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On the directions of Borel of meromorphic functions of finite order $> \frac{1}{2}$

by

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The object of this paper is to prove the following:

THEOREM IV. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions (E) ¹⁾. Suppose that, in an angle A of vertex 0 and of measure $\frac{\pi}{k}$ ($\frac{1}{2} < k < \varrho$), we have*

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, a, A)}{V(r)} = \beta > 0,$$

for a value of a .

There exists, in an arbitrary angle A' containing A and of vertex 0 , at least one semi-infinite line D such that for an arbitrary angle Ω of vertex 0 and of bisector D , we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{n(r, \pi, \Omega)}{V(r)} > 0,$$

for all elements π , except at the most two, in the family $K(\eta, f)$, where $K(\eta, f)$ denotes the aggregate of all the distinct constants and the meromorphic functions $\pi(z)$ satisfying

$$T(r, \pi) < \eta(r)V(r), \quad r > r_0(\pi), \quad \lim_{r \rightarrow \infty} \eta(r)V(r) = \infty,$$

where $\eta(r)$ is an infinitesimal.

In reality, the foregoing theorem is a complement to the theorem due to Valiron as follows:

THEOREM of Valiron ²⁾. *$f(z)$ is a meromorphic function of*

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¹⁾ See p. [3] . . .

²⁾ Acta Math. 47 (1926), 137—138.

finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions (E). Suppose that, in an angle A of vertex 0 and of measure $\frac{\pi}{k}$ ($\frac{1}{2} < k < \varrho$), we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

for a value of a , and let A' be an arbitrary angle containing A and of vertex 0 .

There exist three positive finite numbers h, h_1, h_2 and an infinite sequence of positive numbers (R_m) , such that

$$\lim_{m \rightarrow \infty} \frac{\log T(R_m, f)}{\log R_m} = \varrho, \quad R_{m+1} > hR_m$$

in relation with the following property: in the region Δ_m , being the common region of A' and the circular ring $R_m < |z| < hR_m$, the function $f(z)$ takes $kT(R_m, f)$ times all values c except at the most two provided that $m > m_c$ where $h_1 < k < h_2$.

In the whole paper, $n(r, \varphi, \Omega)$ denotes the number of zeros of the function $f(z) - \varphi(z)$ in the common part of the region Ω and the circle $|z| \leq r$; and $N(r, \varphi, \Omega)$ denotes the corresponding density

$$\int_0^r \frac{n(r, \varphi, \Omega) - n(0, \varphi, \Omega)}{r} dr + n(0, \varphi, \Omega) \log r.$$

1. The present work is based principally upon the following THEOREM of Rauch ³. Let $f(z), P(z), Q(z), R(z)$ be four distinct meromorphic functions in a region (Δ) and (D) a region contained in (Δ) . Divide the region (D) into p partial regions (D_i) and let (Γ_i) be the circle concentric to the smallest circle containing (D_i) and radius 20 times larger. Suppose that the following conditions are satisfied:

(C₁) $\left\{ \begin{array}{l} \text{The number of the zeros of } f(z) - c \text{ in } (D) \text{ is superior to} \\ M \text{ for all values of } c \text{ in a certain circle of radius } \frac{1}{2} \text{ on the} \\ \text{Riemann sphere.} \end{array} \right.$

(C₂) $\left\{ \begin{array}{l} \frac{1}{\text{area of } (\Gamma_i)} \int \int_{(\Gamma_i)}^+ \log \left(|P(z) + Q(z)| + \frac{1}{|P(z) - Q(z)|} + \right. \\ \left. + \frac{1}{|Q(z) - R(z)|} + \frac{1}{|R(z) - P(z)|} \right) d\sigma < \frac{M}{p}. \end{array} \right.$

³) Journ. de Math. (9) 12 (1933), 133

(C₃) $\left\{ \begin{array}{l} \text{The number of the zeros and poles of the functions } P(z), \\ Q(z), R(z), P(z) - Q(z), Q(z) - R(z), R(z) - P(z) \text{ in} \\ \text{in } (\Delta) \text{ is inferior to } \varepsilon \frac{M}{p}, \text{ where } \varepsilon \text{ is a numerical constant} \\ < \alpha < 1. \end{array} \right.$

Then there exists at least one circle (Γ_i) in which the number of the zeros of the function $f(z) - \pi(z)$ is superior to $c_1 \frac{M}{p}$ for at least one function $\pi(z)$ out of the three $P(z), Q(z), R(z)$, where c_1 is a numerical constant.

This theorem is valid when $\frac{M}{p}$ is superior to a numerical constant.

It is to be remarked that, for the condition (C₃), the numbers of the zeros and the poles of the constants 0 and ∞ are counted as zero.

2. Let $f(z)$ be a meromorphic function of positive finite order ϱ . We are going to show that there exist continuous functions $V(r)$ adjoined to the characteristic function $T(r, f)$ of $f(z)$, satisfying the following conditions:

$$(E) \left\{ \begin{array}{l} \lim_{r \rightarrow \infty} \frac{V(hr)}{V(r)} = h^\varrho, \quad \text{for every } h > 0, \\ \lim_{r \rightarrow \infty} \frac{\log V(r)}{\log r} = \varrho, \\ T(r, f) \leq V(r), \quad r > r_0, \\ \overline{\lim}_{r \rightarrow \infty} \frac{T(r, f)}{V(r)} = 1. \end{array} \right.$$

In fact, after Valiron⁴), there exist continuous functions $\varrho(r)$ differentiable in adjacent closed intervals of which the end points are finite in number at finite distance, satisfying the following conditions:

$$(1) \quad \begin{array}{l} \lim_{r \rightarrow \infty} \varrho(r) = \varrho, \quad \lim_{r \rightarrow \infty} (r\varrho'(r) \log r) = 0, \\ T(r, f) \leq r^{\varrho(r)}, \quad \overline{\lim}_{r \rightarrow \infty} \frac{T(r, f)}{r^{\varrho(r)}} = 1. \end{array}$$

$\varrho(r)$ is known as a proximate order of $f(z)$. If we put $V(r) = r^{\varrho(r)}$, evidently, the last three conditions in (E) are satisfied. Let us show that it satisfies also the first condition.

⁴) C. R. 194 (1932), 1305—1306.

The case $h = 1$ is trivial, and the case $h < 1$ follows immediately from the case $h > 1$. Consider this case. Put

$$r = e^x, \quad V(r) = e^{x\omega(x)}, \quad \omega(x) = \varrho(e^x),$$

then

$$\lim_{x \rightarrow \infty} \omega(x) = \varrho,$$

and from (1) the first condition in (E) is equivalent to

$$(2) \quad \lim_{x \rightarrow \infty} [(H + x)\omega(H + x) - x\omega(x)] = \varrho H, \quad H = \log h.$$

But

$$\begin{aligned} \lim_{x \rightarrow \infty} [(H + x)\omega(H + x) - x\omega(x)] &= \\ &= \lim_{x \rightarrow \infty} \int_x^{x+H} (x\omega(x))' dx = \lim_{x \rightarrow \infty} \int_x^{x+H} x\omega'(x) dx + H\varrho, \end{aligned}$$

and from (1)

$$\lim_{x \rightarrow \infty} \int_x^{x+H} x\omega'(x) dx = 0,$$

hence we have (2) and we see that the function $V(r) = r^{\varrho(r)}$ satisfies all the conditions (E).

3. In this section, we shall prove the theorem as follows which is analogous to the foregoing theorem of Valiron and also of fundamental importance:

THEOREM I. *$f(z)$ is a meromorphic function of finite order $\varrho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions (E). Suppose that for an angle A of vertex 0 and of measure $\frac{\pi}{k}$ ($\frac{1}{2} < k < \varrho$), we have*

$$(3) \quad \overline{\lim}_{r \rightarrow \infty} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

for a value of a .

There exists at least one sequence (R_n) of values of r , $\lim_{n \rightarrow \infty} R_n = \infty$, such that

$$(4) \quad n[S(n, B), f = c] > K(\varrho, k')V(R_n)$$

for all values of c in a certain circle (C_n) of radius $\frac{1}{2}$ on the Riemann sphere, where $n[S(n, B), f = c]$ denotes the number of the zeros of $f(z) - c$ in the common part $S(n, B)$ of the ring $\frac{R_n}{1+S} \leq |z| \leq R_n$

and an arbitrary angle B containing A , of vertex 0 and of measure $\frac{\pi}{k'}$ ($\frac{1}{2} < k' < k$), and s is a suitably chosen positive constant.

This theorem is established in modifying a method due to Valiron as follows.

From the second fundamental theorem of R. Nevanlinna, we see easily that for every value of a except at the most two, there exists at least one angle A of vertex 0 and of measure $\frac{\pi}{k}$ ($\frac{1}{2} < k < \rho$), such that (3) is satisfied. Hence, that hypothesis is possible.

It follows from (3),

$$(5) \quad n(r, a, A) < (1 + \varepsilon)\rho eV(r)$$

for $r > r_0$, and

$$(6) \quad n(r, a, A) > \beta H(\rho)V(r)$$

for a sequence of values of r tending to infinity.

Let B be an arbitrary angle containing A , of vertex 0 and of measure $\frac{\pi}{k}$ ($\frac{1}{2} < k < \rho$). Without loss of generality, we may suppose that the bisectors of A and B coincide with the positive real axis. Make the transformation $Z = z^{-k'}$, where Z is real when z is so, and then the transformation $Z = 1 - z$, so that the function $f(z)$ in the angle B corresponds to a meromorphic function $F(z)$ in the unit circle $|z| < 1$. From (6) we have

$$(7) \quad n(r, F = a) > \beta H_1(\rho)V(r)$$

for a sequence of values of r tending to 1. Hence

$$(8) \quad T\left(r, \frac{1}{F-a}\right) \geq N(r, a) > (1-r)n(2r-1, F=a) > \\ > \beta H_2(\rho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right]$$

for a sequence of values of r tending to 1. But

$$T\left(r, \frac{1}{F-a}\right) = T(r, F) + h(r, a)$$

$h(r, a)$ being bounded when a is fixed, therefore

$$(9) \quad T(r, F) > \beta H_3(\rho)(1-r)V\left[\left(\frac{1}{1-r}\right)^{\frac{1}{k'}}\right]$$

for a sequence of values of r tending to 1. On the other hand, it is known that

$$(10) \quad T(r, F) < H_4(\varrho) (1-r)V \left[\left(\frac{1}{1-r} \right)^{\frac{1}{k'}} \right]^5$$

for $r > r_0$, which shows that the hypothesis is essential and that $F(z)$ is of finite order.

The foregoing calculations are due to Valiron.

Now, from another well known theorem of Valiron⁶⁾, we deduce that there exists at least one circle $C(r)$ of radius $\frac{1}{2}$ on the Riemann sphere, such that for all values c in that circle, we have

$$(11) \quad N(r, F = c) > \frac{\beta}{4} H_3(\varrho)(1-r)V \left[\left(\frac{1}{1-r} \right)^{\frac{1}{k'}} \right]$$

for all values of $r > r_0^1$ in the sequence for which (9) is satisfied, and that for all values of $R > R_0$,

$$(12) \quad T\left(R, \frac{1}{F-c}\right) < T(R, F) + C(F),$$

$$(13) \quad T\left(R, \frac{1}{f-c}\right) < T(R, f) + C_1(f)$$

$C(F)$, $C_1(f)$ being constants depending only upon $F(z)$ and $f(z)$ respectively.

From (9), (10), (12), we have for all values of c in $C(r)$

$$(14) \quad n(r, F = c) > H(\varrho, k')V(r)$$

for at least one sequence of values of r tending toward ∞ . In passing back to $f(z)$, it follows from (14), for all values of c in (C_n) ,

$$(15) \quad n(R_n, c, B) > H_1(\varrho, k')V(R_n)$$

for at least one sequence (R_n) tending toward infinity with n .

Moreover, from (13) we have

$$(16) \quad n\left(\frac{R_n}{1+s}, c, B\right) < \mu \left(\frac{2}{1+s}\right)^\varrho V(R_n) \quad \left(\frac{R_n}{1+s} > R_0\right)$$

for all values of c in (C) .

It follows from (15) and (16), by choosing s such that $H_1(\varrho, k') - \mu \left(\frac{2}{1+s}\right)^\varrho > K(\varrho, k')$,

⁵⁾ VALIRON, l. c. ²⁾, 136.

⁶⁾ VALIRON, l. c. ²⁾, 123—124.

$$n(S(n, B), f = c) > K(\varrho, k')V(R_n),$$

where $n(S(n, B), f - c)$ denotes the number of zeros of the function $f(z) - c$ in the region $S(n, B)$ common to B and the ring $\frac{R_n}{1+S} < |z| \leq R_n$, and for all values of c in (C_n) .

Thus the foregoing theorem is proved.

4. For the sake of convenience, we shall employ the following notation. Let $T^*(r, \varphi)$ be $T(r, \varphi)$ if $\varphi(z) \not\equiv \infty$, and be 0 if $\varphi(z) \equiv \infty$. Let $C^*(\varphi)$ be $C(\varphi) = T\left(r, \frac{1}{\varphi}\right) - T(r, \varphi)$ if $\varphi(z) \not\equiv 0, \infty$, and be 0 if $\varphi(z) \equiv 0, \infty$. $T(r, \varphi)$ is the characteristic function of $\varphi(z)$.

THEOREM II. $f(z)$ is a meromorphic function of finite order $\rho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions (E).

Suppose that for an angle A of vertex 0 and of measure $\frac{\pi}{k}\left(\frac{1}{2} < k < \varrho\right)$, we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

for a value of a , and let (R_n) be the sequence of values of r in Theorem I.

There exists, in an arbitrary angle A' containing A and of vertex 0, at least one sequence of circles $\Gamma(n)$

$$(17) \quad |z - x(n)| = \alpha |x(n)|, \quad \frac{R_n}{1+S} < |x(n)| < R_n,$$

such that if $P(z), Q(z), R(z)$ are any three distinct meromorphic functions satisfying the conditions

$$(18) \quad \begin{cases} T^*[(1 + \alpha)R_n, \varphi] < \alpha^4 V(R_n), & \varphi(z) \equiv P(z), Q(z), R(z); \\ C^*(\varphi) > -\alpha^4 V(R_n), \\ \varphi(z) \equiv P(z), Q(z), R(z), P(z) - Q(z), Q(z) - R(z), R(z) - P(z), \end{cases}$$

then we have

$$(19) \quad n(\Gamma(n), f - \pi) > \alpha^3 V(R_n)$$

for at least one function out of the three $P(z), Q(z), R(z)$, where $n(\Gamma(n), f - \pi)$ denotes the number of the zeros of $f(z) - \pi(z)$ in $\Gamma(n)$.

This theorem is valid when $\frac{1}{\alpha}$ and $\alpha^4 V(r)$ are greater than a certain constant.

Let B be an angle having the properties given in Theorem I, and contained in A' .

We are going to apply the theorem of Rauch. Let the region $S(n, B)$ defined in theorem I be the region (D) and let $M = K(\rho, k')V(R_n)$. Then by Theorem I, the condition (C_1) in the Theorem of Rauch is satisfied. Divide the angle B into equal sectors of measure $\frac{\pi}{\alpha_1 k'}$ by semi-infinite lines issued from the origin and describe the circles

$$|z| = \frac{R_n}{1+S} (1 + \alpha_1)^i \quad \left(\frac{R_n}{1+S} (1 + \alpha_1)^q = R_n; i = 1, 2, \dots, q \right).$$

The region (D) is thus divided into

$$p = c(k') \frac{\log(1+S)}{\alpha_1^2}$$

similar curvilinear rectangles $D_i(n)$, where α_1 is sufficiently small and $c(k')$ is a constant depending only upon k' . Let $D_i(n)$ be the partial regions (D_i) in the theorem of Rauch and $\Gamma_i(n)$ the corresponding circles Γ_i . Then the circles $\Gamma_i(n)$ are contained in the ring

$$(1 - 15\alpha_1) \frac{R_n}{1+S} < |z| < (1 + 15\alpha_1) R_n$$

which is taken for each n as the region (Δ) .

Let $\rho_i(n)$ be the modulus of the center of $\Gamma_i(n)$, then its radius is $k_i(n)\alpha_1\rho_i(n)$, $k_i(n)$ lying between two numerical constants h_1 and h_2 .

In modifying suitably a method due to Rauch ⁷⁾, we see that the conditions (C_2) , (C_3) in his theorem given above are satisfied if the conditions (18) have been imposed, where α is taken to be the largest of $16\alpha_1$ and $h_2\alpha_1$, and when $\frac{1}{\alpha}$ and $\alpha^4 V(R_n)$ are greater than a certain constant.

Hence, by the theorem of Rauch, for each large integer n , there exists at least one circle defined by (17), such that (19) is satisfied.

It is evident that when α is less than a certain constant, all the circles $\Gamma(n)$ are contained in A' .

⁷⁾ RAUCH, l. c., 133—138.

5. Let $H(\alpha, f)$ be the family of meromorphic functions $\pi(z)$ satisfying the condition

$$T^*[(1 + \alpha)r, \pi] < \alpha^4 V(r), \quad r > r_0(\pi).$$

Consider a certain infinite sequence of circles $\Gamma(n)$ in theorem II. We are going to show that for all functions $\pi(z)$ of the family $H(\alpha, f)$, except at the most two, we have

$$n(\Gamma(n'), f - \pi) > \alpha^3 V(R_{n'}), \quad n' > n_0(\pi).$$

Suppose that there are two functions $P(z)$, $Q(z)$ in the family $H(\alpha, f)$, such that

$$\begin{aligned} n(\Gamma(n'), f - P) &\leq \alpha^3 V(R_{n'}), \\ n(\Gamma(n'), f - Q) &\leq \alpha^3 V(R_{n'}), \end{aligned}$$

for an infinite sequence of values of n' . Let $R(z)$ be a function in the family $H(\alpha, f)$ distinct from $P(z)$ and $Q(z)$. Evidently when $n' > n_0(R)$ the conditions (18) are satisfied, hence by theorem II, we have

$$n(\Gamma(n'), f - R) > \alpha^3 V(R_{n'}), \quad n' > n_0(R).$$

The statement is therefore proved and we have the following

THEOREM III. *$f(z)$ is a meromorphic function of finite order $\rho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions (E). Suppose that in an angle A of vertex 0 and of measure $\frac{\pi}{k}$ ($\frac{1}{2} < k < \rho$), we have*

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, a, A)}{V(r)} = \beta > 0$$

for a value of a . Let $H(\alpha, f)$ be the family of meromorphic functions satisfying the condition

$$T^*[(1 + \alpha)r, \pi] < \alpha^4 V(r), \quad r > r_0(\pi).$$

There exists, in an arbitrary angle A' containing A and of vertex 0 , at least one sequence of circles $\Gamma(n)$ defined by (17) such that

$$(20) \quad n(\Gamma(n), f - \pi) > \alpha^3 V(r), \quad n > n_0(\pi),$$

for all functions $\pi(z)$ of the family $H(\alpha, f)$ except at the most two.

This theorem is valid when $\frac{1}{\alpha}$ and $\alpha^4 V(r)$ are greater than a certain constant.

6. Let A_α be the smallest of the angles contained in A' and of vertex 0 in which there are an infinite number of the circles $\Gamma(n)$ in theorem III. Then

$$(21) \quad \overline{\lim}_{r \rightarrow \infty} \frac{n(r, \pi, A_\alpha)}{V(r)} \geq \frac{\alpha^3}{2(1+\alpha)^2}$$

from (20).

Let (α_n) be a sequence of values of α , tending to 0 with $\frac{1}{n}$. Let (D_{α_n}) be the bisector of A_{α_n} and D a limit-line of the semi-lines D_{α_n} . An arbitrary angle Ω of vertex 0 and of bisector D contains then an infinity of the angles A_{α_n} .

Let $K(r, f)$ be the family of meromorphic functions $\pi(z)$ satisfying the condition

$$T^*(r, \pi) \leq \eta(r)V(r), \quad r > r_0(\pi), \quad \lim \eta(r)V(r) = \infty,$$

where $\eta(r)$ is an infinitesimal. It is evident that the family $K(\eta, f)$ is contained in the family $H(\alpha, f)$ for every fixed value of α . Hence from (21) we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{n(r, \pi, \Omega)}{V(r)} > 0$$

for all elements $\pi(z)$ in the family $K(\eta, f)$ except at the most two.

It is also evident that the family $K(\eta, f)$ is the aggregate of all the distinct constants, and the meromorphic functions (non-degenerated to constants) $\pi(z)$ satisfying

$$T(r, \pi) \leq \eta(r)V(r), \quad r \geq r_0(\pi).$$

We have therefore the following

THEOREM IV. ⁸⁾ $f(z)$ is a meromorphic function of finite order $\rho > \frac{1}{2}$. Let $V(r)$ be a continuous function satisfying the conditions (E). Suppose that, in an angle A of vertex 0 and of measure $\frac{\pi}{k}$ ($\frac{1}{2} < k < \rho$), we have

$$\overline{\lim}_{r \rightarrow \infty} \frac{N(r, a, A)}{V(r)} = \beta > 0,$$

for a value of a .

There exists, in an arbitrary angle A' containing A and of vertex 0 at least one semi-line (D) issued from 0 , such that for an arbitrary angle Ω of vertex 0 and of bisector D , we have

⁸⁾ This theorem has been stated in a Note in *Comptes Rendus* **206** (1938), 811–812.

$$\overline{\lim}_{r \rightarrow \infty} \frac{n(r, \pi, \Omega)}{V(r)} > 0$$

for all elements π of the family $K(r, f)$ except at the most two.

It is to be remarked that the foregoing theorem and the theorem IX ⁹⁾ in the Thèse of Rauch do not contain each other and that the family $K(\alpha, f)$ in the latter must be stated analogously to the family $K(\eta, f)$ in our theorem IV.

Finally, the writer wishes to thank Prof. Valiron for his useful criticisms.

(Received April 12th, 1938.)

⁹⁾ RAUCH, l. c. ³⁾, 157.