COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 5 (1938), p. 292-298

<http://www.numdam.org/item?id=CM_1938__5__292_0>

On a theorem in the theory of dimensionality

by

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1. In his note „Zum allgemeinen Dimensionsproblem" [1]) Professor Alexandroff proved the following theorem:

A compact set A of the n-dimensional space R^n has a dimensionality $\leqq r$ if and only if for every $\varepsilon > 0$ it is ε-removable from an arbitrary $(n-r-1)$-dimensional polyhedron.

In this theorem a set A is called ε-removable from a set B (where A and B are subsets of the same metric space R) when there exists an ε-transformation f of A with the property:

$$f(A) \cdot B = 0.$$

The main purpose of this note is to generalize and to make more precise the above result established by Prof. Alexandroff:

THEOREM: *Any subset A of the n-dimensional euclidean space R^n has a dimension $\leqq r$ then and only then, when for any $\varepsilon > 0$ it is ε-removable from every $(n-r-1)$-dimensional plane.* [2])

Remark I. The theorem holds if the word „plane" is replaced by the word „simplex".

Remark II. An r-dimensional set A is ε-removable even from every countable complex of the dimension $\leqq n - r - 1$.

Remark III. [3]) If a set A is r-dimensional there is an $(n-r)$-dimensional plane (or simplex), parallel to a certain $(n-r)$-dimensional coordinate plane of R^n, from which A is not ε-removable.

Consequently, any space R is r-dimensional then and only then, when its topological image in euclidian space of a sufficiently large dimension answers the conditions described. The

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[1]) Gött. Nachrichten **1928**, 37.

[2]) r-dimensional plane of R^n is an r-dimensional euclidian subspace of R^n.

[3]) This remark is due to Prof. Pontrjagin.

problem of such a generalization of his theorem was proposed to me by Prof. Alexandroff to whom I wish to express here my best thanks for the kind attention he has shown me.

Here we shall dwell for a while on some theorems concerning the ε-transformations which have been proved only as to compact or, in the best cases, as to totally bounded spaces. This will enable us to give a more direct proof of our theorem.

2. The theorem we start from is the following:

Any r-dimensional subset A of a metric separable space R may be covered by arbitrarily small open sets, each $r+2$ of which has a vacuous intersection; their number being finite in the case of the subset being totally bounded and countable [4]) *in the opposite case.*

The proof of the theorem in the case when the set is not necessarily totally bounded is the same as when it is totally bounded [5]) except for the number (finite or infinite) of the covering sets involved.

With every countable system of sets $\mathfrak{S} = \{U_1, U_2, \ldots, U_i, \ldots\}$ we associate, as in the case of the finite system, a certain countable complex N, which is said to be the nerve of this system. Vertices of N are in (1-1)-correspondence with the sets of the system $\mathfrak{S}$, and some subset of them form the vertices of a simplex then and only then when the corresponding sets do not have a null intersection. The nerve N is *realized in the field of vertices* E, if all the vertices belong to this field. N is *realized in* $\mathfrak{S}$ or *near* $\mathfrak{S}$, if all the elements of $\mathfrak{S}$ belong to the same space R, and each vertex of N is a point of its corresponding element or of a definite neighborhood of that element. [6])

If $R = R^n$, then N may be considered as a geometrical complex. If n is sufficiently great and the vertices of N are in a general position then the interiors of the simplexes of N do not intersect each other.

Such a realization of a nerve, which is always possible in R^{2r+1}, when $\mathfrak{S}$ has an order equal to r, i.e. if $\dim N = r$, is called an euclidian realization of the nerve N of $\mathfrak{S}$.

Having an arbitrarily small finite or countable covering of the order r of an r-dimensional space R, and the realization of

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[4]) but locally finite, i.e. any element can intersect only a finite number of other elements of the covering.

[5]) See K. Menger's Dimensionstheorie [1928], 158.

[6]) P. Alexandroff & H. Hopf: Topologie I [1935], Neuntes Kapitel, § 3.

the nerve of this covering, we can, as in the finite case, construct a single-valued continuous transformation of the space R in $\bar{N}$; especially we can apply the so-called Kuratowski-transformation $X(p)$ which in the euclidian case gives:

$$X^{-1}[b_{i_0} \dots b_{i_h}] = U_{i_0} \cdots U_{i_h} - \sum_{i \neq i_j} U_i;$$

here b_{i_j} is the vertex of N corresponding to the element U_{i_j} of $\mathfrak{S}$, and $[b_{i_0} \dots b_{i_h}]$ is the interior of the simplex $b_{i_0} \dots b_{i_h} \subset N$. It follows that if a system $\mathfrak{S}$ is an ε-covering of a space R, then $X(p)$ is an ε-mapping of R on $\bar{N}$, i.e. on an r-dimensional complex.

If f is a single-valued continuous transformation of a space R in R^n, then, as in the case when R is compact we get that for a sufficiently small covering of R and for a suitable realization of the nerve of this covering in R^n, $X(p)$ represents an ε-approximation of the given transformation f:

$$\varrho(f(p), X(p)) < \varepsilon.$$

Supposing that $A \subset R^n$ and f is an identical transformation, we get an $\frac{\varepsilon}{2}$-deformation of the set A into the polyhedron $\bar{N}$, i.e. into an r-dimensional complex. On the other hand it may be shown, using our chief theorem[7] (from which the above is independent), that by an arbitrarily small deformation of an r-dimensional set A it is impossible to transform A into a set of a dimension less than r. In fact it would be possible otherwise to remove A by an arbitrarily small deformation of it from every $(n-r)$-dimensional plane, but that contradicts the assumption that A is of the dimension r.

Thus we get the following theorem, the first part of which will be wanted later:

An r-dimensional set A of the space R^n is ε-deformable into an r-dimensional polyhedron (finite or countable according as the set A is bounded or not), but not into a polyhedron (not into any set) of a lower dimension.

3. We shall require further the following

LEMMA: *If a set A does not intersect the simplex S $(A, S \subset R^n)$ it is possible to get a positive distance between the set A and the simplex S by an arbitrarily small deformation of the set A.*

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[7]) Prof. Ephrämowitsch has been so kind as to indicate to me this consequence of our theorem.

Furthermore, at the end of § 4 it will be shown that, if a set A is removable by an arbitrarily small deformation from each simplex of a given finite complex K, then a positive distance between the polyhedron $\bar{K}$ and the set A may be established by an arbitrarily small deformation of the latter.

Proof of the lemma: given $\varepsilon > 0$ let us choose such a number d, that: $0 < d < \varepsilon$ and consider the set F consisting of all points whose distance from S is less than or equal to d. Especially, let $T \subset F$ be the set of those points whose distance from S is equal to d. Let us connect by segments each point of S with all the points of the set T which are at the distance d from this point. We shall call the points of these segments which belong to the simplex S, s-points and those which belong to T, t-points. It is not difficult to show that the set of all these segments fills the whole set $F - S$. No one of these segments intersects any other at a point which does not belong to S. Suppose the contrary: let the segments P and Q intersect in the point o, $o \not\subset S$. Denoting the s-points of the segments P and Q by s_P and s_Q and the t-points by t_P and t_Q respectively, we have: $s_P \neq s_Q$ (for, otherwise, P and Q would coincide with each other) and $t_P \neq t_Q$ (as, otherwise, the point $t_P = t_Q$ being equally distant from the two points s_P, s_Q of S would be less distant from S than from these points, which is impossible). Suppose that

$$\varrho(t_P, o) \leqq \varrho(t_Q, o)\,[8];$$

then

$$\varrho(o, s_P) \geqq \varrho(o, s_Q);$$

for, otherwise, the length of P would be less than that of Q. Therefore we have:

$$\varrho(t_P, o) + \varrho(o, s_Q) \leqq d.$$

But that is impossible. The impossibility of the inequality is evident; but no equality can exist either, since, as already mentioned above, a point which is at distance d from a simplex cannot be at this distance from two different points of the simplex. Thus, the segments do not intersect one another. Let us shift each point of the set $A(F-S)$ with uniform velocity in unit time along the segment on which it lies to the t-point of this segment. We obtain in this way a single-valued continuous

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[8]) If $\varrho(t_Q, o) < \varrho(t_P, o)$, then, in the following argument P and Q must replace each other.

transformation f of $A(F-S)$ into $R^n - U(S, d)$; we define moreover f as the identical transformation in all points of $A[R^n - U(S, d)]$. Then f is a continuous ε-transformation of A all over and $f(A)$ has a positive distance from S, q.e.d.

4. We get now the first part of our theorem direct from what has been said in § 2: by an arbitrarily small deformation of the set A, we transform it into an r-dimensional complex and, by arbitrarily small shifts of the vertices of the latter remove it from any $(n-r-1)$-dimensional finite or even countable polyhedron, in particular from any $R^{n-r-1} \subset R^n$, and, moreover from any $(n-r-1)$-dimensional element.

In order to prove the second part, let us prove first of all that by sufficiently small deformation of the given bounded set A of dimension r it is impossible to remove it from a certain $(n-r)$-dimensional finite polyhedron. From here naturally follows an analoguous statement as to an unbounded set. Let $\varepsilon > 0$ be so small that at a finite ε-covering of the set A by sets closed in it there should be at least one point belonging to $r+1$ elements of the covering. Let us, following Lebesgue [9]), decompose the space R^n in cubes with the side $\eta < \frac{\varepsilon}{3n}$ so that the points belonging to, at least, s, $1 \leqq s \leqq n+1$, cubes lie on $(n-s+1)$-dimensional sides of these cubes.

Let $Q_1, Q_2, \ldots, Q_t$ be all the cubes of the polyhedral neighborhood of that polyhedron [10]) of this decomposition whose cubes intersect the set A.

Let us denote by K the $(n-r)$-dimensional polyhedron formed by all $(n-r)$-dimensional sides of these cubes. It is clear that

$$\varrho\left(A, R^n - \sum_{i=1}^{t} Q_i\right) \geqq \eta.$$

Suppose that by η-deformation of the set A which transforms A into A', it is possible to remove A from K. Sets $Q_i A'$, $1 \leqq i \leqq t$, closed in A', form $\frac{\varepsilon}{3}$-covering of the set A' of the order r at most. Let the sets A_i be the originals („Urbild") of the sets $Q_i A'$ of the deformation in question. As originals of sets closed in A' the sets A_i are closed in A; their aggregate covers A; their

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[9]) Fund. Math. 2 (1921), 256—285.

[10]) i.e. the aggregate of all the cubes of the decomposition in question intersecting that polyhedron.

diameters are less than ε, and each $r+1$ of them has a null set in its intersection; but all this contradicts the choice of the number ε.

It remains to prove the impossibility of removing the given set from a certain $(n-r)$-dimensional simplex and, therefore from the $(n-r)$-dimensional plane which is determined by this simplex.

Suppose that it is possible: A may be ε-removed, by arbitrarily small $\varepsilon > 0$, from every $(n-r)$-dimensional simplex. Let us have any $(n-r)$-dimensional polyhedron

$$K = \sum_{i=1}^{k} S_i, \ \dim S_i \leqq n-r,$$

and any positive number ε. Let us choose ε_1, so that $0 < \varepsilon_1 < \frac{\varepsilon}{k}$ and by an ε_1-deformation f_1 of A establish a positive distance between $f_1(A) = A_1$ and S_1:

$$\varrho(A_1, S_1) = d_1.$$

This is possible in virtue of the above assumption and the lemma of § 3 from which evidently follows: if by an arbitrarily small deformation of a set A it is possible to remove the latter from a simplex S, then it is possible to establish a positive distance between A and S by an arbitrarily small deformation of A.

Let

$$\varepsilon_i, \ f_i, \ A_i, \ d_i; \ i = 1, 2, \ldots, j-1,$$

be already constructed.

Let us choose ε_j so that

$$0 < \varepsilon_j < \min\left(\frac{\varepsilon}{k}, \frac{d_i}{k-i}\right), \ i = 1, 2, \ldots, j-1,$$

and, by an ε_j-deformation f_j of the set A_{j-1}, say $f_j(A_{j-1}) = A_j$, establish a positive distance between A_j and S_j:

$$\varrho(A_j, S_j) = d_j > 0.$$

We shall get

$$\varrho(A_j, S_t) > \frac{k-j}{k-t} d_t \geqq 0, \ t = 1, 2, \ldots, j,$$

for when $j = t$ we had

$$\varrho(A_t, S_t) = d_t > 0$$

and with the following $j - t$ deformations all the shifts, in virtue of the properties of ε_τ, $\tau = t + 1, \ldots, j$, were less than $\frac{d_t}{k-t}$.

Let us perform the same construction for $j = 1, 2, \ldots, k$ and consider the mapping

$$f^*(A) = f_k \cdots f_2 f_1(A) = A^*.$$

f^* being the result of k successive $\frac{\varepsilon}{k}$-deformations of A is an ε-deformation of this set. A^* does not intersect the polyhedron $\bar{K}$, moreover, they are at a positive distance from each other, since

$$A^* = f_k(A_{k-1}) = A_k$$

and

$$\varrho(A_k, S_t) > 0, \ t = 1, 2, \ldots, k. \text{[11]}$$

The contradiction of the fact just established with our former statement proves our theorem completely. The above considerations prove also the generalizations of the lemma mentioned in § 3.

Remarks I and II are already proved. It is obvious that remark III holds too. In fact, in Lebesgue's decomposition of R^n every $(n-r)$-dimensional side is parallel to a certain $(n-r)$-dimensional coordinate plane; but, on the other hand, it was proved already that the set A cannot be removed by an arbitrarily small deformation from one of these sides.

(Received, February 10th, 1937.)

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[11]) The expression $\varrho(A_k, S_t) > \dfrac{k-k}{k-t} d_t$, for $t = k$, is not undetermined, as, according to the definition of f_k:

$$\varrho(A_k, S_k) = d_k > 0.$$