{+}\overline{V}(\bar{y}) = \overline{G}^{-1}\Big(\bar{y},\, M\Big(x,\, G(y,\, {}^{+}V(y))\Big)\Big)$$

where $\overline{G}^{-1}$ is the inverse function of $\overline{G}$. But (2.12) must hold in particular for ${}^{+}Z(y)$ and since ${}^{+}Z(y)$ is arbitrary we have by (2.10) that (2.12) must reduce to

$$ {}^{+}\overline{V}(\bar{y}) = M(p,\, {}^{+}V(y)).$$

§ 3. *The differentials of $G(y, X_0)$ and its inverse $G^{-1}(x, \overline{X}_0)$.*

To develop the theory of normal vector forms, and tensor extensions of multilinear forms, explicit expressions for the differentials of $G(y, \overline{X}_0)$ and of its inverse are necessary. However since the method of obtaining these differentials is essentially the same as that developed in Michal-Hyers [1] (p. 11—13), I shall merely state the results here.

Let us define the functions $K_m(x, V, \xi_1, \ldots, \xi_m)$ by the following recurrence relation

$$K_2(x, V, \xi_1, \xi_2) = \frac{1}{2} P\Big\{K(x, V, \xi_1; \xi_2) - K(x, K(x,V,\xi_2),\xi_1) - K(x, V, \Gamma(x, \xi_1, \xi_2))\Big\},$$

$$(3.1) \quad K_m(x, V, \xi_1, \ldots, \xi_m) = \frac{1}{m} P\Big\{K_{(m-1)}(x, V, \xi_1, \ldots, \xi_{m-1}; \xi_m) - K_{(m-1)}(x, K(x, V, \xi_m), \xi_1, \ldots, \xi_{m-1}) - \sum_{i=1}^{m-1} (K_{(m-1)}(x, V, \xi_1, \ldots, \xi_{i-1}, \Gamma(x, \xi_i, \xi_m), \xi_{i-1}, \ldots, \xi_{m-2})\Big\}$$

where $P\{\ldots\}$ means the sum of terms obtained by a cyclic permutation of the ξ's.

With these definitions we obtain the following results

$$(3.2) \quad G(0, X_0) = X_0,$$

$$(3.3) \quad G(0, X_1; \delta_1 y) = -K(p, X_0, \delta_1 x),$$

$$(3.4) \quad G(0, X_0; \delta_2 y; d_2 y; \ldots; \delta_m y) = -K_{(m)}(p, X_0, \delta_1 x, \ldots, \delta_m x) \qquad (m \leqq r+1).$$

To obtain the differentials of the inverse function we have the identity $G^{-1}(x, G(y, X_0)) = X_0$. From this we can obtain

$$(3.5) \qquad G^{-1}(p, X_0) = X_0,$$

$$(3.6) \qquad G^{-1}(p, X_0; \delta x) = K(p, X_0, \delta x).$$

From the normal representations of $V(x)$ and $V(x \mid \delta x)$ we can verify that

$$(3.7)\quad K(x, V, \delta x) = G\big(y, {}^{+}K(y, {}^{+}V(y), \delta y)\big) - G(y, {}^{+}V(y); \delta y).$$

Evaluating (3.7) at $y = 0$, and using (3.3), we find

$$(3.8)\qquad {}^{+}K(0, V_0, \delta y) = 0.$$

§ 4. *Tensor extensions of multilinear forms and normal vector forms.*

Let $F(x, V_1, \ldots, V_s, \xi_1, \ldots, \xi_n)$ be a non-holonomic contravariant vector field valued multilinear form in the arbitrary non-holonomic contravariant vectors $V_1, \ldots, V_s$, and the arbitrary contravariant vectors $\xi_1, \ldots, \xi_n$.

Def. 4.1. The k^{th} extension

$$F(x, V_1, \ldots, V_s, \xi_1, \ldots, \xi_n \mid \xi_{n+1}, \ldots, \xi_{n+k})$$

of the form F is defined by

$$(4.1)\quad \begin{aligned} &F(p, V_1, \ldots, V_s, \xi_1, \ldots, \xi_n \mid \xi_{n+1}, \ldots, \xi_{n+k}) \\ &= {}^{+}F(0, {}^{+}V_1, \ldots, {}^{+}V_s, {}^{+}\xi_1, \ldots, {}^{+}\xi_n; {}^{+}\xi_{n+1}; \ldots; {}^{+}\xi_{n+k}) \end{aligned}$$

where $p = x(P_0)$ is any point of the coordinate domain of the coordinate system $x(P)$ and P_0 is the center of the normal representation.

Theorem 4.1. The k^{th} extension of $F(x, V_1, \ldots, V_s, \xi_1, \ldots, \xi_n)$ is again a non-holonomic contravariant vector field valued multilinear form in $\xi_1, \ldots, \xi_{n+k}, V_1, \ldots, V_s$.

Proof. Under a change of representation $\bar{x} = \bar{x}(x)$, $\overline{V}(\bar{x}) = M(x, V(x))$ the normal representations of F and $\overline{F}$ are related by

$$(4.2)\qquad \begin{aligned} &{}^{+}\overline{F}(\bar{y}, {}^{+}\overline{V}_1, \ldots, {}^{+}\overline{V}_s, {}^{+}\bar{\xi}_1, \ldots, {}^{+}\bar{\xi}_n) \\ &= M(p, {}^{+}F(y, {}^{+}V, \ldots, {}^{+}V_s, {}^{+}\xi_1, \ldots, {}^{+}\xi_n)). \end{aligned}$$

Taking differentials of (4.2) and evaluating at $y = 0$ we obtain

$$(4.3)\qquad \begin{aligned} &\overline{F}(\bar{p}, \overline{V}_1, \ldots, \overline{V}_s, \xi_1, \ldots, \xi_n \mid \xi_{n+1}, \ldots, \xi_{n+k}) \\ &= M(p, F(p, V_1, \ldots, V_s, \xi_1, \ldots, \xi_n \mid \xi_{n+1}, \ldots, \xi_{n+k})). \end{aligned}$$

Since p is any point of the coordinate domain, the theorem is proven.

Theorem 4.2. The first extension of a non-holonomic contravariant vector field valued multilinear form $F(x, V_1, \ldots, V_s, \xi_1, \ldots, \xi_n)$ is equal to its covariant differential.

This follows by taking the differential of

$$(4.4)\qquad F(x, V_1, \ldots, V_s, \xi_1, \ldots, \xi_n) = G\big(y, {}^+F(y, {}^+V_1(y), \ldots, {}^+V_s(y), {}^+\xi_1(y), \ldots, {}^+\xi_n(y))\big)$$

and evaluating at $y = 0$.

Def. 4.2. The function $C_m(x, V, \xi_1, \ldots, \xi_{m+1})$ defined by

$$(4.5)\quad C_{(m)}(p, V, \xi_1, \ldots, \xi_{m+1}) = {}^+K(0, {}^+V, {}^+\xi_1; {}^+\xi_2; \ldots; {}^+\xi_{m+1})$$

is called the m^{th} non-holonomic normal vector form.

Theorem 4.3. The m^{th} non-holonomic normal vector form is a non-holonomic contravariant vector field valued multilinear form in V, $\xi_1, \ldots, \xi_n$.

The proof is similar to that of Theorem 4.1.

Theorem 4.4. The first non-holonomic normal vector form is given by

$$(4.6)\qquad C_{(1)}(x, V, \xi_1, \xi_2) = \frac{1}{2} H(x, V, \xi_1, \xi_2).$$

Proof. By taking the differential of (3.7) and evaluating at $y = 0$ we find

$$(4.7)\qquad C_{(1)}(p, V, \xi_1, \xi_2) = K(p, V, \xi_1; \xi_2) - K_{(2)}(p, V, \xi_1, \xi_2) - K(p, V, \Gamma(p, \xi_1, \xi_2)).$$

Substituting for $K_{(2)}(p, V, \xi_1, \xi_2)$ we obtain relation (4.6).

Theorem 4.5. Let $F(x, V_1, \ldots, V_m)$ be a non-holonomic contravariant vector field valued multilinear form in the arbitrary non-holonomic contravariant vectors $V_1, \ldots, V_m$ and assume $F(x, V_1, \ldots, V_m; \delta_1 x, \delta_2 x)$ exists continuous in x, then

$$(4.8)\qquad F(x, V, \ldots, V_m \mid \xi_1, \xi_2) = \frac{1}{2}\Big\{F(x, V_1, \ldots, V_m \mid \xi_1 \mid \xi_2) + F(x, V_1, \ldots, V_m \mid \xi_2 \mid \xi_1)\Big\}.$$

The proof for the general case does not differ in principle from the proof for $m = 1$.

Proof for $m = 1$.

The normal representation of $F(x, V \mid \xi_1 \mid \xi_2)$ is at $y = 0$

$${}^+F(0, {}^+V \mid {}^+\xi_1 \mid {}^+\xi_2) = {}^+F(0, {}^+V; {}^+\xi_1; {}^+\xi_2) - {}^+F(0, {}^+K(0, {}^+V, {}^+\xi_1; {}^+\xi_2)) + {}^+K(0, {}^+F(0, {}^+V), {}^+\xi_1; {}^+\xi_2).$$

By (4.5) and (4.6) this leads to

$$(4.9)\qquad {}^+F(0, {}^+V \mid {}^+\xi_1 \mid {}^+\xi_2) = {}^+F(0, {}^+V; {}^+\xi_1; {}^+\xi_2) - \frac{1}{2} {}^+F(0, H(p, V, \xi_1, \xi_2)) + \frac{1}{2} H(p, F(p, V), \xi_1, \xi_2).$$

Since $H(p, V, \xi_1, \xi_2)$ is skew-symmetric in ξ_1, ξ_2 and ${}^+F(0, {}^+V; {}^+\xi_1; {}^+\xi_2)$ is symmetric [13]) in ${}^+\xi_1$, ${}^+\xi_2$ we have

$$ {}^+F(0, {}^+V; {}^+\xi_1; {}^+\xi_2) = \frac{1}{2}\left\{ {}^+F(0, {}^+V \mid {}^+\xi_1 \mid {}^+\xi_2) + {}^+F(0, {}^+V \mid {}^+\xi_2 \mid {}^+\xi_1)\right\}. $$

This relation implies (4.8).

Replacement Theorem.

Consider a functional $R_{\alpha\beta\ldots\delta}[f_1(\alpha_1, \alpha_2), f_2(\beta_1, \beta_2, \beta_3), \ldots, f_l(\varrho_1, \ldots, \varrho_{l+1}), g_1(T_1, \gamma_1), g_2(T_2, \mu_1, \mu_2), \ldots, g_r(T_2, \delta_1, \ldots, \delta_r) \,\|\, \xi_1, \ldots, \xi_n, V_1, \ldots, V_n]$ whose arguments are multilinear functions $f_1, \ldots, g_r$ and whose value is a multilinear function of $\xi_1, \ldots, \xi_n, V_1, \ldots, V_n$.

Def. 4.3. A non-holonomic contravariant vector field valued multilinear form

$F(x, \xi_1, \ldots, \xi_n, V_1, \ldots, V_m)$
$= R_{\alpha\beta\ldots\delta}[\Gamma(x, \alpha_1, \alpha_2), \Gamma(x, \beta_1, \beta_2; \beta_3), \ldots, \Gamma(x, \varrho_1, \varrho_2, \varrho_3; \ldots; \varrho_{l+1}),$
$K(x, T_1, \gamma_1), K(x, T_2, \mu_1; \mu_2), \ldots, K(x, T_2, \delta_1; \delta_2; \ldots; \delta_r) \,\|\, \xi_1, \ldots,$
$\xi_n, V_1, \ldots, V_s]$

will be called a differential invariant of order (l, r) if R as a functional retains its form under a change of representation.

Theorem 4.6. Every differential invariant can be expressed in terms of the normal vector forms [14]) $A_k(x, \gamma_1, \ldots, \gamma_{l+2})$ and the non-holonomic normal vector forms $C_{(r)}(x, T, \varrho_1, \ldots, \varrho_{l+1})$ by the following process.

(I) $\Gamma(x, \alpha_1, \alpha_2)$ is replaced by zero.

(II) $\Gamma(x, \gamma_1, \gamma_2; \gamma_3; \ldots; \gamma_{j+2})$ is replaced by $A_j(x, \gamma_1, \gamma_2, \ldots, \gamma_{j+2})$.

(III) $K(x, T, \mu)$ is replaced by zero.

(IV) $K(x, T, \delta_1; \delta_2; \ldots; \delta_{k+1})$ is replaced by $C_{(k)}(x, T, \delta_1, \ldots, \delta_{k+1})$.

The proof is obvious from the normal representation theory. As an example of this process we take

$$ \begin{aligned} H(x, V, \xi_1, \xi_2) &= K(x, V, \xi_1; \xi_2) - K(x, V, \xi_2; \xi_1) \\ &+ K(x, K(x, V, \xi_1), \xi_2) - K(x, K(x, V, \xi_2), \xi_1). \end{aligned} $$

By our replacement process we have

$$ H(x, V, \xi_1, \xi_2) = C_{(1)}(x, V, \xi_1, \xi_2) - C_{(1)}(x, V, \xi_2, \xi_1). \tag{4.10} $$

(4.10) is obviously another form of the identify (4.6).

§ 5. *An infinite dimensional example.*

We take E to be the Banach space of continuous functions $x(m)$ on closed interval (a, b) as defined in Michal-Hyers [2]

[13]) Kerner [2], 549.

[14]) See Michal-Hyers [1], 13—17.

(p. 329—332), and take E_1 to be the Banach space of continuous functions $V(\mu)$ defined on a closed interval (c, d). We introduce the following notation.

1) Latin letters $m, n, \ldots$ shall be variables ranging over (a, b).

2) Greek letters $\mu, \nu, \ldots$ shall be variables ranging over (c, d).

3) An element $x(m)$ of E is written by the Michal convention as x^m or x_m, and similarly $V(\mu)$ of E_1 shall be written V^μ or V_μ.

4) A repetition of an index once as a superscript and once as a subscript shall mean integration over the corresponding interval.

With these conventions we take $M(x, V)$ to have the form

$$M(x, V) = M^{(\alpha)}V^\alpha + M^\alpha_\beta[x^\tau]V^\beta \tag{5.1}$$

where $M^{(\alpha)}[x^\tau]$, $M^\alpha_\beta[x^\tau]$ are functionals on E to E_1 with the following properties:

(I) M^α, M^α_β possess continuous Fréchet differentials of order $(r+1)$.

(II) The Fredholm determinant $D\left[\dfrac{M^\alpha_\beta}{M^\alpha}\right] \neq 0$.

Thus $M(x, V)$ is solvable linear in V. The linear connection [15]) is taken to be of the form

$$K(x, V, \xi) = K^\alpha_{\beta m}[x^\tau]\, V^\beta \xi^m + K^\alpha_m[x^\tau]\xi^m \tag{5.2}$$

where the functionals $K^\alpha_{\beta m}$, K^α_m possess Fréchet differentials of order r.

With these definitions and assumptions the theory of the paper applies to these infinitely dimens onal function spaces.

In concluding this paper I would like to point out that these results are in the main generalisations of results obtained for the n-dimensional space by Michal and Botsford [1]. However in their treatment of normal representation they assumed analyticity of the functions involved. The treatment I have given is patterned after that given by Michal-Hyers for abstract spaces.

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[15]) See Michal [7], 212.

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